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Optimization on Metric and Normed Spaces

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OPTIMIZATION ON METRIC AND NORMED SPACES

By

ALEXANDER J. ZASLAVSKI
Technion - Israel Institute of Technology
Israel

Springer

Alexander J. Zaslavski
Department of Mathematics
Technion - Israel Institute of Technology
32000 Haifa
Israel
ajzasl@techunix.technion.ac.il

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Preface

The monograph is devoted to the study of optimization problems on complete metric spaces and Banach spaces. It contains a number of recent results obtained by the author in the last ten years. It is a well-known fact that solutions exist for minimization problems on compact metric spaces with lower semicontinuous objective functions. Solutions also exist for minimization problems on metric spaces such that all their bounded closed subsets are compact if objective functions are lower semicontinuous and satisfy a coercivity growth condition. Since in the book we do not use compactness assumptions on spaces where optimization problems are considered the situation becomes more difficult. We overcome this difficulty in the following way: in Chapters 2 and 3 we deal with approximate solutions of optimization problems which always exist and in Chapters 4–10 we use the so-called generic approach which is applied fruitfully in many areas of Analysis (see, for example, [3, 74, 130] and the references mentioned there).

According to the generic approach we say that a property holds for a generic (typical) element of a complete metric space (or the property holds generically) if the set of all elements of the metric space possessing this property contains a G_δ everywhere dense subset of the metric space which is a countable intersection of open everywhere dense sets. In Chapters 4–10 we use this approach in order to establish a generic existence of solutions for various classes of minimization problems.

We begin the study of minimization problems on normed spaces and metric spaces with a discussion of penalty methods in constrained optimization (Chapters 2 and 3). Penalty methods were one of the first methods employed for solving constrained minimization problems about 60 years ago. According to these methods instead of solving a constrained minimization problem we solve a penalized unconstrained minimization problem. It is well-known that when penalty coefficients tend to infinity the solutions of the perturbed unconstrained problems converge to a solution of the original constrained problem.

In Chapters 2 and 3 we study an exact penalty property in constrained optimization — the existence of a penalty coefficient for which any solution of a penalized unconstrained problem is a solution of the original unconstrained problem. Exact penalty properties are of great interest (see, for example, [14, 16, 33] and the references mentioned there) and the results known in the literature show that the existence of exact penalty is related to calmness of the perturbed constraint mapping. In our recent research we established simple sufficient conditions of a different nature for the existence of exact penalty. This research was begun with the work [122] where a sufficient condition for the exact penalty property was established for an equality-constraint problem and for an inequality-constrained problem with one constraint in Banach spaces with locally Lipschitz constraint and objective functions. In our series of subsequent papers the main result of [122] was generalized for different classes of constrained minimization problems and Chapters 2 and 3 are based on these papers. Since in these chapters we consider optimization problems in general Banach spaces and metric spaces, not necessarily finite-dimensional, the existence of solutions of original constrained problems and corresponding penalized unconstrained problems is not guaranteed. By this reason we deal with approximate solutions and establish an approximate exact penalty property which contains the classical exact penalty property as a particular case.

In Chapters 4–10 we use the generic approach in order to show that solutions of minimization problems exist generically for different classes of problems which are identified with corresponding spaces of functions with natural complete metrics. It should be mentioned that the author obtained his first generic existence result in [103] for general Bolza and Lagrange optimal control problems without usual convexity assumptions. This result was published in [108].

The Tonelli classical existence theorem in the calculus of variations [93] is based on two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex (or exhibit a more special convexity property in case of a multiple integral with vector-valued functions) with respect to the last variable. In [103, 108] it was shown that the convexity condition is not needed generically, and not only for the existence but also for the uniqueness of a solution and even for well-posedness of the problem (with respect to some natural topology in the space of integrands). More precisely, in [103, 108] we considered a class of optimal control problems (with the same system of differential equations, the same functional constraints and the same boundary conditions) which is identified with the corresponding complete metric space of cost functions (integrands), say F. We did not impose any convexity assumptions. These integrands are only assumed to satisfy the Cesari growth condition. The main result in [103, 108] establishes the existence of an everywhere dense G_δ-set $F' \subset F$ such that for each integrand in F' the corresponding optimal control problem has a unique solution. The next step was done in a joint paper by Alexander Ioffe and the author [52]. There we introduced a general variational principle having its prototype in the variational principle of Deville, Godefroy and Zizler [31]. A generic existence result in the calculus of variations without convexity assumptions was then obtained as a realization of this variational principle. It was also shown in [52] that some other generic well-posedness results in optimization theory known in the literature and their modifications are obtained as a realization of this variational principle.

The paper [52] became a starting point of the author's research on optimization problems which has been continued in the last ten years. Many results obtained during this period of time are presented in Chapters 4–10 of the book. Among them generic existence results for classes of constrained minimization problems with different type of constraints, for classes of parametric minimization problems, for classes of problems with increasing objective functions, classes of vector minimization problems and infinite horizon optimization problems. Any of these classes of problems is identified with a space of functions equipped with a natural complete metric and it is shown that there exists a G_δ everywhere dense subset of the space of functions such that for any element of this subset the corresponding minimization problem possesses a unique solution and that any minimizing sequence converges to this unique solution. These results are obtained as realizations of variational principles which are extensions or concretization of the variational principle established in [52]. Instead of considering the existence of a solution for a single minimization problem, we investigate it for a class (space) of problems and show that a unique solution exists for most of the problems in the class. A reader can ask how these results can help if one needs to solve a single (individual) problem. It turns out that our results provide a proper explanation of what happens with individual problems in practice. The thing is that because of computational errors and errors in data which are always present, instead of solving an individual problem with certain objective (constraint) function we actually solve a similar problem with an approximation of the objective (constraint) function. Probably this new problem possesses desirable properties which hold for most of the problems. This explains, for example, the fact that sometimes in practice we can get results which are better than their theoretical expectations. This happens when an algorithm works under some conditions which hold for "almost all" problems.

Haifa, December 31, 2009 *Alexander J. Zaslavski*

Contents

1

Introduction

1.1 Penalty methods

Penalty methods are an important and useful tool in constrained optimization. See, for example, [5, 6, 14, 15, 16, 20, 21, 28, 29, 30, 33, 34, 38, 39, 42, 44, 47, 48, 49, 66, 68, 72, 73, 75, 86, 87, 99] and the references mentioned there. In this book we use the penalty approach in order to study constrained minimization problems in infinite-dimensional spaces.

Assume that X is a Banach space with the norm $||\cdot||$ and consider an inequality-constrained problem

$$\text{minimize } f(x) \text{ subject to } g(x) \le c,$$

where $f, g: X \to R^1$ are continuous functions and the set $\{x \in X: \ g(x) \le c\}$ is nonempty, and an equality-constrained problem

$$\text{minimize } f(x) \text{ subject to } g(x) = c,$$

where $f, g: X \to R^1$ are continuous functions and the set $\{x \in X: \ g(x) = c\}$ is nonempty. Here c is a real number.

We associate with these two constrained minimization problems the corresponding penalized unconstrained problems

$$\text{minimize } f(x) + \lambda \max\{g(x) - c, \ 0\} \text{ subject to } x \in X$$

and

$$\text{minimize } f(x) + \lambda|g(x) - c| \text{ subject to } x \in X,$$

where $\lambda > 0$.

It is well known that if the space X is finite-dimensional and the objective function f satisfies a coercivity growth condition, then all these problems possess solutions and solutions of penalized unconstrained problems converge to a solution of the corresponding constrained problem. Indeed, assume that

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X is finite-dimensional, $\lim_{||x|| \to \infty} f(x) = \infty$ and consider the inequality-constrained minimization problem. By the growth condition on f the original inequality-constrained problem possesses a solution x_* and for each $\lambda > 0$ the corresponding penalized unconstrained problem with penalty λ possesses a solution x_λ. It is clear that for each $\lambda > 0$,

$$f(x_\lambda) + \lambda \max\{g(x_\lambda) - c, 0\} \le f(x_*).$$

Assume that a sequence of positive numbers $\{\lambda_k\}_{k=1}^\infty$ tends to infinity. Then by the inequality above and the growth condition on f,

$$\lim_{k \to \infty} \max\{g(x_{\lambda_k}) - c, 0\} = 0$$

and the sequence $\{x_{\lambda_k}\}_{k=1}^\infty$ is bounded. Extracting a subsequence and re-indexing if necessary we may assume that there exists $\bar{x} = \lim_{k \to \infty} x_{\lambda_k}$. Now it is not difficult to see that $g(\bar{x}) \le c$ and $f(\bar{x}) \le f(x_*)$. Hence $\bar{x}$ is a solution of the original inequality-constrained problem. Analogous arguments work for the equality-constrained problem and for constrained minimization problems with several mixed constraints.

In Chapters 2 and 3 we study the existence of an exact penalty for constrained minimization problems. A penalty function is said to have the exact penalty property [14, 16, 33, 44] if there is a penalty coefficient for which a solution of an unconstrained penalized problem is a solution of the corresponding constrained problem.

The notion of exact penalization was introduced by Eremin [38] and Zangwill [99] for use in the development of algorithms for nonlinear constrained optimization. Since that time, exact penalty functions have continued to play a key role in the theory of mathematical programming [5, 6, 15, 20, 21, 28, 29, 34, 39, 42, 47, 48, 49, 66, 68, 72, 73, 86, 87]. For historical reviews of the literature on exact penalization see [14, 16, 33].

Chapters 2 and 3 are based on our recent research where we obtained a number of existence results on the exact penalty with a different nature than previous results known in the literature which usually relate the existence of exact penalty to calmness of the perturbed constraint mapping.

The starting point of this research was the paper [122] where we considered problems with one constraint (an equality-constrained problem and an inequality-constrained problem) in Banach spaces with locally Lipschitz objective and constraint functions stated above. The results established in [122] are discussed in Sections 2.1–2.3.

Here in order to demonstrate the idea of the proof of the main result of [122] we consider a simplified version of the problem

$$\text{minimize } f(x) \text{ subject to } g(x) \le c,$$

where the Banach space X is finite-dimensional, the functions $f, g : X \to R^1$ are continuously differentiable, $\lim_{||x|| \to \infty} f(x) = \infty$ and the set $\{x \in X : g(x) \le c\}$ is nonempty.

Assume that the exact penalty does not exist and that a sequence of positive numbers $\{\lambda_k\}_{k=1}^{\infty}$ tends to infinity as $k \to \infty$. Then for any natural number k there exists a solution x_k of the penalized unconstrained problem with the penalty coefficient λ_k which is not a solution of the original constrained problem. We have already shown that the sequence $\{x_k\}_{k=1}^{\infty}$ is bounded. We may assume without loss of generality that there exists $\bar{x} = \lim_{k\to\infty} x_k$. We have shown above that $\bar{x}$ is a solution of the original inequality-constrained minimization problem.

Let k be a natural number. If $g(x_k) \le c$, then

$$f(x_k) = f(x_k) + \lambda_k \max\{g(x_k) - c, 0\} \le f(\bar{x}) + \lambda_k \max\{g(\bar{x}) - c, 0\} = f(\bar{x})$$

and x_k is a solution of the inequality-constrained problem. The contradiction we have reached proves that $g(x_k) > c$. Then there exists an open neighborhood V_k of x_k in X such that $g(z) > c$ for all $z \in V_k$. Now it is clear that for all $z \in V_k$

$$f(z) + \lambda_k(g(z) - c) = f(z) + \lambda_k \max\{g(z) - c, 0\}$$

$$\ge f(x_k) + \lambda_k \max\{g(x_k) - c, 0\} = f(x_k) + \lambda_k(g(x_k) - c).$$

This implies that

$$0 = \nabla f(x_k) + \lambda_k \nabla g(x_k) \text{ and } \nabla g(x_k) = -\lambda_k^{-1} \nabla f(x_k).$$

Evidently,

$$\nabla g(\bar{x}) = \lim_{k\to\infty} \nabla g(x_k) = -\lim_{k\to\infty} \lambda_k^{-1} \nabla f(x_k) = 0.$$

Since $g(x_k) > c$ for all natural numbers k we obtain that $g(\bar{x}) \ge c$ and since $\bar{x}$ is an admissible point of the original constrained problem we have $g(\bar{x}) = c$. Therefore assuming that the exact penalty does not exist we obtain that the set $\{x \in X : \; g(x) = c \text{ and } \nabla g(x) = 0\}$ is nonempty, or in other words c is a critical value of the function g. Thus if c is not a critical value of the function g, then the exact penalty exists. The main results of [122] stated in Section 2.1 and proved in Section 2.2 are actually a generalization of the result obtained above for an equality-constrained problem and an inequality-constrained problem in infinite-dimensional Banach spaces with locally Lipschitz objective and constraint functions. In [122] we used the notion of a critical point with respect to the Clarke's generalized gradient [21] and a version of Palais–Smale condition [8, 76, 100] because the Banach space X is infinite-dimensional. Since the existence of solutions of optimization problems in the infinite-dimensional Banach space X is not guaranteed we deal in Sections 2.1 and 2.2 with approximate solutions and employ Ekeland's variational principle [37]. In Sections 2.1 and 2.2 we establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. This is a novel approach in the penalty-type methods.

Consider a problem minimize $h(z)$ subject to $z \in X$ where $h : X \to R^1$ is a continuous bounded from below function. We say that $x \in X$ is a δ-approximate solution of the problem minimize $h(z)$ subject to $z \in X$, where $\delta > 0$, if $h(x) \le \inf\{h(z) : \ z \in X\} + \delta$.

The main result of Section 2.1 which is established for the equality-constrained problem and for the inequality-constrained problem stated at the beginning of this section implies that if c is not a critical value of g, then for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ which depends only on ϵ such that the following property holds:

If $\lambda \ge \bar{\lambda}$ and x is a $\delta(\epsilon)$-approximate solution of the penalized unconstrained problem, then there exists a $(\bar{\lambda}\epsilon)$-approximate solution of the original penalized problem such that $||y - x|| \le \epsilon$.

Here $\bar{\lambda}$ is a positive constant which does not depend on ϵ.

This property implies that any exact solution of the unconstrained penalized problem whose penalty coefficient is larger than $\bar{\lambda}$, is an exact solution of the corresponding constrained problem. Indeed, let x be a solution of the unconstrained penalized problem whose penalty coefficient is larger than $\bar{\lambda}$. Then for any $\epsilon > 0$ the point x is also a $\delta(\epsilon)$-approximate solution of the same unconstrained penalized problem and in view of the property above there is a $(\bar{\lambda}\epsilon)$-approximate solution y_ϵ of the corresponding constrained problem such that $||x - y_\epsilon|| \le \epsilon$. Since ϵ is an arbitrary positive number we can easily deduce that x is an exact solution of the corresponding constrained problem. Therefore the result also includes the classical penalty result as a particular case.

In Chapters 2 and 3 we establish several extensions and generalizations of the main result of Section 2.1 for various classes of constrained minimization problems. In Sections 2.4–2.6 we establish the exact penalty property for a large class of inequality-constrained minimization problems

$$\text{minimize } f(x) \text{ subject to } x \in A$$

where

$$A = \{x \in X : \ g_i(x) \le c_i \text{ for } i = 1, \dots, n\}.$$

Here X is a Banach space, c_i, $i = 1, \dots, n$ are real numbers, the constraint functions $g_i : X \to R^1 \cup \{\infty\}$, $i = 1, \dots, n$ are convex and lower semicontinuous and the objective function $f : X \to R^1\{\infty\}$ belongs to a certain space of functions. This space of functions includes the set of all convex bounded from below semicontinuous functions $f : X \to R^1$ which satisfy the growth condition $\lim_{||x|| \to \infty} f(x) = \infty$ and the set of all functions f on X which satisfy the growth condition above and which are Lipschitzian on all bounded subsets of X. We associate with the inequality-constrained minimization problem above the corresponding family of unconstrained minimization problems

$$\text{minimize } f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \text{ subject to } z \in X$$

where $\gamma > 0$ is a penalty. Under the assumption that there is $\tilde{x} \in X$ such that

$$g_j(\tilde{x}) < c_j \text{ for all } j = 1, \dots, n \text{ and } f(\tilde{x}) < \infty$$

we establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem.

The existence of exact penalty is established in Sections 2.7 and 2.8 for constrained minimization problems with mixed nonconvex nonsmooth constraints in a Banach space. In Sections 2.9 and 2.10 we study an equality-constrained problem and an inequality-constrained problem with one constraint in a Hilbert space assuming that an objective function and a constraint function are Frechet differentiable. We establish the following sufficient condition for the existence of exact penalty which involves the second Frechet derivative:

In the case of the inequality-constrained problem (the equality-constrained problem, respectively) the set $\{x \in X : \ g(x) = c\}$ does not contain a critical point of g whose second derivative generates a positively semidefinite (positively semidefinite or negatively semidefinite, respectively) quadratic form.

The existence of exact penalty is obtained in Sections 2.11–2.13 for constrained minimization problems with locally Lipschitzian mixed constraints in complete metric spaces and in Sections 2.14 and 2.15 we establish an extension of the main result of Section 2.1 using Mordukhovich's basic subdifferential [71, 73].

In Chapter 3 we study the stability of the exact penalty. It is shown that if a sufficient condition for the existence of exact penalty holds for a constrained minimization problem, then the same exact penalty exists for a family of constrained minimization problems which are obtained from the original problem by perturbations. These results show that exact penalty is stable under influence of computational errors and this stability is important for practice from the computational point of view.

The stability results are established for equality-constrained problems and inequality-constrained problems with one constraint and with locally Lipschitz objective and constrained functions in Sections 3.1–3.3, for inequality-constrained problems with a lower semicontinuous objective function and convex lower semicontinuous constraint functions in Sections 3.4–3.7, and for inequality-constrained problems with locally Lipschitz objective and constraint functions in Sections 3.8 and 3.9.

1.2 Generic existence of solutions of minimization problems

In the second part of the monograph (Chapters 4–10) we use the generic approach in order to study the existence of solutions of various optimization problems in Banach spaces and complete metric spaces and show that for most problems (in the sense of Baire category) all minimizing sequences converge to a unique solution. Here we demonstrate simple applications of this generic approach in optimization.

Let (X, ρ) be a bounded complete metric space. Put

$$\text{diam }(X) = \sup\{\rho(x, y) :\ x, y \in X\}$$

and denote by $\mathcal{M}$ the space of all bounded continuous functions $f : X \to R^1$ equipped with a metric

$$d(f_1, f_2) = \sup\{|f_1(x) - f_2(x)| :\ x \in X\},\ f_1, f_2 \in \mathcal{M}.$$

Clearly, $(\mathcal{M}, d)$ is a complete metric space. It is well-known that if the space X is compact, then for every $\in \mathcal{M}$, the problem minimize $f(x)$ subject to $x \in X$ possesses a solution. Since in our book we do not assume the compactness of X the situation becomes more difficult and less understood. Nevertheless, there exists a subset $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that for each element of $\mathcal{F}$ the corresponding minimization problem has a unique solution.

Denote by $\mathcal{L}$ the set of all $f \in \mathcal{M}$ for which the problem minimize $f(x)$ subject to $x \in X$ possesses a solution. It turns out that $\mathcal{L}$ is an everywhere dense subset of $\mathcal{M}$.

Indeed, let $f \in \mathcal{M}$ and ϵ be a positive number. There exists $x_\epsilon \in X$ such that

$$f(x_\epsilon) \le f(x) + \epsilon \text{ for all } x \in X.$$

Put

$$f_\epsilon(x) = \max\{f(x),\ f(x_\epsilon)\},\ x \in X.$$

Clearly, $f_\epsilon \in \mathcal{M}$, x_ϵ is a point of minimum of the function f_ϵ and therefore $f_\epsilon \in \mathcal{L}$. It is not difficult to see that for all $x \in X$,

$$f(x) \le f_\epsilon(x) \le f(x) + \epsilon$$

and that $d(f, f_\epsilon) \le \epsilon$. Thus $\mathcal{L}$ is an everywhere dense subset of $\mathcal{M}$.

For any $f \in \mathcal{L}$ let $x_f \in X$ be a point of minimum of the function f such that

$$f(x_f) \le f(x) \text{ for all } x \in X.$$

Let $f \in \mathcal{L}$ and n be a natural number. Define $f_n \in \mathcal{M}$ by

$$f_n(x) = f(x) + (4n)^{-1}(\text{diam}(X) + 1)^{-1}\rho(x, x_f),\ x \in X.$$

It is easy to see that
$$d(f, f_n) \leq (4n)^{-1}.$$
Choose a number
$$\delta(f, n) \in (0, (12n^2)^{-1}(\mathrm{diam}(X) + 1)^{-1})$$
and set
$$V(f, n) = \{g \in \mathcal{M} : \ d(f_n, g) < \delta(f, n)\}.$$
Assume that
$$g \in V(f, n), \ x \in X \text{ and } g(x) \leq \inf\{g(z) : \ z \in X\} + \delta(f, n).$$
It follows from the relations above, the definitions of f_n, $V(f, n)$, x_f and $\delta(f, n)$ that
$$f(x) + (4n)^{-1}(\mathrm{diam}(X) + 1)^{-1}\rho(x, x_f)$$
$$= f_n(x) \leq g(x) + \delta(f, n) \leq g(x_f) + 2\delta(f, n)$$
$$\leq f_n(x_f) + 3\delta(f, n) = f(x_f) + 3\delta(f, n) \leq f(x) + 3\delta(f, n)$$
and that
$$\rho(x, x_f) \leq 3\delta(f, n)4n(\mathrm{diam}(X) + 1) < 1/n.$$
Thus for each $g \in V(f, n)$ the following property holds:

if $x \in X$ satisfies $g(x) \leq \inf\{g(z) : \ z \in X\} + \delta(f, n)$, then $\rho(x, x_f) < 1/n$.

Set
$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{V(f, k) : \ f \in \mathcal{L} \text{ and an integer } k \geq n\}.$$
Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}$.

Assume that $g \in \mathcal{F}$ and that n is a natural number. By definition of $\mathcal{F}$, there exist a natural number $k \geq n$ and $f \in \mathcal{L}$ such that $g \in V(f, k)$. Combined with the property above this implies that the following property holds:

for each $x \in X$ satisfying $g(x) \leq \inf\{g(z) : \ z \in X\} + \delta(f, k)$ the inequality $\rho(x, x_f) < 1/n$ holds.

Since n is an arbitrary natural number we conclude that any minimizing sequence for the problem minimize $g(z)$ subject to $z \in X$ converges to its unique solution, where g is an arbitrary element of $\mathcal{F}$.

There is also another way to prove the result obtained above.

Let $f \in \mathcal{M}$ and n be a natural number. Choose a positive number
$$\delta(f, n) < (32n^2)^{-1}(\mathrm{diam}(X) + 1)^{-1}$$
and choose $x_f \in X$ such that
$$f(x_f) \leq \inf\{f(z) : \ z \in X\} + \delta(f, n).$$
Put

$$f_n(z) = f(z) + (4n)^{-1}(\mathrm{diam}(X) + 1)^{-1}\rho(z, x_f),\ z \in X.$$

Clearly $f_n \in \mathcal{M}$ and

$$d(f, f_n) \le (4n)^{-1}.$$

Set

$$V(f, n) = \{g \in \mathcal{M} :\ d(f_n, g) < \delta(f, n)\}.$$

Assume that

$$g \in V(f, n),\ x \in X \text{ and } g(x) \le \inf\{g(z) :\ z \in X\} + \delta(f, n).$$

By the relations above, the definitions of f_n, $V(f, n)$, x_f and $\delta(f, n)$,

$$f(x) + (4n)^{-1}(\mathrm{diam}(X) + 1)^{-1}\rho(x, x_f) = f_n(x)$$

$$\le g(x) + \delta(f, n) \le g(x_f) + 2\delta(f, n)$$

$$\le f_n(x_f) + 3\delta(f, n) = f(x_f) + 3\delta(f, n) \le f(x) + 4\delta(f, n)$$

and

$$\rho(x, x_f) \le 4\delta(f, n)4n(\mathrm{diam}(X) + 1) < 1/n.$$

Thus for each $g \in V(f, n)$ the following property holds:

if $x \in X$ satisfies $g(x) \le \inf\{g(z) :\ z \in X\} + \delta(f, n)$, then $\rho(x, x_f) < 1/n$.

Set

$$\mathcal{F} = \cap_{k=1}^{\infty} \cup \{V(f, n) :\ f \in \mathcal{M} \text{ and an integer } n \ge k\}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}$.

Assume that $g \in \mathcal{F}$ and that k is a natural number. By definition of $\mathcal{F}$, there exist a natural number $n \ge k$ and $f \in \mathcal{M}$ such that $g \in V(f, n)$. Combined with the property above this implies that the following property holds:

for each $x \in X$ satisfying $g(x) \le \inf\{g(z) :\ z \in X\} + \delta(f, n)$ the inequality $\rho(x, x_f) < 1/k$ holds.

Since k is an arbitrary natural number we conclude that any minimizing sequence for the problem minimize $g(z)$ subject to $z \in X$ converges to its unique solution, where g is an arbitrary element of $\mathcal{F}$.

We presented above two proofs of the same generic well-posedness result. In both proofs for a given $f \in \mathcal{M}$ and a positive number ϵ we defined a function $\bar{f} \in \mathcal{M}$, $\bar{x} \in X$ and a positive constant δ such that if $x \in X$ is a δ-approximate solution of the problem $g(z) \to \min$, $z \in X$, where g belongs to a δ-neighborhood of $\bar{f}$, then x belong to an ϵ-neighborhood of $\bar{x}$.

Many results of this type for various classes of minimization problems are collected in Chapters 4–10. Here we briefly describe some of them.

In Chapter 4 we consider problems minimize $f(x)$ subject to $x \in X$ where X is a complete metric space and f belongs to a space of lower semicontinuous functions on X satisfying a certain growth condition. The class of minimization problems is identified with this space of functions. We endow the space of functions with an appropriate metric and show that for most functions f in this metric space the corresponding minimization problem has a unique solution and is well-posed.

In Chapter 4 we also consider the following class of minimization problems

$$\text{minimize } f(x) \text{ subject to } x \in A$$

studied in [52, 110, 111], where A is a nonempty closed subset of a complete metric space X and f belongs to a space of lower semicontinuous functions on X. This class of problems is identified with a space of pairs (f, A) which is equipped with appropriate complete uniformities. It is shown that for a typical pair (f, A) the corresponding minimization problem has a unique solution and is well-posed.

Chapter 4 also contains several other interesting generic results in optimization.

In Chapter 5 we continue to consider various classes of minimization problems showing that most problems in these classes are well-posed. In that chapter in order to meet this goal we use a porosity notion. As in Chapter 4 we identified a class of minimization problems with a certain complete metric space of functions, study the set of all functions for which the corresponding minimization problem is well-posed and show that the complement of this set is not only of the first category but also a σ-porous set.

We now recall the concept of porosity [10, 26, 27, 84, 97, 98, 112].

Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous in (Y, d) if there exist $\alpha \in (0, 1]$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

A subset of the space Y is called σ-porous in (Y, d) if it is a countable union of porous subsets in (Y, d).

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is smaller than the class of sets which have measure 0 and are of the first category.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

In Chapter 5 we employ the notion of porosity in order to study well-posedness of convex minimization problems, well-posedness of a certain class of problems arising in crystallography and well-posedness for a class of equilibrium problems.

In Chapter 6 we study a parametric family of the problems minimize $f(b,x)$ subject to $x \in X$ on a complete metric space X with a parameter b which belongs to a Hausdorff compact space $\mathcal{B}$. Here $f(\cdot,\cdot)$ belongs to a space of functions on $\mathcal{B} \times X$ endowed with an appropriate uniform structure. Using the generic approach and the notion of porosity we show that for a typical function $f(\cdot,\cdot)$ the minimization problem has a solution for all parameters $b \in \mathcal{B}$.

Chapter 7 is devoted to the study of problems minimize $f(x)$ subject to $x \in K$ where K is a closed subset of an ordered Banach space X and f belongs to a space of increasing lower semicontinuous functions on K. Using the generic approach and the notion of porosity we show that for most functions f in this space the corresponding minimization problem has a unique solution.

In Chapter 8 we study minimization problems with mixed constraints

$$\text{minimize } f(x) \text{ subject to } G(x) = y,\ H(x) \le z,$$

where f is a continuous (differentiable) finite-valued function defined on a Banach space X, y is an element of a finite-dimensional Banach space Y, z is an element of a Banach space Z ordered by a convex closed cone and $G : X \to Y$ and $H : X \to Z$ are continuous (differentiable) mappings. We consider two classes of these problems and show that most of the problems (in the Baire category sense) are well-posed. Our first class of problems is identified with the corresponding complete metric space of quintets (f, G, H, y, z). We show that for a generic quintet (f, G, H, y, z) the corresponding minimization problem has a unique solution and is well-posed. Our second class of problems is identified with the corresponding complete metric space of triples (f, G, H) while y and z are fixed. We show that for a generic triple (f, G, H) the corresponding minimization problem has a unique solution and is well-posed.

In Chapter 9 we study classes of vector minimization problems on a complete metric space which are identified with the corresponding complete metric spaces of objective functions. We show that for most (in the sense of Baire category) functions the corresponding vector optimization problem has a solution. Using generic approach we also study other interesting properties of such classes of problems.

Chapter 10 is devoted to the study of infinite horizon minimization problems.

1.3 Comments

The book contains two parts. In the first part (Chapters 2 and 3) we study the exact penalty property for constrained minimization problems in Banach spaces and in complete metric spaces and in the second part (Chapters 4–10) using the generic approach we study the existence of solutions of various optimization problems in Banach spaces and complete metric spaces. In Chapter 1 we describe with more details the subject and the structure of the book.

2

Exact Penalty in Constrained Optimization

2.1 Problems with a locally Lipschitzian constraint function

Let $(X, ||\cdot||)$ be a Banach space and $(X^*, ||\cdot||_*)$ its dual space. For each $x \in X$, each $x^* \in X^*$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||y - x|| \le r\}, \ B_*(x^*, r) = \{l \in X^* : \ ||l - x^*||_* \le r\}.$$

Let $f : X \to R^1$ be a locally Lipschitzian function. For each $x \in X$ let

$$f^0(x, h) = \limsup_{t \to 0^+, y \to x} [f(y + th) - f(y)]/t, \ h \in X$$

be the Clarke generalized directional derivative of f at the point x [21], let

$$\partial f(x) = \{l \in X^* : \ f^0(x, h) \ge l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of f at x [21] and set

$$\Xi_f(x) = \inf\{f^0(x, h) : \ h \in X \text{ and } ||h|| = 1\}$$

[100].

A point $x \in X$ is called a critical point of f if $0 \in \partial f(x)$. It is not difficult to see that $x \in X$ is a critical point of f if and only if $\Xi_f(x) \ge 0$.

A real number $c \in R^1$ is called a critical value of f if there is a critical point x of f such that $f(x) = c$.

It is known that $\partial(-f)(x) = -\partial f(x)$ for any $x \in X$ (see Section 2.3 of [21]). This equality implies that $x \in X$ is a critical point of f if and only if x is a critical point of $-f$ and $c \in R^1$ is a critical value of f if and only if $-c$ is a critical value of $-f$.

For each function $f : X \to R^1$ set $\inf(f) = \inf\{f(z) : \ z \in X\}$. For each $x \in X$ and each $B \subset X$ put

A.J. Zaslavski, *Optimization on Metric and Normed Spaces*, Springer Optimization and Its Applications 44, DOI 10.1007/978-0-387-88621-3_2,

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

Let $f : X \to R^1$ be a function which is Lipschitzian on all bounded subsets of X and which satisfies the following growth condition:

$$\lim_{||x|| \to \infty} f(x) = \infty. \tag{2.1}$$

It is easy to see that $\inf\{f(z) : \ z \in X\} > -\infty$.

Let $g : X \to R^1$ be a locally Lipschitzian function which satisfies the following Palais–Smale (P-S) condition [8, 76, 100]:

If $\{x_i\}_{i=1}^{\infty} \subset X$, the sequence $\{g(x_i)\}_{i=1}^{\infty}$ is bounded and if

$$\liminf_{i \to \infty} \Xi_g(x_i) \geq 0,$$

then there is a norm convergent subsequence of $\{x_i\}_{i=1}^{\infty}$.

Let $c \in R^1$ be such that $g^{-1}(c)$ is nonempty.

We consider the constrained problems

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}(c) \tag{P_e}$$

and

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}((-\infty, c]). \tag{P_i}$$

We associate with these two problems the corresponding families of unconstrained minimization problems

$$\text{minimize } f(x) + \lambda|g(x) - c| \text{ subject to } x \in X \tag{$P_{\lambda e}$}$$

and

$$\text{minimize } f(x) + \lambda \max\{g(x) - c, \ 0\} \text{ subject to } x \in X \tag{$P_{\lambda i}$}$$

where $\lambda > 0$.

Set

$$\inf(f; c) = \inf\{f(z) : \ z \in g^{-1}(c)\}, \tag{2.2}$$

$$\inf(f; (-\infty, c]) = \inf\{f(z) : \ z \in X \text{ and } g(z) \leq c\}. \tag{2.3}$$

The next theorem is the main result of this section.

Theorem 2.1. *Assume that the number c is not a critical value of the function g. Then there exist numbers $\lambda_0 > 0$ and $\lambda_1 > 0$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertions hold:*

1. For each $\lambda > \lambda_0$ and each $x \in X$ which satisfies

$$f(x) + \lambda|g(x) - c| \leq \inf\{f(z) + \lambda|g(z) - c| : \ z \in X\} + \delta$$

there exists $y \in g^{-1}(c)$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; c) + \lambda_1 \epsilon.$$

2. For each $\lambda > \lambda_0$ *and each* $x \in X$ *which satisfies*

$$f(x) + \lambda \max\{g(x) - c,\ 0\} \le \inf\{f(z) + \lambda \max\{g(z) - c,\ 0\} :\ z \in X\} + \delta$$

there exists $y \in g^{-1}((-\infty, c])$ *such that*

$$||y - x|| \le \epsilon \text{ and } f(y) \le \inf(f; (-\infty, c]) + \lambda_1 \epsilon.$$

Theorem 2.1 will be proved in Section 2.2. In this section we present several important results which follow from Theorem 2.1. We will prove these results in Section 2.2.

Theorem 2.1 implies the following result.

Theorem 2.2. *Assume that the number* c *is not a critical value of the function* g*. Then there exists a positive number* λ_0 *such that the following assertions hold:*

1. If $\lambda > \lambda_0$ *and if a sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfies*

$$\lim_{i \to \infty} [f(x_i) + \lambda |g(x_i) - c|] = \inf\{f(z) + \lambda |g(z) - c| :\ z \in X\},$$

then there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset g^{-1}(c)$ *such that*

$$\lim_{i \to \infty} f(y_i) = \inf(f; c) \text{ and } \lim_{i \to \infty} ||y_i - x_i|| = 0.$$

2. If $\lambda > \lambda_0$ *and if a sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfies*

$$\lim_{i \to \infty} [f(x_i) + \lambda \max\{g(x_i) - c, 0\}] = \inf\{f(z) + \lambda \max\{g(z) - c, 0\} :\ z \in X\},$$

then there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset g^{-1}((-\infty, c])$ *such that*

$$\lim_{i \to \infty} f(y_i) = \inf(f; (-\infty, c]) \text{ and } \lim_{i \to \infty} ||y_i - x_i|| = 0.$$

Assertion 1 of Theorem 2.2 implies the following result.

Theorem 2.3. *Assume that the number* c *is not a critical value of the function* g *and that there exists* $\bar{x} \in g^{-1}(c)$ *for which the following conditions hold:*

$f(\bar{x}) = \inf(f; c)$*;*

any sequence $\{x_n\}_{n=1}^{\infty} \subset g^{-1}(c)$ *which satisfies* $\lim_{n \to \infty} f(x_n) = \inf(f; c)$ *converges to* $\bar{x}$ *in the norm topology.*

Then there exists a positive number λ_0 *such that for each* $\lambda > \lambda_0$ *the point* $\bar{x}$ *is a unique solution of the minimization problem*

$$\text{minimize } f(z) + \lambda |g(z) - c| \text{ subject to } z \in X.$$

Assertion 2 of Theorem 2.2 implies the following result.

Theorem 2.4. *Assume that the number c is not a critical value of the function g and that there exists $\bar{x} \in g^{-1}((-\infty, c])$ for which the following conditions hold:*

$f(\bar{x}) = \inf(f; (-\infty, c])$;

any sequence $\{x_n\}_{n=1}^{\infty} \subset g^{-1}((-\infty, c])$ *which satisfies* $\lim_{n\to\infty} f(x_n) = \inf(f; (-\infty, c])$ *converges to $\bar{x}$ in the norm topology.*

Then there exists a positive number λ_0 such that for each $\lambda > \lambda_0$ the point $\bar{x}$ is a unique solution of the minimization problem

$$\textit{minimize } f(z) + \lambda \max\{g(z) - c, 0\} \textit{ subject to } z \in X.$$

The next result follows from Theorem 2.2.

Theorem 2.5. *Assume that $X = R^n$ and the number c is not a critical value of the function g. Then there exists a positive number λ_0 such that the following assertions hold:*

1. For each $\lambda > \lambda_0$ and for each solution x of the minimization problem

$$\textit{minimize } f(z) + \lambda |g(z) - c| \textit{ subject to } z \in X$$

the following relations hold:

$$x \in g^{-1}(c) \textit{ and } f(x) = \inf(f; c).$$

2. For each $\lambda > \lambda_0$ and each solution x of the minimization problem

$$\textit{minimize } f(z) + \lambda \max\{g(z) - c, 0\} \textit{ subject to } z \in X$$

the following relations hold:

$$g(x) \le c \textit{ and } f(x) = \inf(f; (-\infty, c]).$$

Example 2.6. Assume that $X = R^n$, $g \in C^1(R^n)$ and that the gradient of g is not zero at any point $x \in R^n$. Then Theorems 2.1, 2.2 and 2.5 hold.

Example 2.7. Assume that $X = R^n$, g is convex and bounded from below and $c > \inf(g)$. Then Theorems 2.1, 2.2 and 2.5 hold.

Theorems 2.1–2.5 have been established in [122].

Now we give an example which shows that exactness fails when c is a critical value of g.

Example 2.8. Let $X = R^1$ and consider the minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in R^1, \; g(x) = 0$$

where $f(x) = (x - 10)^2$, $x \in R^1$ and

$$g(x) = (x - 1)^2, \; x \in [1, \infty),$$

$$g(x) = (x+1)^2,\ x \in (-\infty, -1],\ g(x) = 0,\ x \in (-1, 1).$$

This problem is equivalent to the problem

$$\text{minimize } f(x) \text{ subject to } x \in R^1,\ g(x) \le 0.$$

Clearly zero is a critical value of g and $\bar{x} = 1$ is a unique solution of the problem.

We show that for each $\lambda > 0$, $\inf(f + \lambda g) < f(1) = 81$. Fix $\lambda > 0$. For each $x \in [1, \min\{2, 1 + 4/\lambda\}]$

$$(f+\lambda g)'(x) = 2(x-10)+2\lambda(x-1) \le -16+2\lambda(x-1) \le -16+2\lambda(4/\lambda) = -8.$$

This relation implies that $\inf(f + \lambda g) < f(1)$.

2.2 Proofs of Theorems 2.1–2.4

Proof of Theorem 2.1: We prove Assertions 1 and 2 simultaneously.

Set

$$A = g^{-1}(c) \text{ in the case of Assertion 1}$$

and

$$A = g^{-1}((-\infty, c]) \text{ in the case of Assertion 2.}$$

For each positive number λ define a function $\psi_\lambda : X \to R^1$ by

$$\psi_\lambda(z) = f(z) + \lambda|g(z) - c|,\ z \in X \tag{2.4}$$

in the case of Assertion 1 and by

$$\psi_\lambda(z) = f(z) + \lambda \max\{g(z) - c, 0\},\ z \in X \tag{2.5}$$

in the case of Assertion 2.

Note that the function ψ_λ is locally Lipschitzian for all positive numbers λ.

We show that there exists a positive number λ_0 such that the following property holds:

(P1) For each $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon)$ such that for each $\lambda > \lambda_0$ and each $x \in X$ which satisfies

$$\psi_\lambda(x) \le \inf(\psi_\lambda) + \delta$$

the set $A \cap B(x, \epsilon)$ is nonempty.

Assume the contrary. Then for each integer $k \ge 1$ there exist

$$\epsilon_k \in (0, 1),\ \lambda_k > k \text{ and } x_k \in X \tag{2.6}$$

such that

$$\psi_{\lambda_k}(x_k) \le \inf(\psi_{\lambda_k}) + 2^{-1}\epsilon_k k^{-2}, \tag{2.7}$$

$$d(x_k, A) \ge \epsilon_k. \tag{2.8}$$

Let $k \ge 1$ be an integer. By (2.7) and Ekeland's variational principle [37], there exists $y_k \in X$ for which

$$\psi_{\lambda_k}(y_k) \le \psi_{\lambda_k}(x_k), \tag{2.9}$$

$$||y_k - x_k|| \le 2^{-1}k^{-1}\epsilon_k, \tag{2.10}$$

$$\psi_{\lambda_k}(y_k) \le \psi_{\lambda_k}(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{2.11}$$

It follows from (2.8) and (2.10) that

$$y_k \notin A \text{ for all integers } k \ge 1. \tag{2.12}$$

In the case of Assertion 2 we obtain that

$$g(y_k) > c \text{ for all integers } k \ge 1. \tag{2.13}$$

In the case of Assertion 1 we obtain that for each integer $k \ge 1$

$$\text{either } g(y_k) > c \text{ or } g(y_k) < c.$$

In the case of Assertion 1 extracting a subsequence and reindexing we may assume that either $g(y_k) > c$ for all integers $k \ge 1$ or $g(y_k) < c$ for all integers $k \ge 1$. Replacing g by $-g$ and c by $-c$, if necessary, we may assume without loss of generality that (2.13) is valid in the case of Assertion 1 too. Now (2.13) is valid in both cases.

Let $k \ge 1$ be an integer. In view of (2.11),

$$0 \in \partial\psi_{\lambda_k}(y_k) + B_*(0, 1/k). \tag{2.14}$$

It follows from (2.4), (2.5) and (2.13) that

$$\partial\psi_{\lambda_k}(y_k) = \partial(f + \lambda_k g)(y_k) \subset f(y_k) + \lambda_k \partial g(y_k). \tag{2.15}$$

By (2.14) and (2.15),

$$0 \in \partial g(y_k) + \lambda_k^{-1}\partial f(y_k) + B_*(0, (k\lambda_k)^{-1}). \tag{2.16}$$

By (2.4)–(2.7), (2.9) and (2.13), for each integer $k \ge 1$

$$f(y_k) \le f(y_k) + \lambda(g(y_k) - c) = \psi_{\lambda_k}(y_k) \le \inf(\psi_{\lambda_k}) + 1$$

$$\le \inf\{\psi_{\lambda_k}(z) : \ z \in A\} + 1 = \inf\{f(z) : \ z \in A\} + 1.$$

It follows from this inequality and growth condition (2.1) that the sequence $\{y_k\}_{k=1}^{\infty}$ is bounded. Since f is Lipschitzian on bounded subsets of X it follows from the boundedness of the sequence $\{y_k\}_{k=1}^{\infty}$ that there exists a positive number L such that

$$\partial f(y_k) \subset B_*(0, L) \tag{2.17}$$

for each integer $k \geq 1$.

By (2.13), (2.4)–(2.7) and (2.9), for each integer $k \geq 1$,

$$f(y_k) + \lambda_k(g(y_k) - c) = \psi_{\lambda_k}(y_k) \leq \psi_{\lambda_k}(x_k) \leq \inf(\psi_{\lambda_k}) + 1 \tag{2.18}$$

$$\leq \inf\{\psi_{\lambda_k}(z) : \ z \in A\} + 1 = \inf\{f(z) : \ z \in A\} + 1.$$

It follows from (2.13), (2.18) and (2.6) that for each natural number k

$$0 < g(y_k) - c \leq \lambda_k^{-1}[\inf\{f(z) : \ z \in A\} - \inf(f) + 1] \to 0 \text{ as } k \to \infty.$$

Hence

$$\lim_{k \to \infty} g(y_k) = c. \tag{2.19}$$

In view of (2.16), (2.17) and (2.6), for each natural number k,

$$0 \in \partial g(y_k) + \lambda_k^{-1} B_*(0, L) + B_*(0, k^{-1})$$

$$\subset \partial g(y_k) + B_*(0, k^{-1}(1 + L)).$$

This inclusion implies that

$$\liminf_{k \to \infty} \Xi_g(y_k) \geq 0. \tag{2.20}$$

By (2.20), (2.19) and (P-S) condition, there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that $\{y_{k_j}\}_{j=1}^{\infty}$ converges in the norm topology to $y_* \in X$. Equality (2.19) implies that $g(y_*) = c$. It follows from (2.20) and upper semicontinuity of the Clarke generalized directional derivative $g^0(\xi, \eta)$ with respect to ξ that $\Xi_g(y_*) \geq 0$ and $0 \in \partial g(y_*)$. Since $g(y_*) = c$ we obtain that c is a critical value of g, a contradiction. The contradiction we have reached proves that there exists a positive number λ_0 such that property (P1) holds.

By growth condition (2.1), there exists a number $K_1 > 0$ such that

$$||x|| \leq K_1 \text{ for each } x \in X \text{ satisfying } f(x) \leq \inf\{f(z) : \ z \in A\} + 1. \tag{2.21}$$

Since f is Lipschitzian on bounded subsets of X there is $\lambda_1 > 2$ such that

$$||f(x_1) - f(x_2)|| \leq 2^{-1}\lambda_1 ||x_1 - x_2|| \tag{2.22}$$

for each $x_1, x_2 \in B(0, K_1 + 1)$.

Assume that $\epsilon \in (0, 1)$. Let $\delta \in (0, \epsilon)$ be as guaranteed by property (P1). Now assume that

$$\lambda > \lambda_0, \ x \in X \text{ and } \psi_\lambda(x) \leq \inf(\psi_\lambda) + \delta. \tag{2.23}$$

Property (P1) implies that there exists $y \in X$ such that

$$y \in A \cap B(x, \epsilon). \tag{2.24}$$

By (2.4), (2.5) and (2.23),

$$f(x) \le \psi_\lambda(x) \le \inf(\psi_\lambda) + \delta \le \inf(\psi_\lambda) + 1$$

$$\le \inf\{\psi_\lambda(z) : \ z \in A\} + 1 = \inf\{f(z) : \ z \in A\} + 1. \tag{2.25}$$

In view of (2.21) and (2.25),

$$x \in B(0, K_1). \tag{2.26}$$

Relations (2.24) and (2.26) imply that

$$||y|| \le ||x|| + ||y - x|| \le K_1 + 1. \tag{2.27}$$

It follows from (2.26), (2.27), the choice of λ_1 (see (2.22)) and (2.24) that

$$|f(y) - f(x)| \le 2^{-1}\lambda_1||y - x|| \le 2^{-1}\lambda_1\epsilon. \tag{2.28}$$

By (2.28), (2.4), (2.5), (2.23) and the inequalities $\delta < \epsilon$ and $\lambda_1 > 2$,

$$f(y) \le 2^{-1}\lambda_1\epsilon + f(x) \le 2^{-1}\lambda_1\epsilon + \psi_\lambda(x) \le 2^{-1}\lambda_1\epsilon + \inf(\psi_\lambda) + \delta$$

$$\le \lambda_1\epsilon + \inf\{f(z) : \ z \in A\}.$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2: We prove Assertions 1 and 2 simultaneously. Let numbers $\lambda_0 > 0$ and $\lambda_1 > 0$ be as guaranteed by Theorem 2.1. For each positive number λ define a function $\psi_\lambda : X \to R^1$ by (2.4) in the case of Assertion 1 and by (2.5) in the case of Assertion 2. Set $A = g^{-1}(c)$ in the case of Assertion 1 and $A = g^{-1}((-\infty, c])$ in the case of Assertion 2.

Assume that $\lambda > \lambda_0$ and that a sequence $\{x_i\}_{i=1}^\infty \subset X$ satisfies

$$\lim_{i \to \infty} \psi_\lambda(x_i) = \inf(\psi_\lambda). \tag{2.29}$$

By the choice of λ_0, for each natural number j there is a positive number δ_j such that the following property holds:

(P2) If $x \in X$ satisfies $\psi_\lambda(x) \le \inf(\psi_\lambda) + \delta_j$, then the set $A \cap B(x, 1/j)$ is nonempty.

Equality (2.29) implies that there exists a strictly increasing sequence of natural numbers $\{t_j\}_{j=1}^\infty$ such that for each natural number j

$$\psi_\lambda(x_i) \le \inf(\psi_\lambda) + \delta_j \text{ for all natural numbers } i \ge t_j. \tag{2.30}$$

By property (P2) and (2.30), for each natural number $j \ge 1$,

$$d(x_i, A) \le 1/j \text{ for all natural numbers } i \ge t_j.$$

This implies that there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset A$ such that

$$\lim_{i\to\infty} ||y_i - x_i|| = 0. \tag{2.31}$$

It follows from (2.1) and (2.29) that the sequence $\{x_i\}_{i=1}^{\infty}$ is bounded. Since f is Lipschitzian on bounded subsets of X it follows from (2.4), (2.5), (2.29) and (2.31) that

$$\inf\{f(z) : \ z \in A\} \ge \inf(\psi_\lambda) = \lim_{i\to\infty} \psi_\lambda(x_i) \ge \limsup_{i\to\infty} f(x_i) = \limsup_{i\to\infty} f(y_i).$$

Combined with the inclusion $\{y_i\}_{i=1}^{\infty} \subset A$ this relation implies that

$$\lim_{i\to\infty} f(y_i) = \inf\{f(z) : \ z \in A\}.$$

This completes the proof of Theorem 2.2.

Proofs of Theorems 2.3 and 2.4: We prove Theorems 2.3 and 2.4 simultaneously. Let a number $\lambda_0 > 0$ be as guaranteed by Theorem 2.2. For each positive number λ define a function $\psi_\lambda : X \to R^1$ by (2.4) in the case of Theorem 2.3 and by (2.5) in the case of Theorem 2.4. Set $A = g^{-1}(c)$ in the case of Theorem 2.3 and $A = g^{-1}((-\infty, c])$ in the case of Theorem 2.4.

Let $\lambda > \lambda_0$. Assume that a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfies

$$\lim_{i\to\infty} \psi_\lambda(x_i) = \inf(\psi_\lambda). \tag{2.32}$$

Theorem 2.2 implies that there exists a sequence

$$\{y_i\}_{i=1}^{\infty} \subset A \tag{2.33}$$

such that

$$\lim_{i\to\infty} ||y_i - x_i|| = 0 \tag{2.34}$$

and

$$\lim_{i\to\infty} f(y_i) = \inf\{f(z) : \ z \in A\}. \tag{2.35}$$

By (2.33) and (2.35), $\lim_{i\to\infty} ||y_i - \bar{x}|| = 0$. Combined with (2.34) this equality implies that

$$\lim_{i\to\infty} ||x_i - \bar{x}|| = 0. \tag{2.36}$$

Thus we have shown that if $\{x_i\}_{i=1}^{\infty} \subset X$ satisfies (2.32), then (2.36) is true. This implies that $\bar{x}$ is a unique solution of the minimization problem minimize $\psi_\lambda(z)$ subject to $z \in X$. Theorems 2.3 and 2.4 are proved.

2.3 An optimization problem in a finite-dimensional space

Let $p \geq 0$ be an integer, m, n be natural numbers such that $p \leq m \leq n$, let $X = R^n$ with the Euclidean norm and let

$$I_1 = \{i : \ i \text{ is an integer such that } 1 \leq i \leq p\}, \tag{2.37}$$

$$I_2 = \{i : \ i \text{ is an integer such that } p < i \leq m\}. \tag{2.38}$$

(Note that one of the sets I_1, I_2 may be empty.)

For each $h : R^n \to R^1$ set

$$\inf(h) = \inf\{h(z) : \ z \in R^n\}. \tag{2.39}$$

For each $h \in C^1(R^n)$ put

$$\nabla h(z) = (\partial h/\partial z_1(z), \dots, \partial h/\partial z_n(z)), \ z \in R^n. \tag{2.40}$$

Assume that $f_0 : R^n \to R^1$ is a locally Lipschitzian function which satisfies the following growth condition:

$$\lim_{||x|| \to \infty} f_0(x) = \infty. \tag{2.41}$$

Let $F = (f_1, \dots, f_m) : R^n \to R^m$ and let $f_i \in C^1(R^n)$, $i = 1, \dots, m$. The points of R^n where the rank of F is less than m are called critical points of F. A point $c \in R^m$ such that $F^{-1}(c)$ contains at least one critical point is called a critical value of F.

Let $c = (c_1, \dots, c_m) \in R^m$. In this section we consider the following optimization problem:

$$\text{minimize } f_0(x) \tag{P}$$

$$\text{subject to } x \in R^n, \ f_i(x) = c_i \text{ for all } i \in I_1, \ f_j(x) \leq c_j \text{ for all } j \in I_2.$$

Set

$$A = \{z \in R^n : \ f_i(z) = c_i, \ i \in I_1 \text{ and } f_j(z) \leq c_j, \ j \in I_2\}. \tag{2.42}$$

We assume that $A \neq \emptyset$. Since f_0 satisfies growth condition (2.41) problem (P) has a solution. Set

$$\inf(f_0; A) = \inf\{f_0(z) : \ z \in A\}.$$

For each vector $\gamma = (\gamma_1, \dots, \gamma_m) \in (0, \infty)^m$ define

$$\psi_\gamma(z) = f_0(z) + \sum_{i \in I_1} \gamma_i |f_i(z) - c_i| + \sum_{i \in I_2} \gamma_i \max\{f_i(z) - c_i, 0\}, \ z \in R^n. \tag{2.43}$$

Clearly for each $\gamma \in (0, \infty)^m$ the function ψ_γ is locally Lipschitzian and the problem

$$\text{minimize } \psi_\gamma(z) \text{ subject to } z \in R^n$$

has a solution.

In this section we prove the following result.

Theorem 2.9. *Assume that the following condition holds:*

For each finite strictly increasing sequence of natural numbers $\{j_i\}_{i=1}^q$ *which satisfies*

$$I_1 \subset \{j_1, \dots, j_q\} \subset \{1, \dots, m\}$$

the point $(c_{j_1}, \dots, c_{j_q})$ *is not a critical value of the mapping* $(f_{j_1}, \dots, f_{j_q}) : R^n \to R^q$.

Let $\gamma = (\gamma_1, \dots, \gamma_m) \in (0, \infty)^m$. *Then there exists a positive number* λ_0 *such that for each* $\lambda > \lambda_0$ *if* $x \in R^n$ *satisfies*

$$\psi_{\lambda\gamma}(x) = \inf(\psi_{\lambda\gamma}),$$

then $f_i(x) = c_i$ *for all* $i \in I_1$, $f_j(x) \le c_j$ *for all* $j \in I_2$ *and* $f_0(x) = \inf(f; A)$.

Proof: Assume the contrary. Then there exist a sequence $\{\lambda_k\}_{k=1}^\infty \subset (0, \infty)$ and a sequence $\{x^{(k)}\}_{k=1}^\infty \subset R^n$ such that

$$\lambda_k \ge k \text{ for all integers } k \ge 1, \tag{2.44}$$

$$\psi_{\lambda_k\gamma}(x^{(k)}) = \inf(\psi_{\lambda_k\gamma}) \text{ for all integers } k \ge 1, \tag{2.45}$$

$$x^{(k)} \notin A \text{ for all integers } k \ge 1. \tag{2.46}$$

For each integer $k \ge 1$ set

$$I_{1k+} = \{i \in I_1 : \ f_i(x^{(k)}) > c_i\},\ I_{1k-} = \{i \in I_1 : \ f_i(x^{(k)}) < c_i\}, \tag{2.47}$$

$$I_{2k+} = \{i \in I_2 : \ f_i(x^{(k)}) > c_i\},\ I_{2k-} = \{i \in I_2 : \ f_i(x^{(k)}) < c_i\}.$$

It follows from (2.47), (2.46) and (2.42) that

$$I_{1k+} \cup I_{1k-} \cup I_{2k+} \neq \emptyset \text{ for all natural numbers } k. \tag{2.48}$$

Extracting a subsequence and reindexing we may assume without loss of generality that for all integers $k \ge 1$

$$I_{1k+} = I_{11+},\ I_{1k-} = I_{11-},\ I_{2k+} = I_{21+},\ I_{2k-} = I_{21-}.$$

Put

$$I_{11} = I_1 \setminus (I_{11+} \cup I_{11-}),\ I_{21} = I_2 \setminus (I_{21+} \cup I_{21-}). \tag{2.49}$$

In view of (2.43) and (2.45), for each integer $k \ge 1$

$$\inf\{f_0(z) : \ z \in A\} = \inf\{\psi_{\lambda_k\gamma}(z) : \ z \in A\} \ge \inf(\psi_{\lambda_k\gamma})$$

$$= \psi_{\lambda_k\gamma}(x^{(k)}) = f_0(x^{(k)}) + \sum_{i \in I_1} \lambda_k\gamma_i |f_i(x^{(k)}) - c_i| + \sum_{i \in I_2} \lambda_k\gamma_i \max\{f_i(x^{(k)}) - c_i, 0\}. \tag{2.50}$$

By (2.50),

$$f_0(x^{(k)}) \le \inf\{f_0(z) : \ z \in A\} \text{ for all integers } k \ge 1. \tag{2.51}$$

Since f_0 satisfies growth condition (2.41) it follows from (2.51) that the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ is bounded. Extracting a subsequence and reindexing we may assume without loss of generality that there exists

$$\bar{x} = \lim_{k\to\infty} x^{(k)}. \tag{2.52}$$

Since f_0 is locally Lipschitzian there exists a positive number L such that

$$\partial f_0(x^{(k)}) \subset B(0, L) \text{ for all integers } k \geq 1. \tag{2.53}$$

Let $i \in I_1$. Relations (2.52) and (2.50) imply that

$$|f_i(\bar{x}) - c_i| = \lim_{k\to\infty} |f_i(x^{(k)}) - c_i| \leq \limsup_{k\to\infty} \lambda_k^{-1}\gamma_i^{-1}[\inf(f_0; A) - \inf(f_0)] = 0.$$

Hence

$$f_i(\bar{x}) = c_i \text{ for all } i \in I_1. \tag{2.54}$$

Let $i \in I_2$. By (2.52) and (2.50),

$$\max\{f_i(\bar{x}) - c_i, 0\} = \lim_{k\to\infty} \max\{f_i(x^{(k)}) - c_i, 0\}$$

$$\leq \limsup_{k\to\infty} \lambda_k^{-1}\gamma_i^{-1}[\inf(f_0; A) - \inf(f_0)] = 0.$$

Thus

$$f_i(\bar{x}) \leq c_i \text{ for all } i \in I_2. \tag{2.55}$$

In view of (2.45), (2.43), (2.47) and (2.49) for each natural number k

$$0 \in \partial\psi_{\lambda_k\gamma}(x^{(k)}) \subset \partial f_0(x^{(k)}) + \sum_{i\in I_{11+}} \lambda_k\gamma_i\nabla f_i(x^{(k)})$$

$$- \sum_{i\in I_{11-}} \lambda_k\gamma_i\nabla f_i(x^{(k)}) + \sum_{i\in I_{11}} \lambda_k\gamma_i\{\alpha\nabla f_i(x^{(k)}) :\ \alpha \in [-1, 1]\}$$

$$+ \sum_{i\in I_{21+}} \lambda_k\gamma_i\nabla f_i(x^{(k)}) + \sum_{i\in I_{21}} \lambda_k\gamma_i\{\alpha\nabla f_i(x^{(k)}) :\ \alpha \in [0, 1]\}. \tag{2.56}$$

It follows from (2.56) and (2.53) that for each natural number k

$$\Bigg[\sum_{i\in I_{11+}} \gamma_i\nabla f_i(x^{(k)}) - \sum_{i\in I_{11-}} \gamma_i\nabla f_i(x^{(k)}) + \sum_{i\in I_{21+}} \gamma_i\nabla f_i(x^{(k)})$$

$$+ \sum_{i\in I_{11}} \{\alpha\nabla f_i(x^{(k)}) :\ \alpha \in [-\gamma_i, \gamma_i]\} + \sum_{i\in I_{21}} \{\alpha\nabla f_i(x^{(k)}) :\ \alpha \in [0, \gamma_i]\}\Bigg]$$

$$\cap B(0, L/\lambda_k) \neq \emptyset.$$

Therefore for each integer $k \geq 1$ and each $i \in I_{11} \cup I_{21}$ there exists

$$\alpha_{ki} \in [-\gamma_i, \gamma_i] \tag{2.57}$$

such that

$$\Big\| \sum_{i \in I_{11+}} \gamma_i \nabla f_i(x^{(k)}) - \sum_{i \in I_{11-}} \gamma_i \nabla f_i(x^{(k)}) + \sum_{i \in I_{21+}} \gamma_i \nabla f_i(x^{(k)})$$

$$+ \sum_{i \in I_{11} \cup I_{21}} \alpha_{ki} \nabla f_i(x^{(k)}) \Big\| \le L/\lambda_k. \tag{2.58}$$

Extracting a subsequence and reindexing we may assume without loss of generality that for each $i \in I_{11} \cup I_{21}$ there exists

$$\alpha_i = \lim_{k \to \infty} \alpha_{ki}. \tag{2.59}$$

By (2.58), (2.59), (2.52) and (2.44),

$$\sum_{i \in I_{11+}} \gamma_i \nabla f_i(\bar{x}) - \sum_{i \in I_{11-}} \gamma_i \nabla f_i(\bar{x}) + \sum_{i \in I_{21+}} \gamma_i \nabla f_i(\bar{x})$$

$$+ \sum_{i \in I_{11} \cup I_{21}} \alpha_i \nabla f_i(\bar{x}) = 0. \tag{2.60}$$

It follows from (2.47), (2.49) and (2.55) that

$$f_i(\bar{x}) = c_i, \ i \in I_{21+} \cup I_{21}.$$

Combined with (2.54) this implies that

$$f_i(\bar{x}) = c_i, \ i \in I_1 \cup I_{21+} \cup I_{21}. \tag{2.61}$$

If $I_{21+} \cup I_{21} = \emptyset$ set

$$\tilde{F} = (f_1, \dots, f_p) : R^n \to R^p, \ \tilde{c} = (c_1, \dots, c_p).$$

If $I_{21+} \cup I_{21} \neq \emptyset$ put

$$\tilde{F} = (f_1, \dots, f_p, f_{j_1}, \dots, f_{j_q}) : R^n \to R^{p+q},$$

$$\tilde{c} = (c_1, \dots, c_p, c_{j_1}, \dots, c_{j_q}),$$

where $\{j_i\}_{i=1}^q$ is a strictly increasing sequence of integers such that

$$\{j_1, \dots, j_q\} = I_{21+} \cup I_{21}.$$

By (2.60) and (2.48), $\bar{x}$ is a critical point of $\tilde{F}$. Combined with (2.61) this implies that $\tilde{c}$ is a critical value of $\tilde{F}$, a contradiction. The contradiction we have reached proves Theorem 2.9.

Theorem 2.9 has been established in [122].

2.4 Inequality-constrained problems with convex constraint functions

In this section we begin to study the existence of the exact penalty for a large class of inequality-constrained minimization problems

$$\text{minimize } f(x) \text{ subject to } x \in A,$$

where

$$A = \{x \in X : \ g_i(x) \le c_i \text{ for } i = 1, \dots, n\}.$$

Here X is a Banach space, c_i, $i = 1, \dots, n$ are real numbers, the constraint functions g_i, $i = 1, \dots, n$ are convex and lower semicontinuous and the objective function f belongs to a space of functions $\mathcal{M}$ described below. This space of objective functions $\mathcal{M}$ is a convex cone in the vector space of all functions on X. It includes the set of all convex bounded from below semicontinuous functions $f : X \to R^1$ which satisfy the growth condition $\lim_{||x|| \to \infty} f(x) = \infty$ and the set of all functions f on X which satisfy the growth condition above and which are Lipschitzian on all bounded subsets of X. It should be mentioned that if f belongs to this class of functions and $g : R^1 \to R^1$ is an increasing Lipschitzian function, then $g \circ f$ also belongs to $\mathcal{M}$. Moreover, if $f \in \mathcal{M}$ and if a function $g : X \to R^1$ is Lipschitzian on all bounded subsets of X and the infimum of g is positive, then $f \cdot g \in \mathcal{M}$.

We associate with the inequality-constrained minimization problem above the corresponding family of unconstrained minimization problems

$$\text{minimize } f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \text{ subject to } z \in X$$

where $\gamma > 0$ is a penalty. We establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem.

We use the convention that $\lambda \cdot \infty = \infty$ for all $\lambda \in (0, \infty)$, $\lambda + \infty = \infty$ and $\max\{\lambda, \infty\} = \infty$ for any real number λ and that supremum over empty set is $-\infty$. We use the following notation and definitions.

Let $(X, ||\cdot||)$ be a Banach space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||x - y|| \le r\}.$$

For each function $f : X \to R^1 \cup \{\infty\}$ and each nonempty set $A \subset X$ put

$$\text{dom}(f) = \{x \in X : \ f(x) < \infty\},$$

$$\inf(f) = \inf\{f(z) : \ z \in X\}$$

and

$$\inf(f; A) = \inf\{f(z) : \ z \in A\}.$$

For each $x \in X$ and each $B \subset X$ set

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

Let $n \geq 1$ be an integer. For each $\kappa \in (0, 1)$ denote by Ω_κ the set of all $\gamma = (\gamma_1, \ldots, \gamma_n) \in R^n$ such that

$$\kappa \leq \min\{\gamma_i : \ i = 1, \ldots, n\} \text{ and } \max\{\gamma_i : \ i = 1, \ldots, n\} = 1. \tag{2.62}$$

Let $g_i : X \to R^1 \cup \{\infty\}$, $i = 1, \ldots, n$ be convex lower semicontinuous functions and $c = (c_1, \ldots, c_n) \in R^n$. Set

$$A = \{x \in X : \ g_i(x) \leq c_i \text{ for } i = 1, \ldots, n\}. \tag{2.63}$$

Let $f : X \to R^1 \cup \{\infty\}$ be a bounded from below lower semicontinuous function which satisfies the following growth condition:

$$\lim_{||x|| \to \infty} f(x) = \infty. \tag{2.64}$$

We suppose that there is $\tilde{x} \in X$ such that

$$g_j(\tilde{x}) < c_j \text{ for } j = 1, \ldots, n \text{ and } f(\tilde{x}) < \infty. \tag{2.65}$$

We consider the following constrained minimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in A. \tag{P}$$

By (2.65) $A \neq \emptyset$ and $\inf(f; A) < \infty$.

For each vector $\gamma = (\gamma_1, \ldots, \gamma_n) \in (0, \infty)^n$ define

$$\psi_\gamma(z) = f(z) + \sum_{i=1}^{n} \gamma_i \max\{g_i(z) - c_i, 0\}, \ z \in X. \tag{2.66}$$

It is easy to see that for each $\gamma \in (0, \infty)^n$ the function $\psi_\gamma : X \to R^1 \cup \{\infty\}$ is lower semicontinuous and satisfies $-\infty < \inf(\psi_\gamma) < \infty$. We associate with problem (P) the corresponding family of unconstrained minimization problems

$$\text{minimize } \psi_\gamma(z) \text{ subject to } z \in X \tag{P$_\gamma$}$$

where $\gamma \in (0, \infty)^n$.

In this section we assume that there exists a function $h : X \times \text{dom}(f) \to R^1 \cup \{\infty\}$ such that the following assumptions hold:

(A1) $h(z, y)$ is finite for each $y, z \in \text{dom}(f)$ and $h(y, y) = 0$ for each $y \in \text{dom}(f)$.

(A2) For each $y \in \text{dom}(f)$ the function $h(\cdot, y) \to R^1 \cup \{\infty\}$ is convex.

(A3) For each $z \in \text{dom}(f)$ and each positive number r,

$$\sup\{h(z, y) : \ y \in \text{dom}(f) \cap B(0, r)\} < \infty.$$

(A4) For each positive number M there exists a positive number M_1 such that for each $y \in X$ satisfying $f(y) \leq M$ there exists a neighborhood V of y in X such that $f(z) - f(y) \leq M_1 h(z, y)$ for each $z \in V$.

Remark 2.10. Note that if f is convex, then assumptions (A1)–(A4) hold with $h(z, y) = f(z) - f(y)$, $z \in X$, $y \in \text{dom}(f)$. In this case $M_1 = 1$ for all $M > 0$. If the function f is finite-valued and Lipschitzian on all bounded subsets of X, then assumptions (A1)–(A4) hold with $h(z, y) = ||z - y||$ for all $z, y \in X$.

The next theorem is the main result of the section.

Theorem 2.11. *Let $\kappa \in (0, 1)$. Then there exists a number $\Lambda_0 > 0$ such that for each $\epsilon > 0$ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

If $\gamma \in \Omega_\kappa$, $\lambda \geq \Lambda_0$ and if $x \in X$ satisfies

$$\psi_{\lambda\gamma}(x) \leq \inf(\psi_{\lambda\gamma}) + \delta,$$

then there exists $y \in A$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; A) + \epsilon.$$

Theorem 2.11 will be proved in Section 2.6. In this section we present several important results which easily follow from Theorem 2.11.

Theorem 2.11 implies the following result.

Corollary 2.12. *Let $\kappa \in (0, 1)$. Then there exists a positive number Λ_0 such that for each $\gamma \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$ and each sequence $\{x_i\}_{i=1}^\infty \subset X$ which satisfies*

$$\lim_{i \to \infty} \psi_{\lambda\gamma}(x_i) = \inf(\psi_{\lambda\gamma}),$$

there exists a sequence $\{y_i\}_{i=1}^\infty \subset A$ such that

$$\lim_{i \to \infty} f(y_i) = \inf(f; A) \text{ and } \lim_{i \to \infty} ||y_i - x_i|| = 0.$$

The next result follows from Corollary 2.12.

Corollary 2.13. *Let $\kappa \in (0, 1)$. Assume that there exists $\bar{x} \in A$ for which the following conditions hold:*

$$f(\bar{x}) = \inf(f; A);$$

any sequence $\{x_n\}_{n=1}^\infty \subset A$ which satisfies $\lim_{n \to \infty} f(x_n) = \inf(f; A)$ converges to $\bar{x}$ in the norm topology.

Then there exists a positive number Λ_0 such that for each $\gamma \in \Omega_\kappa$ and each $\lambda \geq \Lambda_0$ the point $\bar{x}$ is a unique solution of the minimization problem

$$\text{minimize } \psi_{\lambda\gamma}(z) \text{ subject to } z \in X.$$

Corollary 2.12 implies the following result.

Corollary 2.14. *Let $\kappa \in (0,1)$. Then there exists $\Lambda_0 > 0$ such that if $\gamma \in \Omega_\kappa$, $\lambda \geq \Lambda_0$ and if x is a solution of the minimization problem*

$$\textit{minimize } \psi_{\lambda\gamma}(z) \textit{ subject to } z \in X,$$

then $x \in A$ and $f(x) = \inf(f; A)$.

It is clear that Corollary 2.14 is the classical exact penalty result.

Denote by $\mathcal{M}$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ satisfying (2.64) which are not identically infinity and for which there exists a function $h_f : X \times \operatorname{dom}(f) \to R^1 \cup \{\infty\}$ such that assumptions (A1)–(A4) hold with $h = h_f$.

We have already mentioned (see Remark 2.10) that $f \in \mathcal{M}$ if at least one of the following conditions holds:

$f : X \to R^1 \cup \{\infty\}$ is a convex bounded from below lower semicontinuous function satisfying (2.64) which is not identically infinity;

$f : X \to R^1$ is a function which satisfies (2.64) and which is Lipschitzian on all bounded subsets of X.

Obviously, if $f \in \mathcal{M}$ and $\lambda > 0$, then $\lambda f \in \mathcal{M}$. It is not difficult to see that the following properties hold:

if $f_1 \in \mathcal{M}$ and $f_2 : X \to R^1$ is a convex lower semicontinuous bounded from below function, then $f_1 + f_2 \in \mathcal{M}$;

if $f_1 \in \mathcal{M}$ and $f_2 : X \to R^1$ is a bounded from below function which is Lipschitzian on all bounded subsets of X, then $f_1 + f_2 \in \mathcal{M}$.

Note that we establish our result (Theorem 2.11) for any $f \in \mathcal{M}$ such that (2.65) holds with some $\tilde{x} \in X$.

In Section 2.5 we will prove the following three results which show that the class $\mathcal{M}$ is essentially larger than the union of the set of all convex bounded from below lower semicontinuous functions satisfying the growth condition (2.64) and the set of all functions satisfying (2.64) which are Lipschitzian on all bounded subsets of X.

Proposition 2.15. *Let $f_1, f_2 \in \mathcal{M}$, $dom(f_1) \subset dom(f_2)$, $h_i : X \times dom(f_i) \to R^1 \cup \{\infty\}$, $i = 1, 2$ and let for $i = 1, 2$ assumptions (A1)–(A4) hold with $f = f_i$, $h = h_i$. Then $f_1 + f_2 \in \mathcal{M}$ and (A1)–(A4) hold with $f = f_1 + f_2$ and with a function $h : X \times dom(f_1) \to R^1 \cup \{\infty\}$ defined by*

$$h(z, y) = \max\{h_1(z, y), 0\} + \max\{h_2(z, y), 0\},\ x \in X,\ y \in \ dom(f_1). \quad (2.67)$$

Corollary 2.16. *Let $f_1 \in \mathcal{M}$ be a convex function and $f_2 \in \mathcal{M}$ be a finite-valued function which is Lipschitzian on all bounded subsets of X. Then $f_1 + f_2 \in \mathcal{M}$ and (A1)–(A4) hold with $f = f_1 + f_2$ and with a function $h : X \times dom(f_1) \to R^1 \cup \{\infty\}$ defined by*

$$h(z, y) = \max\{f_1(z) - f_1(y), 0\} + ||z - y||,\ x \in X,\ y \in \ dom(f_1).$$

Proposition 2.17. *Let $f \in \mathcal{M}$ and $g : R^1 \to R^1$ be an increasing Lipschitzian function such that $\lim_{t \to \infty} g(t) = \infty$. Then $g \circ f \in \mathcal{M}$.*

(Here we assume that $g(\infty) = \infty$.)

Proposition 2.18. *Let $f \in \mathcal{M}$ and let a function $g : X \to R^1$ be Lipschitzian on all bounded subsets of X and satisfy*

$$\inf\{g(x) : \ x \in X\} > 0.$$

Then $f \cdot g \in \mathcal{M}$.

The results of this section have been obtained in [133].

2.5 Proofs of Propositions 2.15, 2.17 and 2.18

Proof of Proposition 2.15: Clearly, the function $f_1 + f_2$ is bounded from below, lower semicontinuous and satisfies

$$\lim_{||x|| \to \infty} (f_1 + f_2)(x) = \infty \text{ and } \operatorname{dom}(f_1 + f_2) = \operatorname{dom}(f_1).$$

There exists a number c_0 such that

$$f_i(x) \geq -c_0 \text{ for all } x \in X \text{ and } i = 1, 2. \tag{2.68}$$

Put $f = f_1 + f_2$. Evidently, (A1)–(A3) hold for the function h defined by (2.67). In order to complete the proof of the proposition it is sufficient to show that (A4) holds for h.

Let M be a positive number. Since for $j = 1, 2$ the assumption (A4) holds with $f = f_j$, $h = h_j$ there exist positive numbers M_1, M_2 such that the following property holds:

(i) For $j = 1, 2$ and each $y \in X$ satisfying $f_j(y) \leq M + c_0$ there exists a neighborhood V of y in X such that $f_j(z) - f_j(y) \leq M_j h_j(z, y)$ for each $z \in V$.

Assume that $y \in X$ satisfies $(f_1 + f_2)(y) \leq M$. Combined with (2.68) this inequality implies that $f_j(y) \leq M + c_0$, $j = 1, 2$.

It follows from this inequality and property (i) that there exists a neighborhood V of y in X such that

$$f_j(z) - f_j(y) \leq M_j h_j(z, y) \tag{2.69}$$

for each $z \in V$ and $j = 1, 2$. By (2.69) and (2.67), for each $z \in V$,

$$(f_1 + f_2)(z) - (f_1 + f_2)(y) \leq M_1 h_1(z, y) + M_2 h_2(z, y)$$

$$\leq (M_1 + M_2)[\max\{h_1(z, y), 0\} + \max\{h_2(z, y), 0\}] = (M_1 + M_2)h(z, y).$$

Proposition 2.15 is proved.

Proof of Proposition 2.17: Clearly, the function $g \circ f$ is bounded from below, lower semicontinuous and satisfies

$$\lim_{||x||\to\infty} g(f(x)) = \infty \text{ and } \mathrm{dom}(g \circ f) = \mathrm{dom}(f) \neq \emptyset.$$

Since $f \in \mathcal{M}$ there exists a function $h : X \times \mathrm{dom}(f) \to R^1 \cup \{\infty\}$ such that assumptions (A1)–(A4) hold.

Since the function $g : R^1 \to R^1$ is Lipschitzian there exists a positive number L such that

$$|g(t_2) - g(t_1)| \le L|t_2 - t_1| \text{ for each } t_2, t_1 \in R^1. \tag{2.70}$$

Set

$$\xi(z, y) = L \max\{h(z, y), 0\} \text{ for all } z \in X \text{ and all } y \in \mathrm{dom}(f). \tag{2.71}$$

Evidently, assumptions (A1)–(A3) hold with $h = \xi$.

In order to complete the proof of the proposition it is sufficient to show that the following property holds:

(ii) For each positive number M there exists a positive number M_0 such that for each $y \in X$ satisfying $g(f(y)) \le M$ there exists a neighborhood V of y in X such that

$$g(f(z)) - g(f(y)) \le M_0 \xi(z, y)$$

for each $z \in V$.

Let M be a positive number. There exists $M_1 > 0$ such that

$$t \le M_1 \text{ for each } t \in R^1 \text{ satisfying } g(t) \le M. \tag{2.72}$$

It follows from (A4) that there exists a positive number M_0 such that the following property holds:

(iii) For each $y \in X$ satisfying $f(y) \le M_1$ there exists a neighborhood V of y in X such that

$$f(z) - f(y) \le M_0 h(z, y)$$

for each $z \in V$.

Assume that $y \in X$ satisfies

$$g(f(y)) \le M. \tag{2.73}$$

By the choice of M (see (2.72))

$$f(y) \le M_1. \tag{2.74}$$

By (2.74) and property (iii), there is a neighborhood V of y in X such that

$$f(z) - f(y) \le M_0 h(z, y) \tag{2.75}$$

for each $z \in V$.

Let $z \in V$. We estimate $g(f(z)) - g(f(y))$. Since g is increasing we have

$$g(f(z)) - g(f(y)) \le 0 \text{ if } f(z) \le f(y). \tag{2.76}$$

Assume that

$$f(z) > f(y). \tag{2.77}$$

Since the function g is increasing we have $g(f(z)) \geq g(f(y))$. Combined with (2.70), (2.77) and (2.75) this inequality implies that

$$0 \leq g(f(z)) - g(f(y)) = |g(f(z)) - g(f(y))|$$

$$\leq L|f(z) - f(y)| = L(f(z) - f(y)) \leq M_0 L h(z, y).$$

Combined with (2.76) and (2.71) this inequality implies that in both cases

$$g(f(z)) - g(f(y)) \leq M_0 L \max\{h(z, y), 0\} = M_0 \xi(z, y).$$

Thus we have shown that for each $y \in X$ satisfying (2.73) there exists a neighborhood V of y in X such that

$$g(f(z)) - g(f(y)) \leq M_0 \xi(z, y)$$

for each $z \in V$. Thus property (ii) holds. This completes the proof of Proposition 2.17.

Proof of Proposition 2.18: There exists a positive number c_0 such that

$$g(x) \geq c_0 \text{ for all } x \in X. \tag{2.78}$$

Clearly, the function $f \cdot g$ is lower semicontinuous and satisfies

$$\lim_{||x|| \to \infty} f(x)g(x) = \infty \text{ and } \mathrm{dom}(fg) = \mathrm{dom}(f). \tag{2.79}$$

Let us show that the function $f \cdot g$ is bounded from below. In view of (2.79) there exists a positive number M_0 such that

$$f(x)g(x) \geq 1 \text{ for all } x \in X \text{ satisfying } ||x|| \geq M_0. \tag{2.80}$$

Since g is Lipschitzian on bounded subsets of X there exists a positive number c_1 such that

$$g(x) \leq c_1 \text{ for all } x \in B(0, M_0). \tag{2.81}$$

Since f is bounded from below there exists a real number c_2 such that

$$f(x) \geq c_2 \text{ for all } x \in X. \tag{2.82}$$

It follows from (2.78), (2,82) and (2.81) that for each $x \in B(0, M_0)$,

$$f(x)g(x) \geq g(x)c_2 \geq -|g(x)c_2| \geq -c_1|c_2|.$$

Together with (2.80) this inequality implies that

$$f(x)g(x) \geq -c_1|c_2| \text{ for each } x \in X.$$

Thus we have shown that the function $f \cdot g$ is lower semicontinuous, bounded from below and satisfies (2.79).

Since $f \in \mathcal{M}$ there exists a function $h : X \times \mathrm{dom}(f) \to R^1 \cup \{\infty\}$ such that assumptions (A1)–(A4) hold. We may assume without loss of generality that $h(z,y) \geq 0$ for all $z \in X$ and all $y \in \mathrm{dom}(f)$. Set

$$\xi(z,y) = h(z,y) + ||z-y|| \text{ for all } z \in X \text{ and all } y \in \ \mathrm{dom}(f). \tag{2.83}$$

It is not difficult to see that assumptions (A1)–(A3) hold with $h = \xi$.

In order to complete the proof of the proposition it is sufficient to show that the following property holds:

(iv) For each positive number M there exists a positive number M_0 such that for each $y \in X$ satisfying $g(y)f(y) \leq M$ there exists a neighborhood V of y in X such that

$$g(z)f(z) - g(y)f(y) \leq M_0\xi(z,y)$$

for each $z \in V$.

Let M be a positive number. By (A4) there exists $M_1 > 0$ such that the following property holds:

(v) For each $y \in X$ satisfying $f(y) \leq M/c_0$ there exists a neighborhood V of y in X such that

$$f(z) - f(y) \leq M_1 h(z,y) \tag{2.84}$$

for each $z \in V$.

It follows from (2.64) that there exists a positive number M_2 such that

$$\text{if } x \in X \text{ satisfies } f(x) \leq Mc_0^{-1}, \text{ then } ||x|| \leq M_2. \tag{2.85}$$

Since the function g is Lipschitzian on bounded subsets of X there exists a positive number M_3 such that

$$g(x) \leq M_3 \text{ for each } x \in B(0, M_2+1), \tag{2.86}$$

$$|g(y_1) - g(y_2)| \leq M_3||y_1 - y_2|| \text{ for each } y_1, y_2 \in B(0, M_2+1). \tag{2.87}$$

Assume that $y \in X$ satisfies

$$g(y)f(y) \leq M. \tag{2.88}$$

By (2.88) and (2.78),

$$f(y) \leq Mg(y)^{-1} \leq Mc_0^{-1}. \tag{2.89}$$

Let a neighborhood V of y in X be as guaranteed by the property (v). Hence (2.84) is valid for each $z \in V$. We may assume without loss of generality that

$$V \subset B(y,1). \tag{2.90}$$

Let $z \in V$. Relations (2.89), (2.85) and (2.90) imply that

$$||y|| \leq M_2 \text{ and } ||z|| \leq M_2 + 1.$$

By these inequalities, (2.84), (2.78), nonnegativity of h, (2.86), (2.89), (2.82), (2.83) and (2.87),

$$g(z)f(z) - f(y)g(y) = g(z)(f(z) - f(y)) + f(y)(g(z) - g(y))$$

$$\leq g(z)M_1h(z,y) + |f(y)||g(z) - g(y)| \leq M_3M_1h(z,y) + (|c_2| + Mc_0^{-1})M_3||z - y||$$

$$\leq [h(z,y) + ||z - y||](M_3M_1 + M_3(|c_2| + Mc_0^{-1}))$$

$$= \xi(z,y)(M_3M_1 + M_3(|c_2| + Mc_0^{-1})).$$

Thus we have shown that for each $y \in X$ satisfying (2.88) there exists a neighborhood V of y in X such that

$$g(z)f(z) - f(y)g(y) \leq \xi(z,y)(M_3M_1 + M_3(|c_2| + Mc_0^{-1}))$$

for each $z \in V$. Thus property (iv) holds. This completes the proof of Proposition 2.18.

2.6 Proof of Theorem 2.11

We show that there exists a positive number Λ_0 such that the following property holds:

(P1) For each $\epsilon \in (0,1)$ there exists $\delta \in (0,\epsilon)$ such that for each $\lambda \geq \Lambda_0$, each $\gamma \in \Omega_\kappa$ and each $x \in X$ which satisfies

$$\psi_{\lambda\gamma}(x) \leq \inf(\psi_{\lambda\gamma}) + \delta \tag{2.91}$$

there is $y \in A$ for which

$$y \in B(x,\epsilon) \text{ and } \psi_{\lambda\gamma}(y) \leq \psi_{\lambda\gamma}(x). \tag{2.92}$$

Assume the contrary. Then for each integer $k \geq 1$ there exist

$$\epsilon_k \in (0,1),\ \gamma^{(k)} = (\gamma_1^{(k)}, \ldots, \gamma_n^{(k)}) \in \Omega_\kappa,\ \lambda_k \geq k \text{ and } x_k \in X \tag{2.93}$$

such that

$$\psi_{\lambda_k\gamma^{(k)}}(x_k) \leq \inf(\psi_{\lambda_k\gamma^{(k)}}) + 2^{-1}\epsilon_k k^{-2} \tag{2.94}$$

and

$$\{y \in B(x_k,\epsilon_k) \cap A :\ \psi_{\lambda_k\gamma^{(k)}}(y) \leq \psi_{\lambda_k\gamma^{(k)}}(x)\} = \emptyset. \tag{2.95}$$

Let $k \geq 1$ be an integer. By (2.94) and Ekeland's variational principle [37], there exists $y_k \in X$ such that

$$\psi_{\lambda_k \gamma^{(k)}}(y_k) \le \psi_{\lambda_k \gamma^{(k)}}(x_k), \tag{2.96}$$

$$y_k \in B(x_k, 2^{-1}k^{-1}\epsilon_k), \tag{2.97}$$

and

$$\psi_{\lambda_k \gamma^{(k)}}(y_k) \le \psi_{\lambda_k \gamma^{(k)}}(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{2.98}$$

It follows from (2.95)–(2.97) that

$$y_k \notin A \text{ for all integers } k \ge 1. \tag{2.99}$$

For each integer $k \ge 1$ we put

$$I_{1k} = \{i \in \{1, \dots, n\} : \ g_i(y_k) > c_i\}, \tag{2.100}$$

$$I_{2k} = \{i \in \{1, \dots, n\} : \ g_i(y_k) = c_i\},$$

$$I_{3k} = \{i \in \{1, \dots, n\} : \ g_i(y_k) < c_i\}.$$

By (2.99),

$$I_{1k} \ne \emptyset \text{ for all natural numbers } k. \tag{2.101}$$

Extracting a subsequence and reindexing we may assume without loss of generality that

$$I_{1k} = I_{11}, \ I_{2k} = I_{21}, \ I_{3k} = I_{31} \text{ for all integers } k \ge 1. \tag{2.102}$$

By (2.63), (2.66), (2.94), (2.93), (2.96), (2.100) and (2.102) for each integer $k \ge 1$,

$$\inf\{f(z) : \ z \in A\} = \inf\{\psi_{\lambda_k \gamma^{(k)}}(z) : \ z \in A\} \ge \inf(\psi_{\lambda_k \gamma^{(k)}}) \tag{2.103}$$

$$\ge \psi_{\lambda_k \gamma^{(k)}}(x_k) - 1 \ge \psi_{\lambda_k \gamma^{(k)}}(y_k) - 1 = f(y_k) + \sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)}(g_i(y_k) - c_i) - 1.$$

Relations (2.100), (2.102) and (2.103) imply that

$$f(y_k) \le \inf\{f(z) : \ z \in A\} + 1 \text{ for all integers } k \ge 1. \tag{2.104}$$

In view of (2.104) and (2.64) the sequence $\{y_k\}_{k=1}^{\infty}$ is bounded. Fix a number $M > 2$ such that

$$y_k \in B(0, M - 2) \text{ for all integers } k \ge 1. \tag{2.105}$$

Assumption (A4) implies that there exists a positive number M_1 such that the following property holds:

(P2) for each $y \in X$ satisfying $f(y) \le 1 + |\inf(f; A)|$ there exists a neighborhood V of y in X such that

$$f(z) - f(y) \le M_1 h(z, y)$$

for each $z \in V$.

By (P2) and (2.104) for each integer $k \geq 1$ there exists a neighborhood V_k of y_k in X such that

$$f(z) - f(y_k) \leq M_1 h(z, y_k) \tag{2.106}$$

for each $z \in V_k$. It follows from (2.103), (2.102), (2.100) and (2.93) that for each $i \in I_{11}$ and each natural number k

$$0 < g_i(y_k) - c_i \leq [1 + \inf(f; A) - \inf(f)](\gamma_i^{(k)})^{-1} k^{-1}. \tag{2.107}$$

Let k be a natural number. Since the functions g_i, $i = 1, \dots, n$ are lower semicontinuous it follows from (2.100) and (2.102) that there exists $r_k \in (0, 1)$ such that for each $y \in B(y_k, r_k)$

$$g_i(y) > c_i \text{ for each } i \in I_{11}. \tag{2.108}$$

By (2.108), (2.98), (2.100), (2.102), (2.96), (2.94) and (2.66) for each $z \in B(y_k, r_k) \cap \mathrm{dom}(f)$

$$\sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} (g_i(z) - c_i) + \sum_{i \in I_{21} \cup I_{31}} \lambda_k \gamma_i^{(k)} \max\{g_i(z) - c_i, 0\}$$

$$- \sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} (g_i(y_k) - c_i) - \sum_{i \in I_{21} \cup I_{31}} \lambda_k \gamma_i^{(k)} \max\{g_i(y_k) - c_i, 0\}$$

$$= \psi_{\lambda_k \gamma^{(k)}}(z) - \psi_{\lambda_k \gamma^{(k)}}(y_k) - f(z) + f(y_k) \geq -k^{-1}||y_k - z|| - f(z) + f(y_k).$$

It follows from this inequality that for each $z \in B(y_k, r_k)$

$$\sum_{i \in I_{11}} \gamma_i^{(k)} g_i(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(z) - c_i, 0\}$$

$$- \sum_{i \in I_{11}} \gamma_i^{(k)} g_i(y_k) - \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(y_k) - c_i, 0\}$$

$$+ \lambda_k^{-1}(f(z) - f(y_k)) \geq -\lambda_k^{-1} k^{-1} ||y_k - z||.$$

By the inequality above and (2.106) for each $z \in B(y_k, r_k) \cap V_k$

$$\sum_{i \in I_{11}} \gamma_i^{(k)} g_i(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(z) - c_i, 0\}$$

$$+ \lambda_k^{-1} M_1 h(z, y_k) + \lambda_k^{-1} k^{-1} ||z - y_k||$$

$$\geq \sum_{i \in I_{11}} \gamma_i^{(k)} g_i(y_k) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(y_k) - c_i, 0\}. \tag{2.109}$$

(A2) implies that the function

$$\sum_{i \in I_{11}} \gamma_i^{(k)} g_i(x) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(x) - c_i, 0\}$$

$$+\lambda_k^{-1} M_1 h(x, y_k) + \lambda_k^{-1} k^{-1} ||x - y_k||, \ x \in X$$

is convex. Combined with the equality $h(y_k, y_k) = 0$ (see (A1)) this implies that (2.109) holds for all $z \in X$.

Extracting a subsequence and reindexing we may assume without loss of generality that for each $i \in \{1, \dots, n\}$ there exists

$$\gamma_i = \lim_{k \to \infty} \gamma_i^{(k)} \in [0, 1]. \tag{2.110}$$

Evidently $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$. Assume that

$$z \in \ \mathrm{dom}(f) \cap [\cap_{i=1}^n \ \mathrm{dom}(g_i)]. \tag{2.111}$$

Relations (2.105) and (2.93) imply that

$$\lim_{k \to \infty} \lambda_k^{-1} k^{-1} ||z - y_k|| = 0. \tag{2.112}$$

It follows from (2.93), (2.105), (2.104) and (A3) that

$$\lim_{k \to \infty} \lambda_k^{-1} M_1 h(z, y_k) = 0.$$

By this equality, (2.111), (2.110), (2.112), (2.109), (2.100), (2.102), (2.107), (2.93) and the definition of Ω_κ (see (2.62)),

$$\sum_{i \in I_{11}} \gamma_i g_i(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i \max\{g_i(z) - c_i, 0\}$$

$$= \lim_{k \to \infty} [\sum_{i \in I_{11}} \gamma_i^{(k)} g_i(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(z) - c_i, 0\}$$

$$+\lambda_k^{-1} k^{-1} ||z - y_k|| + \lambda_k^{-1} M_1 h(z, y_k)]$$

$$\geq \limsup_{k \to \infty} [\sum_{i \in I_{11}} \gamma_i^{(k)} g_i(y_k) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i(y_k) - c_i, 0\}]$$

$$= \limsup_{k \to \infty} \sum_{i \in I_{11}} \gamma_i^{(k)} g_i(y_k) = \sum_{i \in I_{11}} (\lim_{k \to \infty} \gamma_i^{(k)}) \lim_{k \to \infty} g_i(y_k) = \sum_{i \in I_{11}} \gamma_i c_i.$$

Thus we have shown that for each $z \in X$ satisfying (2.111) the following inequality holds:

$$\sum_{i \in I_{11}} \gamma_i g_i(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i \max\{g_i(z) - c_i, 0\} \geq \sum_{i \in I_{11}} \gamma_i c_i.$$

Combined with (2.65), (2.111) and the inclusion $\gamma \in \Omega_\kappa$ this inequality implies that

$$\sum_{i \in I_{11}} \gamma_i c_i \leq \sum_{i \in I_{11}} \gamma_i g_i(\tilde{x}) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i \max\{g_i(\tilde{x}) - c_i, 0\}$$

$$= \sum_{i \in I_{11}} \gamma_i g_i(\tilde{x}) < \sum_{i \in I_{11}} \gamma_i c_i.$$

The contradiction we have reached proves that there is a positive number Λ_0 such that property (P1) holds.

Let $\epsilon \in (0, 1)$ and let $\delta \in (0, \epsilon)$ be as guaranteed by property (P1). Assume that $\lambda \geq \Lambda_0$, $\gamma \in \Omega_\kappa$ and that $x \in X$ satisfies

$$\psi_{\lambda\gamma}(x) \leq \inf(\psi_{\lambda\gamma}) + \delta. \tag{2.113}$$

Then by (P1) and the choice of δ there exists $y \in A$ such that

$$y \in B(x, \epsilon) \text{ and } \psi_{\lambda\gamma}(y) \leq \psi_{\lambda\gamma}(x). \tag{2.114}$$

By (2.66), (2.114) and (2.113),

$$f(y) \leq \psi_{\lambda\gamma}(y) \leq \psi_{\lambda\gamma}(x) \leq \inf(\psi_{\lambda\gamma}) + \delta$$

$$\leq \inf(\psi_{\lambda\gamma}; A) + \epsilon = \inf(f; A) + \epsilon.$$

This completes the proof of Theorem 2.11.

2.7 Optimization problems with mixed nonsmooth nonconvex constraints

Let $(X, ||\cdot||)$ be a Banach space and $(X^*, ||\cdot||_*)$ be its dual space. For each $x \in X$, each $x^* \in X^*$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||y - x|| \leq r\}, \ B_*(x^*, r) = \{l \in X^* : \ ||l - x^*||_* \leq r\}.$$

Assume that $f : U \to R^1$ is a Lipschitzian function which is defined on a nonempty open set $U \subset X$. For each $x \in U$ let

$$f^0(x, h) = \limsup_{t \to 0^+, y \to x} [f(y + th) - f(y)]/t, \ h \in X$$

be the Clarke generalized directional derivative of f at the point x [21], let

$$\partial f(x) = \{l \in X^* : \ f^0(x, h) \geq l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of f at x [21] and put

$$\Xi_f(x) = \inf\{f^0(x, h) : \ h \in X \text{ and } ||h|| \leq 1\}[100].$$

Recall that a point $x \in U$ is called a critical point of f if $0 \in \partial f(x)$. A real number $c \in R^1$ is called a critical value of f if there is a critical point $x \in U$ of f such that $f(x) = c$.

In Section 2.1 we considered the constrained problems

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}(c)$$

and

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}((-\infty, c])$$

where c is a real number, $f : X \to R^1$ is a function which is Lipschitzian on all bounded subsets of X and which satisfies the following growth condition

$$\lim_{||x|| \to \infty} f(x) = \infty$$

and $g : X \to R^1$ is a locally Lipschitzian function which satisfies the Palais–Smale (P-S) condition [8, 76, 100]:

If $\{x_i\}_{i=1}^{\infty} \subset X$, the sequence $\{g(x_i)\}_{i=1}^{\infty}$ is bounded and if

$$\liminf_{i \to \infty} \Xi_g(x_i) \geq 0,$$

then there is a norm convergent subsequence of $\{x_i\}_{i=1}^{\infty}$.

The Palais–Smale condition for differentiable functions was introduced in [76]. Its version for Lipschitzian functions was given in [100]. This condition is very useful when we consider problems in infinite-dimensional Banach spaces whose bounded subsets are not compact.

We showed in Section 2.1 that the constrained problems given above have the exact penalty property if c is not a critical value of g. In order to generalize this result for a constrained problem with mixed constraints we need to introduce a notion of a critical point for a Lipschitzian mapping $F : X \to R^n$ and a version of (P-S) condition for F.

Assume that n is a natural number, U is a nonempty open subset of X and $F = (f_1, \dots, f_n) : U \to R^n$ is a locally Lipschitzian mapping.

Let $\kappa \in (0, 1)$. For each $x \in U$ set

$$\Xi_{F,\kappa}(x) = \inf\{||\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i2}\eta_{i2})|| : \tag{2.115}$$

$$\eta_{i1},\ \eta_{i2} \in \partial f_i(x),\ \alpha_{i1}, \alpha_{i2} \in [0, 1],\ i = 1, \dots, n$$

$$\text{and there is } j \in \{1, \dots, n\} \text{ such that } \alpha_{j1}\alpha_{j2} = 0,\ |\alpha_{j1}| + |\alpha_{j2}| \geq \kappa\}.$$

It is known (see Chapter 2, Section 2.3 of [21]) that for each $x \in U$ and $i = 1, \dots, n$

$$\partial(-f_i)(x) = -\partial f_i(x). \tag{2.116}$$

This equality implies that

$$\Xi_{-F,\kappa}(x) = \Xi_{F,\kappa}(x) \text{ for each } x \in U. \tag{2.117}$$

In the sequel we assume that $U = X$.

A point $x \in X$ is called a critical point of F with respect to κ if $\Xi_{F,\kappa}(x) = 0$.

A vector $c = (c_1, \dots, c_n) \in R^n$ is called a critical value of F with respect to κ if there is a critical point $x \in X$ of F with respect to κ such that $F(x) = c$.

Remark 2.19. Let $n = 1$. Then $x \in X$ is a critical point of F with respect to κ if and only if $0 \in \partial F(x)$. Therefore x is a critical point of F in our sense if and only if x is a critical point of F in the sense of [100, 122]. It is clear that in this case the notion of a critical point does not depend on κ.

Remark 2.20. Assume that $f_i \in C^1$, $i = 1, \dots, n$ and $Df_i(x)$ is the Frechet derivative of f_i at $x \in X$, $i = 1, \dots, n$. If $x \in X$ is a critical point of F with respect to κ, then $Df_i(x)$, $i = 1, \dots, n$ are linearly dependent.

In the sequel we use the following proposition.

Proposition 2.21. *Assume that* $\{x_k\}_{k=1}^{\infty} \subset X$, $x = \lim_{k\to\infty} x_k$ *in the norm topology and*

$$\liminf_{k\to\infty} \Xi_{F,\kappa}(x_k) = 0.$$

Then $\Xi_{F,\kappa}(x) = 0$.

Proof: We may assume without loss of generality that

$$\lim_{k\to\infty} \Xi_{F,\kappa}(x_k) = 0.$$

For each natural number k and each $i \in \{1, \dots, n\}$ there exist

$$\alpha_{i1}^{(k)},\ \alpha_{i2}^{(k)} \in [0,1],\ \eta_{i1}^{(k)},\ \eta_{i2}^{(k)} \in \partial f_i(x_k)$$

and $j_k \in \{1, \dots, n\}$ such that

$$\alpha_{j_k 1}^{(k)}\alpha_{j_k 2}^{(k)} = 0,\ |\alpha_{j_k 1}^{(k)}|+|\alpha_{j_k 2}^{(k)}| \geq \kappa,\ ||\sum_{i=1}^{n}(\alpha_{i1}^{(k)}\eta_{i1}^{(k)} - \alpha_{i2}^{(k)}\eta_{i2}^{(k)})|| \leq \Xi_{F,\kappa}(x_k) + 1/k. \tag{2.118}$$

Extracting a subsequence and reindexing if necessary we may assume that for each $i \in \{1, \dots, n\}$ there exist

$$\alpha_{i1} = \lim_{k\to\infty} \alpha_{i1}^{(k)},\ \alpha_{i2} = \lim_{k\to\infty} \alpha_{i2}^{(k)} \tag{2.119}$$

and $j_k = j_1$ for all natural numbers k.

Since the unit ball in the space X^* with the weak-star topology is compact there exists

$$\{\eta_{11}, \eta_{12}, \dots, \eta_{i1}, \eta_{i2}, \dots, \eta_{n1}, \eta_{n2}\} \in (X^*)^{2n}$$

which is a cluster point of

$$\{\eta_{11}^{(k)}, \eta_{12}^{(k)}, \dots, \eta_{i1}^{(k)}, \eta_{i2}^{(k)}, \dots, \eta_{n1}^{(k)}, \eta_{n2}^{(k)}\}_{k=1}^{\infty} \in (X^*)^{2n}$$

in the weak-star topology. It follows from the upper semicontinuity of the Clarke generalized directional derivative $f_i^0(\xi, \eta)$ with respect to ξ that

$$\eta_{i1},\ \eta_{i2} \in \partial f_i(x),\ i = 1, \dots, n. \tag{2.120}$$

We will show that $\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i2}\eta_{i2}) = 0$. Let ϵ be a positive number and let $l \in B_*(0,1)$. There exists an integer $k \geq 1$ such that

$$1/k < \epsilon/8, \; \Xi_{F,\kappa}(x_k) < \epsilon/8, \tag{2.121}$$

$$|\alpha_{ij}^{(k)} - \alpha_{ij}||\eta_{ij}| < (8n)^{-1}\epsilon \text{ and}$$

$$|l(\eta_{ij} - \eta_{ij}^{(k)})| < (8n)^{-1}\epsilon \text{ for } i = 1, \dots, n \text{ and } j = 1, 2.$$

Combined with (2.118) these relations imply that

$$|l(\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i2}\eta_{i2}))| \leq |l(\sum_{i=1}^{n}(\alpha_{i1}^{(k)}\eta_{i1}^{(k)} - \alpha_{i2}^{(k)}\eta_{i2}^{(k)}))|$$

$$+|l(\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i1}^{(k)}\eta_{i1}^{(k)}))| + |l(\sum_{i=1}^{n}(\alpha_{i2}\eta_{i2} - \alpha_{i2}^{(k)}\eta_{i2}^{(k)}))|$$

$$\leq \Xi_{F,\kappa}(x_k) + 1/k + \sum_{i=1}^{n}(\alpha_{i1}^{(k)}|l(\eta_{i1} - \eta_{i1}^{(k)})|) + \sum_{i=1}^{n}|\alpha_{i1} - \alpha_{i1}^{(k)}|||\eta_{i1}||$$

$$+\sum_{i=1}^{n}(\alpha_{i2}^{(k)}|l(\eta_{i2} - \eta_{i2}^{(k)})|) + \sum_{i=1}^{n}|\alpha_{i2} - \alpha_{i2}^{(k)}|||\eta_{i2}|| < \epsilon.$$

Since l is an arbitrary element of $B_*(0,1)$ and ϵ is an arbitrary positive number we conclude that $\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i2}\eta_{i2}) = 0$. This completes the proof of Proposition 2.21.

Let M be a nonempty subset of X. We say that the mapping $F : X \to R^n$ satisfies (P-S) condition on M with respect to κ if for each norm-bounded sequence $\{x_i\}_{i=1}^{\infty} \subset M$ such that $\{F(x_i)\}_{i=1}^{\infty}$ is bounded and $\liminf_{i\to\infty} \Xi_{F,\kappa}(x_i) = 0$ there exists a convergent subsequence of $\{x_i\}_{i=1}^{\infty}$ in X in the norm topology.

For each function $h : X \to R^1$ and each nonempty set $A \subset X$ set

$$\inf(h) = \inf\{h(z) : \; z \in X\}, \; \inf(h; A) = \inf\{h(z) : \; z \in A\}. \tag{2.122}$$

For each $x \in X$ and each $B \subset X$ set

$$d(x, B) = \inf\{||x - y|| : \; y \in B\}. \tag{2.123}$$

We assume that the sum over an empty set is zero.

Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t\to\infty} \phi(t) = \infty \tag{2.124}$$

and $\bar{a}$ be a positive number.

Denote by $\mathcal{M}$ the set of all continuous functions $h : X \to R^1$ such that

$$h(x) \geq \phi(||x||) - \bar{a} \text{ for all } x \in X. \tag{2.125}$$

We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\begin{aligned}\mathcal{E}(M, q, \epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} :\\ |f(x) - g(x)| \leq \epsilon \text{ for all } x \in B(0, M)\}\\ \cap\{(f, g) \in \mathcal{M} \times \mathcal{M} :\ |(f - g)(x) - (f - g)(y)|\\ \leq q||x - y|| \text{ for each } x, y \in B(0, M)\},\end{aligned} \tag{2.126}$$

where M, q, ϵ are positive numbers. This uniform space is metrizable and complete.

Let $n \geq 1$ be an integer, $G = (g_1, \dots, g_n) :\ X \to R^1$ be a locally Lipschitzian mapping and let $c = (c_1, \dots, c_n) \in R^n$.

Assume that an integer p satisfies $0 \leq p \leq n$ and set

$$I_1 = \{i \text{ is an integer: } 1 \leq i \leq p\},\ I_2 = \{i \text{ is an integer : } p < i \leq n\}. \tag{2.127}$$

(Note that one of the sets I_1, I_2 may be empty.)

Set

$$A = \{x \in X :\ g_i(x) = c_i \text{ for all } i \in I_1;\ g_j(x) \leq c_j \text{ for all } j \in I_2\}. \tag{2.128}$$

We assume that $A \neq \emptyset$.

We consider the following constrained minimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in A \tag{P}$$

where $f \in \mathcal{M}$.

For each $f \in \mathcal{M}$ we associate with problem (P) the corresponding family of unconstrained minimization problems

$$\text{minimize } f(x) + \sum_{i \in I_1} \lambda_i |g_i(x) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(x) - c_i, 0\} \text{ subject to } x \in X \tag{P$_\lambda$}$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$.

For each $\kappa \in (0, 1)$ set

$$\Omega_\kappa = \{x = (x_1, \dots, x_n) \in R^n :\ x_i \geq \kappa,\ i = 1, \dots, n,\ \max_{i=1,\dots,n} x_i = 1\}. \tag{2.129}$$

Assume that $f_0 \in \mathcal{M}$ is Lipschitzian on all bounded subsets of X.

Fix $\theta \in A$. It follows from (2.124) that there exists a number M_0 such that

$$M_0 > 2 + ||\theta||,\ \phi(M_0 - 2) > f_0(\theta) + \bar{a} + 4. \tag{2.130}$$

Fix $\kappa \in (0, 1)$. We use the following assumptions.

(A1) If $x \in A$, $I_1 \subset \{i_1, \dots, i_q\} \subset \{i \in \{1, \dots, n\} : \ f_i(x) = c_i\}$ where the sequence $\{i_1, \dots, i_q\}$ is strictly increasing and if x is a critical point of the mapping $(f_{i_1}, \dots, f_{i_q}) : X \to R^q$ with respect to κ, then $f_0(x) > \inf(f_0; A)$.

(A2) There exists a positive number γ_* such that for each finite strictly increasing sequence of natural numbers $\{i_1, \dots, i_q\}$ which satisfies

$$I_1 \subset \{i_1, \dots, i_q\} \subset \{1, \dots, n\}$$

the mapping $(g_{i_1}, \dots, g_{i_q}) : \ X \to R^q$ satisfies (P-S) condition on the set

$$\cap_{j \in I_1} (g_j^{-1}([c_j - \gamma_*, c_j + \gamma_*])) \cap_{j \in \{i_1, \dots, i_q\} \setminus I_1} (g_j^{-1}([c_j, c_j + \gamma_*]))$$

with respect to κ.

We show that if assumptions (A1) and (A2) hold, then the problem (P) has exact penalty property for all objective functions f which belong to a certain neighborhood $\mathcal{U}$ of f_0 in $\mathcal{M}$ with an exact penalty which depends only on f_0.

The next theorem is the main result of this section.

Theorem 2.22. *Let (A1), (A2) hold and let q be a positive number. Then there exist real numbers $\Lambda_0 > 0$, $\Lambda_1 > 0$ and $r > 0$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

If $f \in \mathcal{M}$ satisfies

$$(f, f_0) \in \mathcal{E}(M_0, q, r),$$

$$\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa, \ \lambda \geq \Lambda_0$$

and if $x \in X$ satisfies

$$f(x) + \sum_{i \in I_1} \lambda \gamma_i |g_i(x) - c_i| + \sum_{i \in I_2} \lambda \gamma_i \max\{g_i(x) - c_i, 0\} \leq$$

$$\inf\{f(z) + \sum_{i \in I_1} \lambda \gamma_i |g_i(z) - c_i| + \sum_{i \in I_2} \lambda \gamma_i \max\{g_i(z) - c_i, 0\} : z \in X\} + \delta,$$

then there is $y \in A$ such that

$$||x - y|| \leq \epsilon, \ f(y) \leq \inf(f; A) + \Lambda_1 \epsilon.$$

Theorem 2.22 implies the following result.

Theorem 2.23. *Let (A1), (A2) hold and let q be a positive number. Then there exist real numbers $\Lambda_0 > 0$ and $r > 0$ such that for each $f \in \mathcal{M}$ satisfying*

$$(f, f_0) \in \mathcal{E}(M_0, q, r),$$

$$\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa, \ \lambda \geq \Lambda_0$$

and each sequence $\{x_k\}_{k=1}^\infty \subset X$ which satisfies

$$\lim_{k\to\infty}[f(x_k)+\sum_{i\in I_1}\lambda\gamma_i|g_i(x_k)-c_i|+\sum_{i\in I_2}\lambda\gamma_i\max\{g_i(x_k)-c_i,0\}]$$

$$=\inf\{f(z)+\sum_{i\in I_1}\lambda\gamma_i|g_i(z)-c_i|+\sum_{i\in I_2}\lambda\gamma_i\max\{g_i(z)-c_i,0\}:z\in X\}$$

there exists a sequence $\{y_k\}_{k=1}^{\infty}\subset A$ *such that*

$$\lim_{k\to\infty} f(y_k)=\inf(f;A),\ \lim_{k\to\infty}||y_k-x_k||=0.$$

Corollary 2.24. *Let (A1), (A2) hold and let* q *be a positive number. Then there exists a number* $\Lambda_0>0$ *such that if* $f\in\mathcal{M}$ *satisfies*

$$(f,f_0)\in\mathcal{E}(M_0,q,r),$$

$$\gamma=(\gamma_1,\dots,\gamma_n)\in\Omega_\kappa,\ \lambda\geq\Lambda_0$$

and if $x\in X$ *satisfies*

$$f(x)+\sum_{i\in I_1}\lambda\gamma_i|g_i(x)-c_i|+\sum_{i\in I_2}\lambda\gamma_i\max\{g_i(x)-c_i,0\}$$

$$=\inf\{f(z)+\sum_{i\in I_1}\lambda\gamma_i|g_i(z)-c_i|+\sum_{i\in I_2}\lambda\gamma_i\max\{g_i(z)-c_i,0\}:z\in X\},$$

then $x\in A$ *and* $f(x)=\inf(f;A)$.

Corollary 2.24 follows from Theorem 2.23. Theorem 2.23 also implies the following result.

Theorem 2.25. *Let (A1), (A2) hold,* $\bar{x}\in A$ *and let the following conditions hold:*

$$f_0(\bar{x})=\inf(f_0;A);$$

if $\{x_i\}_{i=0}^{\infty}\subset A$ *satisfies* $\lim_{i\to\infty}f_0(x_i)=\inf(f_0;A)$, *then* $\lim_{i\to\infty}||x_i-\bar{x}||=0$.

Then there exists a number $\Lambda_0>0$ *such that for each*

$$\gamma=(\gamma_1,\dots,\gamma_n)\in\Omega_\kappa,\ \lambda\geq\Lambda_0$$

$\bar{x}\in X$ *is a unique solution of the minimization problem*

$$\textit{minimize}\ f_0(z)+\sum_{i\in I_1}\lambda\gamma_i|g_i(z)-c_i|+\sum_{i\in I_2}\lambda\gamma_i\max\{g_i(z)-c_i,0\}$$

$$\textit{subject to}\ z\in X.$$

The results of this section have been established in [126].

2.8 Proof of Theorem 2.22

For each $f \in \mathcal{M}$ and each $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$ define a function

$$\psi_{f,\lambda}(x) = f(x) + \sum_{i \in I_1} \lambda_i |g_i(x) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(x) - c_i, 0\},\ x \in X. \quad (2.131)$$

It is easy to see that the function $\psi_{f,\lambda} \in \mathcal{M}$ for each $f \in \mathcal{M}$ and each $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$.

We show that there exist numbers $\Lambda_0 > 0$ and $r > 0$ such that the following property holds:

(P1) For each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that for each $f \in \mathcal{M}$ satisfying

$$(f, f_0) \in \mathcal{E}(M_0, q, r),$$

each $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$ and each $x \in X$ satisfying

$$\psi_{f,\lambda\gamma}(x) \leq \inf(\psi_{f,\lambda\gamma}) + \delta$$

the set $A \cap B(x, \epsilon)$ is nonempty.

Assume the contrary. Then for each integer $k \geq 1$ there exist

$$\epsilon_k \in (0,1),\ f_k \in \mathcal{M},\ \gamma^{(k)} = (\gamma_1^{(k)}, \dots, \gamma_n^{(k)}) \in \Omega_\kappa, \quad (2.132)$$

$$\lambda_k \geq k \text{ and } x_k \in X$$

such that

$$(f_0, f_k) \in \mathcal{E}(M_0, q, k^{-1}), \quad (2.133)$$

$$\psi_{f_k, \lambda_k \gamma^{(k)}}(x_k) \leq \inf(\psi_{f_k, \lambda_k \gamma^{(k)}}) + (2k^2)^{-1} \epsilon_k, \quad (2.134)$$

$$d(x_k, A) \geq \epsilon_k. \quad (2.135)$$

Since f_0 is Lipschitz on bounded subsets of X there exists a number $L_0 > 1$ such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0 ||z_1 - z_2|| \text{ for each } z_1, z_2 \in B(0, M_0). \quad (2.136)$$

Let $k \geq 1$ be an integer. By (2.134) and Ekeland's variational principle [37] there exists $y_k \in X$ such that

$$\psi_{f_k, \lambda_k \gamma^{(k)}}(y_k) \leq \psi_{f_k, \lambda_k \gamma^{(k)}}(x_k), \quad (2.137)$$

$$||y_k - x_k|| \leq 2^{-1} k^{-1} \epsilon_k, \quad (2.138)$$

$$\psi_{f_k, \lambda_k \gamma^{(k)}}(y_k) \leq \psi_{f_k, \lambda_k \gamma^{(k)}}(z) + k^{-1} ||z - y_k|| \text{ for all } z \in X. \quad (2.139)$$

Relations (2.135) and (2.138) imply that

$$y_k \notin A \text{ for all integers } k \geq 1. \quad (2.140)$$

For each integer $k \geq 1$ put

$$I_{1k+} = \{i \in I_1 :\ g_i(y_k) > c_i\},\ I_{1k-} = \{i \in I_1 :\ g_i(y_k) < c_i\}, \tag{2.141}$$

$$I_{2k+} = \{i \in I_2 :\ g_i(y_k) > c_i\},\ I_{2k-} = \{i \in I_2 :\ g_i(y_k) < c_i\}.$$

It follows from (2.140) and (2.128) that

$$I_{1k+} \cup I_{1k-} \cup I_{2k+} \neq \emptyset \text{ for all natural numbers } k. \tag{2.142}$$

Extracting a subsequence and reindexing we may assume without loss of generality that for all integers $k \geq 1$

$$I_{1k+} = I_{11+},\ I_{1k-} = I_{11-},\ I_{2k+} = I_{21+}. \tag{2.143}$$

Put

$$I_{11} = I_1 \setminus (I_{11+} \cup I_{11-}),\ I_{21} = I_2 \setminus (I_{21+} \cup I_{21-}),\ I = \{1, \dots, n\} \setminus I_{2k-}. \tag{2.144}$$

We show that $y_k \in B(0, M_0 - 2)$ for all integers $k \geq 1$. It follows from (2.132), (2.125), (2.131), (2.137), (2.134), (2.128) and the inclusion $\theta \in A$ that for all integers $k \geq 1$

$$\phi(||y_k||) - \bar{a} \leq f_k(y_k) \leq \psi_{f_k, \lambda_k \gamma^{(k)}}(y_k) \leq \psi_{f_k, \lambda_k \gamma^{(k)}}(x_k)$$

$$\leq \inf(\psi_{f_k, \lambda_k \gamma^{(k)}}) + (2k^2)^{-1} \leq \inf(\psi_{f_k, \lambda_k \gamma^{(k)}}; A) + (2k^2)^{-1}$$

$$= \inf(f_k; A) + (2k^2)^{-1} \leq f_k(\theta) + (2k^2)^{-1}.$$

Together with (2.130), (2.133) and (2.126) this relation implies that for all integers $k \geq 1$

$$\phi(||y_k||) \leq \bar{a} + (2k^2)^{-1} + f_k(\theta) \leq \bar{a} + (2k^2)^{-1} + f_0(\theta) + k^{-1}.$$

By this inequality and the choice of M_0 (see (2.130))

$$y_k \in B(0, M_0 - 2) \text{ for all integers } k \geq 1. \tag{2.145}$$

In view of (2.145), (2.133) and (2.126),

$$\partial f_k(y_k) \subset \partial f_0(y_k) + B_*(0, q) \text{ for all integers } k \geq 1. \tag{2.146}$$

We show that

$$\lim_{k \to \infty} (\inf(f_k; A)) = \inf(f_0; A). \tag{2.147}$$

Let k be a natural number. It follows from (2.130), (2.126), (2.133) and the inclusion $\theta \in A$ that

$$|f_k(\theta) - f_0(\theta)| \leq 1/k \tag{2.148}$$

and

$$\inf(f_k; A) \le f_k(\theta) \le f_0(\theta) + 1/k < \phi(M_0 - 2) - \bar{a} - 3. \tag{2.149}$$

Inclusion $\theta \in A$ and (2.130) imply that

$$\inf(f_0; A) \le f_0(\theta) < \phi(M_0 - 2) - \bar{a} - 4. \tag{2.150}$$

In view of (2.149), (2.150), (2.130) and (2.125)

$$\inf(f_0; A) = \inf\{f_0(z) : \ z \in A \cap B(0, M_0)\}$$

and

$$\inf(f_k; A) = \inf\{f_k(z) : \ z \in A \cap B(0, M_0)\}.$$

It follows from these equalities and (2.133) that

$$|\inf(f_0; A) - \inf(f_k; A)|$$

$$= |\inf\{f_0(z) : \ z \in A \cap B(0, M_0)\} - \inf\{f_k(z) : \ z \in A \cap B(0, M_0)\}| \le 1/k.$$

Since this relation holds for all natural numbers k we obtain that (2.147) is valid.

By (2.131), (2.142), (2.132), (2.125), (2.137), (2.134), (2.148) and the inclusion $\theta \in A$, for each $i \in I_1$, each $j \in I_2$ and each natural number k,

$$-\bar{a} + \lambda_k \gamma_i^{(k)} |g_i(y_k) - c_i|, \ -\bar{a} + \lambda_k \gamma_j^{(k)} \max\{g_j(y_k) - c_j, 0\}$$

$$\le \psi_{f_k, \lambda_k \gamma^{(k)}}(y_k) \le f_k(\theta) + (2k^2)^{-1} \le f_0(\theta) + 1/k + (2k^2)^{-1} \le f_0(\theta) + 2.$$

It follows from the relation above, (2.132) and (2.129) that for each $i \in I_1$, each $j \in I_2 \setminus I_{21-}$ and each natural number k that

$$|g_i(y_k) - c_i|, \ \max\{g_j(y_k) - c_j, 0\} \le k^{-1}\kappa^{-1}(f_0(\theta) + 2 + \bar{a}).$$

Then for all sufficiently large natural numbers k

$$|g_i(y_k) - c_i| \le \gamma_*, \ i \in I_1, \tag{2.151}$$

$$0 \le g_i(y_k) - c_i \le \gamma_*, \ i \in I_2 \setminus I_{21-}. \tag{2.152}$$

In view of (2.136) and (2.145)

$$\partial f_0(y_k) \subset B_*(0, L_0) \text{ for all natural numbers } k. \tag{2.153}$$

Let k be a natural number. By (2.141), (2.143) and (2.145) there exists an open neighborhood V of y_k in X such that for each $y \in V$,

$$g_i(y) > c_i, \ i \in I_{11+}, \ g_i(y) < c_i, \ i \in I_{11-}, \tag{2.154}$$

$$g_i(y) > c_i, \ i \in I_{21+}, \ g_i(y) < c_i, \ i \in I_{21-},$$
$$V \subset B(0, M_0 - 1). \tag{2.155}$$

By (2.131), (2.139), (2.133) and (2.155), for each $z \in V$,

$$\begin{aligned}
\psi_{f_0,\lambda_k\gamma^{(k)}}(y_k) &= \psi_{f_k,\lambda_k\gamma^{(k)}}(y_k) + f_0(y_k) - f_k(y_k) \\
&\le \psi_{f_k,\lambda_k\gamma^{(k)}}(z) + k^{-1}||z - y_k|| + f_0(y_k) - f_k(y_k) \\
&= \psi_{f_0,\lambda_k\gamma^{(k)}}(z) - f_0(z) + f_k(z) + k^{-1}||z - y_k|| + f_0(y_k) - f_k(y_k) \\
&= \psi_{f_0,\lambda_k\gamma^{(k)}}(z) + k^{-1}||z - y_k|| + (f_k - f_0)(z) - (f_k - f_0)(y_k) \\
&\le \psi_{f_0,\lambda_k\gamma^{(k)}}(z) + k^{-1}||z - y_k|| + q||z - y_k||. \qquad (2.156)
\end{aligned}$$

It is easy to see that the function $\psi_{f_0,\lambda_k\gamma^{(k)}}$ is locally Lipschitz. In view of (2.156),

$$0 \in \partial\psi_{f_0,\lambda_k\gamma^{(k)}}(y_k) + (q + k^{-1})B_*(0, 1). \qquad (2.157)$$

It follows from (2.157), (2.131), (2.141), (2.143), (2.154), (2.144) and the properties of Clarke's generalized gradient (see Chapter 2, Section 2.3 of [21]) that

$$\begin{aligned}
0 \in \partial f_0(y_k) &+ \sum_{i\in I_{11+}} \lambda_k\gamma_i^{(k)}\partial g_i(y_k) - \sum_{i\in I_{11-}} \lambda_k\gamma_i^{(k)}\partial g_i(y_k) \\
&+ \sum_{i\in I_{11}} (\lambda_k\gamma_i^{(k)}) \cup \{\alpha\partial g_i(y_k) + (\alpha - 1)\partial g_i(y_k) :\ \alpha \in [0, 1]\} \\
&+ \sum_{i\in I_{21+}} \lambda_k\gamma_i^{(k)}\partial g_i(y_k) + \sum_{i\in I_{21}} (\lambda_k\gamma_i^{(k)}) \cup \{\alpha\partial g_i(y_k) :\ \alpha \in [0, 1]\} \\
&+(q + k^{-1})B_*(0, 1). \qquad (2.158)
\end{aligned}$$

Relation (2.158) implies that there exists $l_* \in X^*$ satisfying

$$l_* \in B_*(0, 1), \qquad (2.159)$$

$$l_0 \in \partial f_0(y_k),\ l_i \in \partial g_i(y_k),\ i \in I_{11+} \cup I_{11-} \cup I_{21+} \cup I_{21}, \qquad (2.160)$$

$$l_{i1},\ l_{i2} \in \partial g_i(y_k),\ i \in I_{11},$$

$$\alpha_i \in [0, 1],\ i \in I_{11} \cup I_{21} \qquad (2.161)$$

such that

$$\begin{aligned}
0 = (q + k^{-1})\lambda_k^{-1}l_* + \lambda_k^{-1}l_0 &+ \sum_{i\in I_{11+}\cup I_{21+}} \gamma_i^{(k)}l_i - \sum_{i\in I_{11-}} \gamma_i^{(k)}l_i \\
&+ \sum_{i\in I_{11}} \gamma_i^{(k)}[\alpha_i l_{i1} + (\alpha_i - 1)l_{i2}] + \sum_{i\in I_{21}} \gamma_i^{(k)}\alpha_i l_i.
\end{aligned}$$

It follows from this equality, (2.159), (2.132), (2.160) and (2.153) that

$$|| \sum_{i\in I_{11+}\cup I_{21+}} \gamma_i^{(k)}l_i - \sum_{i\in I_{11-}} \gamma_i^{(k)}l_i + \sum_{i\in I_{11}} \gamma_i^{(k)}[\alpha_i l_{i1} + (\alpha_i - 1)l_{i2}] + \sum_{i\in I_{21}} \gamma_i^{(k)}\alpha_i l_i|| \qquad (2.162)$$

$$\leq (\lambda_k^{-1} q)||l_*||_* + \lambda_k^{-1}||l_0||_* + (k\lambda_k)^{-1}||l_*||_* \leq k^{-1}q + k^{-1}L_0 + k^{-2}.$$

Put

$$\tilde{I} = I_1 \cup I_{21+} \cup I_{21}. \tag{2.163}$$

It is easy to see that

$$\{1, \ldots, p\} = I_1 \subset \tilde{I}.$$

There exists a finite strictly increasing sequence of natural numbers $i_1 < \cdots < i_q$ where q is a natural number such that

$$\tilde{I} = \{i_1, \ldots, i_q\}. \tag{2.164}$$

Consider a mapping $G = (g_{i_1}, \ldots, g_{i_q}) : X \to R^q$. It follows from (2.162), (2.115), (2.132), (2.129), (2.161), (2.142) and (2.143) that

$$\Xi_{G,\kappa}(y_k) \leq k^{-1}(q + L_0) + k^{-2} \text{ for each integer } k \geq 1. \tag{2.165}$$

By (2.165), (A2), (2.164), (P-S) condition, (2.145), (2.151) and (2.152) there exists a subsequence $\{y_{k_p}\}_{p=1}^{\infty}$ of the sequence $\{y_k\}_{k=1}^{\infty}$ which converges to $y_* \in X$ in the norm topology:

$$\lim_{p \to \infty} ||y_{k_p} - y_*|| = 0. \tag{2.166}$$

Relations (2.166) and (2.165) and Proposition 2.21 imply that

$$\Xi_{G,\kappa}(y_*) = 0. \tag{2.167}$$

It follows from (2.151), (2.152), (2.166), (2.164), (2.163) and (2.141) that

$$g_{i_s}(y_*) = c_{i_s}, \; s = 1, \ldots, q, \; y_* \in A. \tag{2.168}$$

By (2.166), (2.145), (2.133), (2.131), (2.137) and (2.147),

$$f_0(y_*) = \lim_{p \to \infty} f_0(y_{k_p}) \leq \limsup_{p \to \infty} [f_{k_p}(y_{k_p}) + k_p^{-1}]$$

$$\leq \limsup_{p \to \infty} \psi_{f_{k_p}, \lambda_{k_p} \gamma^{(k_p)}}(y_{k_p}) \leq \limsup_{p \to \infty} \inf(\psi_{f_{k_p}, \lambda_{k_p} \gamma^{(k_p)}})$$

$$\leq \limsup_{p \to \infty} \inf(f_{k_p}; A) = \inf(f_0; A). \tag{2.169}$$

Relations (2.167)–(2.169) contradict (A1). The contradiction we have reached proves that there exist positive numbers Λ_0, r for which property (P1) holds.

We may assume that $r < 1$. Since f_0 is Lipschitz on bounded subsets of X there exists a positive number L_0 such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0||z_1 - z_2|| \text{ for each } z_1, z_2 \in B(0, M_0). \tag{2.170}$$

Let $\epsilon \in (0, 1)$ and let $\delta \in (0, \epsilon)$ be as guaranteed by property (P1). Assume that $f \in \mathcal{M}$ satisfies

$$(f, f_0) \in \mathcal{E}(M_0, q, r), \tag{2.171}$$

$$\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa, \ \lambda \geq \Lambda_0, \tag{2.172}$$

$x \in X$ satisfies

$$\psi_{f,\lambda\gamma}(x) \leq \inf(\psi_{f,\lambda\gamma}) + \delta. \tag{2.173}$$

In view of the choice of Λ_0, r and property (P1) there exists $y \in A$ which satisfies

$$y \in B(x, \epsilon). \tag{2.174}$$

It follows from (2.125), (2.131), (2.173), the inclusion $\theta \in A$, the inequality $r < 1$ and (2.130) that

$$\phi(||x||) - \bar{a} \leq f(x) \leq \psi_{f,\lambda\gamma}(x) \leq \inf(\psi_{f,\lambda\gamma}) + 1$$

$$\leq \inf(f; A) + 1 \leq f(\theta) + 1 \leq f_0(\theta) + 2.$$

Combined with (2.130) and (2.174) this inequality implies that

$$x \in B(0, M_0 - 2), \ y \in B(0, M_0 - 1). \tag{2.175}$$

By (2.175), (2.170) and (2.171),

$$|f(y) - f(x)| \leq |f_0(x) - f_0(y)| + |(f - f_0)(x) - (f - f_0)(y)|$$

$$\leq L_0||x - y|| + q||x - y||.$$

It follows from this inequality, (2.131), (2.173) and (2.174) that

$$f(y) \leq f(x) + (L_0 + q)||x - y|| \leq \inf(\psi_{f,\lambda\gamma}) + \epsilon + (L_0 + q)\epsilon$$

$$\leq \inf(f; A) + (L_0 + q + 1) + \epsilon.$$

This completes the proof of Theorem 2.22.

2.9 Optimization problems with smooth constraint and objective functions

In this section we study two constrained nonconvex minimization problems with smooth cost functions. The first problem is an equality-constrained problem in a Hilbert space with a smooth constraint function and the second problem is an inequality-constrained problem in a Hilbert space with a smooth constraint function. For these problems we study the existence of exact penalty. Note that the classes of minimization problems with a smooth objective function and a smooth constraint are considered in the book by Cesari on optimal control [18].

Let $(X, < \cdot, \cdot >)$ be a Hilbert space with the inner product $< \cdot, \cdot >$ and the norm $||x|| = < x, x >^{1/2}, \ x \in X$. Denote by $C^1(X; R^1)$ the set of all Frechet differentiable functions $f : X \to R^1$ such that the mapping $x \to f'(x)$, $x \in X$ is continuous. Here $f'(x) \in X$ is a Frechet derivative of f at $x \in X$.

Denote by $C^2(X; R^1)$ the set of all Frechet differentiable functions $f \in C^1(X; R^1)$ such that the mapping $x \to f'(x)$, $x \in X$ is also Frechet differentiable and that the mapping $x \to f''(x)$, $x \in X$ is continuous. Here $f''(x)$ is a Frechet second-order derivative of f at $x \in X$. It is a linear continuous self-mapping of X.

For each $x \in X$ and each positive number r put

$$B(x, r) = \{y \in X : \ ||x - y|| \leq r\}.$$

For each function $f : X \to R^1$ and each $A \subset X$ put

$$\inf(f) = \inf\{f(z) : \ z \in X\}$$

and

$$\inf(f; A) = \inf\{f(x) : \ x \in A\}.$$

For each $x \in X$ and each $B \subset X$ set

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

Let $g \in C^2(X; R^1)$. A point $x \in X$ is called a critical point of g if $g'(x) = 0$. Denote by $Cr(g)$ the set of all critical points of g. A real number c is a critical value of g if there exists $x \in Cr(g)$ such that $g(x) = c$. Denote by $Cr(g, +)$ the set of all $x \in Cr(g)$ such that

$$< g''(x)u, u > \geq 0 \text{ for all } u \in X$$

and by $Cr(g, -)$ the set of all $x \in Cr(g)$ such that

$$< g''(x)u, u > \leq 0 \text{ for all } u \in X.$$

Let $c \in R^1$, $\gamma > 0$ and let $g^{-1}(c) \neq \emptyset$. We assume that g satisfies the following (P-S) condition [76] on the set $g^{-1}([c - \gamma, c + \gamma])$:

(P-S) If $\{x_i\}_{i=1}^{\infty} \subset g^{-1}([c-\gamma, c+\gamma])$ is a bounded (with respect to the norm of X) sequence and if $\lim_{i \to \infty} ||g'(x_i)|| = 0$, then there is a norm convergent subsequence of $\{x_i\}_{i=1}^{\infty}$.

Let a function $f \in C^1(X; R^1)$ be Lipschitzian on all bounded subsets of X and satisfy the growth condition

$$\lim_{||x|| \to \infty} f(x) = \infty \qquad (2.176)$$

and let the mapping $f' : X \to X$ be locally Lipschitzian. Clearly f is bounded from below.

We consider the constrained problems

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}(c) \qquad (\mathrm{P}_e)$$

and

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}((-\infty, c]). \tag{P$_i$}$$

We associate with these two problems the corresponding families of unconstrained minimization problems

$$\text{minimize } f(x) + \lambda|g(x) - c| \text{ subject to } x \in X \tag{P$_{\lambda e}$}$$

and

$$\text{minimize } f(x) + \lambda \max\{g(x) - c,\ 0\} \text{ subject to } x \in X \tag{P$_{\lambda i}$}$$

where λ is a positive number.

The main results of this section imply that exact penalty exists for the problem (P$_e$) if $g^{-1}(c) \cap (Cr(g, +) \cup Cr(g, -)) = \emptyset$ and that exact penalty exists for the problem (P$_i$) if $g^{-1}(c) \cap Cr(g, +) = \emptyset$.

The next two theorems are the main results of the section.

Theorem 2.26. *Assume that*

$$\{x \in g^{-1}(c) :\ f(x) = \inf(f; g^{-1}(c))\} \cap (Cr(g, +) \cup Cr(g, -)) = \emptyset. \tag{2.177}$$

Then there exists a number $\Lambda_0 > 0$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:

If $\lambda \geq \Lambda_0$ and if $x \in X$ satisfies

$$f(x) + \lambda|g(x) - c| \leq \inf\{f(z) + \lambda|g(z) - c| :\ z \in X\} + \delta,$$

then there exists $y \in g^{-1}(c)$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; g^{-1}(c)) + \epsilon.$$

Theorem 2.27. *Assume that*

$$\{x \in g^{-1}(c) :\ f(x) = \inf(f; g^{-1}((-\infty, c])\} \cap Cr(g, +) = \emptyset. \tag{2.178}$$

Then there exists a positive number Λ_0 such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:

If $\lambda \geq \Lambda_0$ and if $x \in X$ satisfies

$$f(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{f(z) + \lambda \max\{g(z) - c, 0\} :\ z \in X\} + \delta,$$

then there exists $y \in g^{-1}((-\infty, c])$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; g^{-1}((-\infty, c]))) + \epsilon.$$

Theorems 2.26 and 2.27 will be proved in Section 2.10. In this section we present several important results which easily follow from Theorems 2.26 and 2.27.

Theorems 2.26 and 2.27 imply the following result.

Theorem 2.28. *1. Assume that*

$$\{x \in g^{-1}(c) : \ f(x) = \inf(f; g^{-1}(c))\} \cap (Cr(g,+) \cup Cr(g,-)) = \emptyset.$$

Then there exists a positive number Λ_0 such that for each $\lambda \geq \Lambda_0$ and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies

$$\lim_{i\to\infty} [f(x_i) + \lambda|g(x_i) - c|] = \inf\{f(z) + \lambda|g(z) - c| : \ z \in X\}$$

there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset g^{-1}(c)$ such that

$$\lim_{i\to\infty} f(y_i) = \inf(f; g^{-1}(c)) \text{ and } \lim_{i\to\infty} ||y_i - x_i|| = 0.$$

2. Assume that

$$\{x \in g^{-1}(c) : \ f(x) = \inf(f; g^{-1}((-\infty, c]))\} \cap Cr(g,+) = \emptyset.$$

Then there exists a positive number Λ_0 such that for each $\lambda \geq \Lambda_0$ and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies

$$\lim_{i\to\infty} [f(x_i) + \lambda \max\{g(x_i) - c, 0\}] = \inf\{f(z) + \lambda \max\{g(z) - c, 0\} : \ z \in X\}$$

there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset g^{-1}((-\infty, c])$ such that

$$\lim_{i\to\infty} f(y_i) = \inf(f; g^{-1}((-\infty, c])) \text{ and } \lim_{i\to\infty} ||y_i - x_i|| = 0.$$

The next result easily follows from Theorem 2.28.

Theorem 2.29. *1. Assume that (2.177) holds. Then there exists a positive number Λ_0 such that if $\lambda \geq \Lambda_0$ and if $x \in X$ satisfies*

$$f(x) + \lambda|g(x) - c| = \inf\{f(z) + \lambda|g(z) - c| : \ z \in X\},$$

then $g(x) = c$ and $f(x) = \inf(f; g^{-1}(c))$.

2. Assume that (2.178) holds. Then there exists a positive number Λ_0 such that if $\lambda \geq \Lambda_0$ and if $x \in X$ satisfies

$$f(x) + \lambda \max\{g(x) - c, 0\} = \inf\{f(z) + \lambda \max\{g(z) - c, 0\} : \ z \in X\},$$

then $g(x) \leq c$ and $f(x) = \inf(f; g^{-1}((-\infty, c]))$.

Theorem 2.28 implies the following result.

Theorem 2.30. *1. Assume that (2.177) holds and that $\bar{x} \in g^{-1}(c)$ satisfies the following conditions:*

$f(\bar{x}) = \inf(f; g^{-1}(c))$;

any sequence $\{x_n\}_{n=1}^{\infty} \subset g^{-1}(c)$ which satisfies

$$\lim_{n\to\infty} f(x_n) = \inf(f; g^{-1}(c))$$

converges to $\bar{x}$ in the norm topology.

Then there exists a positive number Λ_0 such that for each $\lambda \geq \Lambda_0$ the point $\bar{x}$ is a unique solution of the minimization problem

$$\textit{minimize } f(z) + \lambda|g(z) - c| \textit{ subject to } z \in X.$$

2. Assume that (2.178) holds and that $\bar{x} \in g^{-1}((-\infty, c])$ satisfies the following conditions:

$f(\bar{x}) = \inf(f; g^{-1}((-\infty, c]))$;

any sequence $\{x_n\}_{n=1}^{\infty} \subset g^{-1}((-\infty, c])$ which satisfies $\lim_{n\to\infty} f(x_n) = \inf(f; g^{-1}((-\infty, c]))$ converges to $\bar{x}$ in the norm topology.

Then there exists a positive number Λ_0 such that for each $\lambda \geq \Lambda_0$ the point $\bar{x}$ is a unique solution of the minimization problem

$$\textit{minimize } f(z) + \lambda \max\{g(z) - c, 0\} \textit{ subject to } z \in X.$$

The results of this section have been established in [136].

2.10 Proofs of Theorems 2.26 and 2.27

We prove Theorems 2.26 and 2.27 simultaneously. Put

$$A = g^{-1}(c) \text{ in the case of Theorem 2.26} \tag{2.179}$$

and

$$A = g^{-1}((-\infty, c]) \text{ in the case of Theorem 2.27.} \tag{2.180}$$

For each positive number λ we define a function $\psi_\lambda : X \to R^1$ by

$$\psi_\lambda(z) = f(z) + \lambda|g(z) - c|, \; z \in X \tag{2.181}$$

in the case of Theorem 2.26 and by

$$\psi_\lambda(z) = f(z) + \lambda \max\{g(z) - c, 0\}, \; z \in X \tag{2.182}$$

in the case of Theorem 2.27.

It is easy to see that the function ψ_λ is locally Lipschitzian for all positive numbers λ.

We show that there exists a positive number Λ_0 such that the following property holds:

(P1) For each $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon)$ such that for each $\lambda \geq \Lambda_0$ and each $x \in X$ which satisfies

$$\psi_\lambda(x) \leq \inf(\psi_\lambda) + \delta$$

there is $y \in A$ for which

$$y \in B(x, \epsilon) \text{ and } \psi_\lambda(y) \le \inf(\psi_\lambda) + \epsilon. \tag{2.183}$$

Assume the contrary. Then for each integer $k \ge 1$ there exist

$$\epsilon_k \in (0, 1),\ \lambda_k \ge k,\ x_k \in X \tag{2.184}$$

such that

$$\psi_{\lambda_k}(x_k) \le \inf(\psi_{\lambda_k}) + (8k^2)^{-1}\epsilon_k^2 \tag{2.185}$$

and

$$\{z \in A \cap B(x_k, \epsilon_k) :\ \psi_{\lambda_k}(z) \le \inf(\psi_{\lambda_k}) + \epsilon_k\} = \emptyset. \tag{2.186}$$

Let $k \ge 1$ be an integer. Consider the function

$$\phi_{\lambda_k}(z) = \psi_{\lambda_k}(z) + (4k^2)^{-1}||z - x_k||^2,\ z \in X. \tag{2.187}$$

It is not difficult to see that the function ϕ_{λ_k} is locally Lipschitzian and bounded from below. It follows from the variational principle of Deville–Godefroy–Zizler [31] that there exist $h_k \in C^2(X; R^1)$ and $y_k \in X$ such that

$$\sup\{||h_k(z)|| + ||h_k'(z)|| + ||h_k''(z)|| :\ z \in X\} \le 32^{-1}k^{-2}\epsilon_k^2, \tag{2.188}$$

$$(\phi_{\lambda_k} + h_k)(z) > (\phi_{\lambda_k} + h_k)(y_k) \text{ for all } z \in X \setminus \{y_k\}. \tag{2.189}$$

We show that $||x_k - y_k|| < \epsilon_k$. Assume the contrary. Then

$$||y_k - x_k|| \ge \epsilon_k. \tag{2.190}$$

Relations (2.187) and (2.185) imply that

$$\phi_{\lambda_k}(x_k) = \psi_{\lambda_k}(x_k) \le \inf(\psi_{\lambda_k}) + (8k^2)^{-1}\epsilon_k^2. \tag{2.191}$$

It follows from (2.187), (2.190) and (2.191) that

$$\phi_{\lambda_k}(y_k) = \psi_{\lambda_k}(y_k) + (4k^2)^{-1}||y_k - x_k||^2 \ge \psi_{\lambda_k}(y_k) + (4k^2)^{-1}\epsilon_k^2$$
$$\ge \inf(\psi_{\lambda_k}) + (4k^2)^{-1}\epsilon_k^2 \ge \phi_{\lambda_k}(x_k) + (8k^2)^{-1}\epsilon_k^2$$

and

$$\phi_{\lambda_k}(y_k) - \phi_{\lambda_k}(x_k) \ge (8k^2)^{-1}\epsilon_k^2.$$

Together with (2.188) this inequality implies that

$$(\phi_{\lambda_k}+h_k)(y_k)-(\phi_{\lambda_k}+h_k)(x_k) \ge \phi_{\lambda_k}(y_k)-\phi_{\lambda_k}(x_k)-(16k^2)^{-1}\epsilon_k^2 \ge (16k^2)^{-1}\epsilon_k^2.$$

This inequality contradicts (2.189). The contradiction we have reached proves that

$$||y_k - x_k|| < \epsilon_k. \tag{2.192}$$

By (2.181), (2.182), (2.187), (2.188), (2.189) and (2.185),

$$f(y_k) \le \psi_{\lambda_k}(y_k) \le \phi_{\lambda_k}(y_k) \le (\phi_{\lambda_k} + h_k)(y_k) + 32^{-1}k^{-2}\epsilon_k^2$$
$$\le \phi_{\lambda_k}(x_k) + h_k(x_k) + (32k^2)^{-1}\epsilon_k^2$$
$$\le \phi_{\lambda_k}(x_k) + (16k^2)^{-1}\epsilon_k^2 = \psi_{\lambda_k}(x_k) + (16k^2)^{-1}\epsilon_k^2 \le \inf(\psi_{\lambda_k}) + (4k^2)^{-1}\epsilon_k^2. \tag{2.193}$$

It follows from (2.192), (2.193) and (2.186) that

$$y_k \notin A. \tag{2.194}$$

By (2.193), (2.181) and (2.182),

$$f(y_k) \le 1 + \inf(\psi_{\lambda_k}; A) = \inf(f; A) + 1. \tag{2.195}$$

By this inequality and the growth condition (2.176) the sequence $\{y_k\}_{k=1}^{\infty}$ is bounded. Since the function f is Lipschitzian on bounded subsets of X it follows from the boundedness of the sequence $\{y_k\}_{k=1}^{\infty}$ that

$$\sup\{||f'(y_k)|| : \ k \text{ is a natural number }\} < \infty. \tag{2.196}$$

It should be mentioned that (2.194) holds for all integers $k \ge 1$.

In the case of Theorem 2.27 we obtain that

$$g(y_k) > c \text{ for all integers } k \ge 1. \tag{2.197}$$

In the case of Theorem 2.26 we obtain that for each integer $k \ge 1$ either $g(y_k) > c$ or $g(y_k) < c$. In the case of Theorem 2.26 extracting a subsequence and reindexing we may assume that either

$$g(y_k) > c \text{ for all integers } k \ge 1$$

or

$$g(y_k) < c \text{ for all integers } k \ge 1. \tag{2.198}$$

Define a function $\tilde{g}$ and a real number $\tilde{c}$ as follows. In the case of Theorem 2.27 put $\tilde{g} = g$, $\tilde{c} = c$. In the case of Theorem 2.26, if (2.197) holds then set $\tilde{g} = g$ and $\tilde{c} = c$, and if (2.198) is valid then put $\tilde{g} = -g$ and $\tilde{c} = -c$. Note that in all these cases we have

$$\tilde{g}(y_k) > \tilde{c} \text{ for all integers } k \ge 1. \tag{2.199}$$

Let $k \ge 1$ be an integer. There exists an open neighborhood W of y_k in X with the norm topology such that

$$\tilde{g}(z) > \tilde{c} \tag{2.200}$$

for each $z \in W$.

It follows from (2.187), (2.181), (2.182), the definition of $\tilde{g}$ and $\tilde{c}$ and the choice of W that for each $z \in W$

$$(\phi_{\lambda_k} + h_k)(z) = \psi_{\lambda_k}(z) + (4k^2)^{-1}||z - x_k||^2 + h_k(z)$$

$$= f(z) + \lambda_k(\tilde{g}(z) - \tilde{c}) + (4k^2)^{-1}||z - x_k||^2 + h_k(z). \quad (2.201)$$

By (2.201), the definition of $\tilde{g}$ and $\tilde{c}$ and the choice of h_k, the function $\phi_{\lambda_k} + h_k$ is Frechet differentiable on the set W. Relations (2.189) and (2.201) imply that

$$0 = (\phi_{\lambda_k} + h_k)'(y_k) = f'(y_k) + \lambda_k \tilde{g}'(y_k) + (4k^2)^{-1}2(y_k - x_k) + h'_k(y_k).$$

Combined with (2.184), (2.192) and (2.188) this equality implies that

$$||\tilde{g}'(y_k)|| = \lambda_k^{-1}||f'(y_k) + (4k^2)^{-1}2(y_k - x_k) + h'_k(y_k)||$$

$$\leq k^{-1}(||f'(y_k)|| + \epsilon_k + ||h'_k(y_k)||) \leq k^{-1}(||f'(y_k)|| + 1 + \epsilon_k^2) \leq k^{-1}(||f'(y_k)|| + 2).$$

Together with (2.196) and the definition of $\tilde{g}$ this relation implies that

$$\lim_{k\to\infty} ||g'(y_k)|| = \lim_{k\to\infty} ||\tilde{g}'(y_k)|| = 0. \quad (2.202)$$

Let $k \geq 1$ be an integer. By (2.181), (2.182), the definition of $\tilde{g}$ and $\tilde{c}$, (2.199), (2.193) and (2.184),

$$f(y_k) + \lambda_k(\tilde{g}(y_k) - \tilde{c}) = \psi_{\lambda_k}(y_k) \leq \inf(\psi_{\lambda_k}) + 1$$

$$\leq \inf(\psi_{\lambda_k}; A) + 1 \leq \inf(f; A) + 1.$$

Combined with (2.199) and (2.184) this relation implies that

$$0 < \tilde{g}(y_k) - \tilde{c} \leq \lambda_k^{-1}[\inf(f; A) + 1 - \inf(f)]$$

$$\leq k^{-1}[\inf(f; A) + 1 - \inf(f)] \to 0 \text{ as } k \to \infty. \quad (2.203)$$

Hence

$$\lim_{k\to\infty} \tilde{g}(y_k) = \tilde{c}. \quad (2.204)$$

It follows from (2.204) and the definition of $\tilde{g}$ and $\tilde{c}$ that

$$c - \gamma \leq g(y_k) \leq c + \gamma \text{ for all sufficiently large natural numbers } k.$$

By this inequality, (2.202), the boundedness of the sequence $\{y_k\}_{k=1}^{\infty}$ and (P-S) condition there exists a norm convergent subsequence of $\{y_k\}_{k=1}^{\infty}$. Extracting a subsequence and reindexing we may assume without loss of generality that the sequence $\{y_k\}_{k=1}^{\infty}$ converges to $y_* \in X$ in the norm topology. Equality (2.204), the definition of $\tilde{g}$ and $\tilde{c}$ and the definition of A imply that

$$g(y_*) = \lim_{i\to\infty} g(y_k) = 0 \text{ and } y_* \in A. \quad (2.205)$$

By (2.202),

$$g'(y_*) = \lim_{k\to\infty} g'(y_k) = 0 \quad (2.206)$$

in the norm topology. By the equality $\lim_{k\to\infty} y_k = y_*$, (2.193), (2.184), (2.181) and (2.182),

$$f(y_*) = \lim_{k\to\infty} f(y_k) \le \limsup_{k\to\infty}[\inf(\psi_{\lambda_k}) + (4k^2)^{-1}\epsilon_k^2]$$

$$= \limsup_{k\to\infty} \inf(\psi_{\lambda_k}) \le \limsup_{k\to\infty} \inf(\psi_{\lambda_k}; A) = \inf(f; A).$$

Combined with (2.205) this relation implies that

$$f(y_*) = \inf(f; A). \tag{2.207}$$

Since the mapping $f' : X \to X$ is locally Lipschitzian there exist positive numbers r_* and L_* such that

$$||f'(z_1) - f'(z_2)|| \le L_*||z_1 - z_2|| \text{ for each } z_1, z_2 \in B(y_*, r_*). \tag{2.208}$$

The equality $\lim_{k\to\infty} ||y_k - y_*|| = 0$ implies that there exists an integer $k_0 \ge 1$ such that

$$||y_k - y_*|| \le r_*/4 \text{ for all integers } k \ge k_0. \tag{2.209}$$

Let $k \ge k_0$ be an integer. Since $h_k, g \in C^2(X; R^1)$ it follows from (2.199) that there exists a positive number

$$r_k < \min\{\epsilon_k/2, r_*/4\} \tag{2.210}$$

such that

$$\tilde{g}(z) > \tilde{c} \text{ for each } z \in B(y_k, r_k), \tag{2.211}$$

$$||h_k''(y_k) - h_k''(y_k + v)|| \le k^{-2} \text{ for each } v \in B(0, r_k) \tag{2.212}$$

and

$$||g''(y_k + v) - g''(y_k)|| \le k^{-2}\lambda_k^{-1} \text{ for each } v \in B(0, r_k). \tag{2.213}$$

It follows from (2.187), (2.181), (2.182), (2.211) and the definition of $\tilde{g}$ and $\tilde{c}$ that for each $z \in B(y_k, r_k)$

$$(\phi_{\lambda_k} + h_k)(z) = f(z) + \lambda_k(\tilde{g}(z) - \tilde{c}) + (4k^2)^{-1}||z - x_k||^2 + h_k(z). \tag{2.214}$$

By (2.210), (2.208) and (2.209) for each $v \in B(0, r_k)$

$$||f'(y_k + v) - f'(y_k)|| \le L_*||v||. \tag{2.215}$$

Since the sequence $\{y_k\}_{k=1}^\infty$ is bounded it follows from (2.192) and (2.184) that the sequence $\{x_k\}_{k=1}^\infty$ is also bounded. Thus there exists a positive number M such that

$$y_k, x_k \in B(0, M) \text{ for all natural numbers } k. \tag{2.216}$$

Let

$$u \in B(0, r_k) \setminus \{0\}. \tag{2.217}$$

In view of the inclusion $h_k \in C^2(X; R^1)$ and the Taylor theorem there exists $t_1 \in [0, 1]$ such that

$$h_k(y_k + u) = h_k(y_k) + < h_k'(y_k), u) > + < h_k''(y_k + t_1 u)u, u > /2. \tag{2.218}$$

Since $g \in C^2(X; R^1)$ it follows from the Taylor theorem that there exists $t_2 \in [0, 1]$ such that

$$g(y_k + u) = g(y_k) + < g'(y_k), u > + < g''(y_k + t_2 u)u, u > /2. \tag{2.219}$$

It follows from the relation $f \in C^1(X; R^1)$ and the Taylor theorem that there exists $t_3 \in [0, 1]$ such that

$$f(y_k + u) = f(y_k) + < f(y_k + t_2 u), u > . \tag{2.220}$$

By (2.212), (2.217) and (2.218),

$$|h_k(y_k + u) - h_k(y_k) - < h_k'(y_k), u > -2^{-1} < h_k''(y_k)u, u > |$$
$$= 2^{-1}| < (h_k''(y_k + t_1 u) - h_k''(y_k))u, u > |$$
$$\leq 2^{-1}||u||^2||h_k''(y_k + t_1 u) - h_k''(y_k)|| \leq k^{-2}||u||^2. \tag{2.221}$$

Relations (2.219), (2.213) and (2.217) imply that

$$|g(y_k + u) - g(y_k) + < g'(y_k), u > - < g''(y_k)u, u > /2|$$
$$= 2^{-1}| < (g'(y_k + t_2 u) - g''(y_k))u, u > |$$
$$\leq 2^{-1}||u||^2||g''(y_k + t_2 u) - g''(y_k)|| \leq ||u||^2 k^{-2}\lambda_k^{-1}. \tag{2.222}$$

By (2.220), (2.215) and (2.217),

$$|f(y_k + u) - f(y_k) - < f'(y_k), u > | = | < f'(y_k + t_3 u) - f'(y_k), u > |$$
$$\leq ||f'(y_k + t_3 u) - f'(y_k)||||u|| \leq L_* t_3 ||u||^2 \leq L_* ||u||^2. \tag{2.223}$$

It follows from (2.189), (2.217), (2.214), (2.223), (2.222) and (2.221) that

$$0 < (\phi_{\lambda_k} + h_k)(y_k + u) - (\phi_{\lambda_k} + h_k)(y_k)$$
$$= f(y_k + u) - f(y_k) + \lambda_k(\tilde{g}(y_k + u) - \tilde{g}(y_k))$$
$$+(4k^2)^{-1}[||y_k + u - x_k||^2 - ||y_k - x_k||^2] + h_k(y_k + u) - h_k(y_k)$$
$$\leq < f'(y_k), u > +L_*||u||^2 + \lambda_k[< \tilde{g}'(y_k), u > + < \tilde{g}''(y_k)u, u > 2^{-1}] + ||u||^2 k^{-2}$$
$$+(4k^2)^{-1}[||u||^2 + 2 < y_k - x_k, u >]$$
$$+ < h_k'(y_k), u > +2^{-1} < h_k''(y_k)u, u > +k^{-2}||u||^2. \tag{2.224}$$

Put

$$F(u) =< f'(y_k), u > +L_*||u||^2 + \lambda_k[< \tilde{g}'(y_k), u > + < \tilde{g}''(y_k)u, u > 2^{-1}]$$
$$+||u||^2k^{-2} + (4k^2)^{-1}[||u||^2 + 2 < y_k - x_k, u >]$$
$$+ < h_k'(y_k), u > +2^{-1} < h_k''(y_k)u, u >, \ u \in X. \quad (2.225)$$

It is easy to see that $F \in C^2(X; R^1)$. Relations (2.225), (2.224) and (2.217) imply that

$$F(u) > F(0) \text{ for all } u \in B(0, r_k) \setminus \{0\}. \quad (2.226)$$

In view of (2.226),

$$F'(0) = 0 \text{ and } < F''(0)v, v >\geq 0 \text{ for all } v \in X. \quad (2.227)$$

Denote by I the identity operator $I : X \to X$ such that $Ix = x$ for all $x \in X$. It follows from (2.225) that

$$F''(0) = 2L_*I + \lambda_k\tilde{g}''(y_k) + (5/2)k^{-2}I + h_k''(y_k). \quad (2.228)$$

By (2.227), (2.228), (2.184) and (2.1888) for each $v \in X$

$$0 \leq< \lambda_k^{-1}F''(0)v, v >= 2L_*\lambda_k^{-1}||v||^2+ < \tilde{g}''(y_k)v, v >$$
$$+(5/2)k^{-2}\lambda_k^{-1}||v||^2 + \lambda_k^{-1} < h_k''(y_k)v, v >$$
$$\leq< \tilde{g}''(y_k)v, v > +2L_*k^{-1}||v||^2 + (5/2)k^{-3}||v||^2 + k^{-1}| < h_k''(y_k)v, v > |$$
$$\leq< \tilde{g}''(y_k)v, v > +2L_*k^{-1}||v||^2 + (5/2)k^{-3}||v||^2 + k^{-3}||v||^2 \to< \tilde{g}''(y_*)v, v >$$

as $k \to \infty$. Hence

$$< \tilde{g}''(y_*)v, v >\geq 0 \text{ for all } v \in X. \quad (2.229)$$

In view of (2.205), (2.206) and (2.207),

$$g(y_*) = c, \ g'(y_*) = 0, \ f(y_*) = \inf(f; A). \quad (2.230)$$

In the case of Theorem 2.26 relations (2.229) and (2.230) contradict (2.177). In the case of Theorem 2.27 $\tilde{g} = g$ and (2.229) and (2.230) contradict (2.178). The contradiction we have reached proves that there exists a positive number Λ_0 such that property (P1) holds.

Let $\epsilon \in (0, 1)$ and let $\delta \in (0, \epsilon)$ be as guaranteed by (P1). Assume that $\lambda \geq \Lambda_0$ and $x \in X$ satisfies

$$\psi_\lambda(x) \leq \inf(\psi_\lambda) + \delta.$$

It follows from the choice of δ and (P1) that there exists $y \in A$ such that

$$y \in B(x, \epsilon), \ \psi_\lambda(y) \leq \inf(\psi_\lambda) + \epsilon.$$

By the relations above, (2.181) and (2.182),

$$f(y) \leq \psi_\lambda(y) \leq \inf(\psi_\lambda) + \epsilon \leq \inf(\psi_\lambda; A) + \epsilon = \inf(f : A) + \epsilon.$$

This completes the proof of Theorems 2.26 and 2.27.

2.11 Optimization problems in metric spaces

In this section we study constrained minimization problems in a complete metric space with locally Lipschitzian mixed constraints and generalize the results of Section 2.1. In order to obtain this generalization we need to define a critical point for a Lipschitzian function defined on a complete metric space.

Let (X, ρ) be a metric space. Assume that U is a nonempty open subset of X. A function $f : U \to R^1$ is called Lipschitzian if there exists a positive number c such that

$$|f(x_1) - f(x_2)| \le c\rho(x_1, x_2) \text{ for each } x_1, x_2 \in U.$$

Let a function $f : U \to R^1$ be Lipschitzian. For each $x \in U$ define

$$\Xi_f(x) = \limsup_{y \to x} [\liminf_{z \to y,\, z \ne y} (f(z) - f(y))(\rho(y, z))^{-1}]. \tag{2.231}$$

Evidently, $\Xi_f(x)$ is well defined for all $x \in U$.

A point $x \in U$ is called a critical point of f if $\Xi_f(x) \ge 0$. A real number $c \in R^1$ is called a critical value of f on U if there is a critical point $x \in U$ of f such that $f(x) = c$.

For each $x \in X$ and each positive number r set

$$B(x, r) = \{y \in X : \rho(x, y) \le r\}.$$

A set $D \subset X$ is called bounded if there exist $x \in X$ and a positive number r such that $D \subset B(x, r)$.

It is not difficult to see that the following proposition holds.

Proposition 2.31. *A point $x \in U$ is a critical point of f if and only if there exist a sequence $\{x_k\}_{k=1}^\infty \subset U$ and a sequence $\{r_k\}_{k=1}^\infty \subset (0, 1)$ such that $\rho(x_k, x) \to 0$ as $k \to \infty$ and that for each natural number k*

$$B(x_k, r_k) \subset U,$$

$$f(z) \ge f(x_k) - \rho(z, x_k)/k \text{ for all } z \in B(x_k, r_k).$$

It follows from the definition (2.231) that the following proposition holds.

Proposition 2.32. *Let $x \in U$, $\{x_i\}_{i=1}^\infty \subset U$ and let $\lim_{i \to \infty} \rho(x_i, x) = 0$. Then*

$$\Xi_f(x) \ge \limsup_{i \to \infty} \Xi_f(x_i).$$

Corollary 2.33. *Let $x \in U$, $\{x_i\}_{i=1}^\infty \subset U$, $\lim_{i \to \infty} \rho(x_i, x) = 0$ and let $\limsup_{i \to \infty} \Xi_f(x_i) \ge 0$. Then x is a critical point of f.*

Let M be a nonempty subset of U. We say that f satisfies (P-S) condition on M [76, 100] if each bounded sequence $\{x_i\}_{i=1}^{\infty} \subset M$ such that the sequence $\{f(x_i)\}_{i=1}^{\infty}$ is bounded and $\liminf_{i\to\infty} \Xi_f(x_i) \geq 0$, possesses a convergent subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ in (X, ρ).

The analogs of the notion of a critical point of f and (P-S) condition introduced above were used in the previous sections in the case when X is a Banach space.

Now we compare the notion of a critical point introduced above and the notion of a critical point used in the previous sections when X is a Banach space.

Assume that $(X, ||\cdot||)$ is a Banach space, $(X^*, ||\cdot||_*)$ is its dual space and that $\rho(x,y) = ||x-y||$, $x, y \in X$.

Let $f : U \to R^1$ be a Lipschitzian function defined on a nonempty open subset U of X. For each $x \in U$ let

$$f^0(x,h) = \limsup_{t\to 0^+, y\to x} [f(y+th) - f(y)]/t, \ h \in X$$

be the Clarke generalized derivative of f at the point x [21], let

$$\partial f(x) = \{l \in X^* : \ f^0(x,h) \geq l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of f at x [21] and set

$$\tilde{\Xi}_f(x) = \inf\{f^0(x,h) : \ h \in X \text{ and } ||h|| = 1\}.$$

Proposition 2.34. *For each $x \in U$, $\tilde{\Xi}_f(x) \geq \Xi_f(x)$.*

Proof: Let $x \in U$. In order to prove the proposition it is sufficient to show that for each $h \in X$ satisfying $||h|| = 1$ the inequality $f^0(x,h) \geq \Xi_f(x)$ is valid. Let

$$h \in X \text{ and } ||h|| = 1.$$

By (2.231) for each positive number r which satisfies $B(x, 2r) \subset U$

$$\Xi_f(x) \leq \limsup_{y\to x} [\liminf_{t\to 0^+} (f(y+th) - f(y))/t]$$

$$\leq \sup_{y\in B(x,r)} \sup_{t\in(0,r]} [f(y+th) - f(y)]/t.$$

Since this inequality holds for any positive number r satisfying $B(x, 2r) \subset U$ we obtain that $\Xi_f(x) \leq f^0(x,h)$. This completes the proof of Proposition 2.34.

In Section 2.1 we say that $x \in U$ is a critical point of f if $\tilde{\Xi}_f(x) \geq 0$ and that a real number c is a critical value of f if there is a critical point $x \in U$ of f such that $f(x) = c$. Proposition 2.34 implies that if $x \in U$ is a critical point of f according to the definition given in this paper, then x is also a critical point of f in the sense of the definition given in Section 2.1.

Proposition 2.34 also implies that if $c \in R^1$ is a critical value of f according to the definition given in this section, then c is also a critical value of f in the sense of the definition given in Section 2.1.

Let M be a nonempty subset of U. In Section 2.1 we consider an analog of the (P-S) condition above for which the inequality $\liminf_{i\to\infty} \Xi_f(x_i) \geq 0$ is replaced by the inequality $\liminf_{i\to\infty} \tilde{\Xi}_f(x_i) \geq 0$.

By Proposition 2.34 (P-S) type condition introduced in this section is weaker than (P-S) type condition in Section 2.1.

Let (X, ρ) be a complete metric space. For each function $f : X \to R^1$ set $\inf(f) = \inf\{f(z) : \ z \in X\}$. For each function $f : X \to R^1$ and each nonempty set $A \subset X$ put

$$\inf(f; A) = \inf\{f(z) : \ z \in A\}.$$

For each $x \in X$ and each $B \subset X$ put

$$\rho(x, B) = \inf\{\rho(x, y) : \ y \in B\}.$$

We assume that the sum over an empty set is zero. If A is a subset of (X, ρ), then we consider a metric space A equipped with the metric ρ.

If $g : Y \to R^1$ where Y is nonempty and if Z is a nonempty subset of Y, then we denote by $g|_Z$ the restriction of g to Z.

Fix $\theta \in X$. Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t\to\infty} \phi(t) = \infty \tag{2.232}$$

and let $\bar{a}$ be a positive number.

Denote by $\mathcal{M}$ the set of all continuous functions $h : X \to R^1$ such that

$$h(x) \geq \phi(\rho(x, \theta)) - \bar{a} \text{ for all } x \in X. \tag{2.233}$$

We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\begin{aligned}\mathcal{E}(M, q, \epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \ |f(x) - g(x)| \leq \epsilon \text{ for each } x \in B(\theta, M)\} \\ \cap\{(f, g) \in \mathcal{M} \times \mathcal{M} : \\ |(f - g)(x) - (f - g)(y)| \leq q\rho(x, y) \text{ for each } x, y \in B(\theta, M)\}\end{aligned} \tag{2.234}$$

where M, q, ϵ are positive numbers. It is not difficult to see that this uniform space is metrizable and complete.

Let $n \geq 1$ be an integer, let $g_i : X \to R^1$, $i = 1, \dots, n$ be locally Lipschitzian functions and let $c = (c_1, \dots, c_n) \in R^n$.

We assume that g_i is Lipschitzian on all bounded subsets of X for each integer i satisfying $2 \leq i \leq n$.

Let I_1 and I_2 be subsets of $\{1, \dots, n\}$ such that

$$I_1 \cap I_2 = \emptyset \text{ and } I_1 \cup I_2 = \{1, \dots, n\}. \tag{2.235}$$

(Note that one of the sets I_1, I_2 may be empty.)

Set

$$A = \{x \in X : \ g_i(x) = c_i \text{ for all } i \in I_1 \text{ and } g_j(x) \le c_j \text{ for all } j \in I_2\}. \tag{2.236}$$

We assume that $A \neq \emptyset$.

In this paper we consider the following constrained minimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in A \tag{P}$$

where $f \in \mathcal{M}$.

For each $f \in \mathcal{M}$ we associate with problem (P) the corresponding family of unconstrained minimization problems

$$\text{minimize } f(x) + \sum_{i \in I_1} \lambda_i |g_i(x) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(x) - c_i, 0\} \text{ subject to } x \in X \tag{P$_\lambda$}$$

where

$$\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n.$$

Assume that $f_0 \in \mathcal{M}$ is Lipschitzian on all bounded subsets of X.

Fix $\theta_1 \in A$. By (2.232) there exists

$$M_0 > 2 + \rho(\theta, \theta_1) \tag{2.237}$$

such that

$$\phi(M_0 - 2) > f_0(\theta_1) + \bar{a} + 4. \tag{2.238}$$

For each $i \in \{1, \dots, n\}$ put

$$A_i = \{x \in X : \ g_i(x) = c_i\} \text{ if } i \in I_1,$$
$$A_i = \{x \in X : \ g_i(x) \le c_i\} \text{ if } i \in I_2. \tag{2.239}$$

Define

$$B_1 = X, \ B_j = \cap_{i=1}^{j-1} A_i \text{ for each integer } j \text{ satisfying } 1 < j \le n. \tag{2.240}$$

We use the following assumptions.

(A1) If $p \in I_1$ and if $x \in A$ satisfies $f_0(x) = \inf(f_0; A)$, then x is not a critical point of the function $g_p|B_p$ and it is not a critical point of the function $-g_p|_{B_p}$. If $p \in I_2$ and if $x \in A$ satisfies $f_0(x) = \inf(f_0; A)$, then x is not a critical point of the function $g_p|_{B_p}$.

(A2) There exists a positive number γ_0 such that for each $j \in I_1$ the functions $g_j|_{B_j}$ and $-g_j|_{B_j}$ satisfy (P-S) condition on the set

$$B_j \cap g_j^{-1}([c_j - \gamma_0, c_j + \gamma_0])$$

and for each $j \in I_2$ the function $g_j|_{B_j}$ satisfies (P-S) condition on the set $B_j \cap g_j^{-1}([c_j, c_j + \gamma_0])$.

The following theorem is the main result of this section.

Theorem 2.35. *Assume that (A1) and (A2) hold and let* q *be a positive number. Then there exist numbers* $\Lambda_0 > 0$, $\Lambda_1 > 0$ *and* $r > 0$ *such that for each positive number* ϵ *there exists* $\delta \in (0, \epsilon)$ *such that the following assertion holds:*

If $f \in \mathcal{M}$ *satisfies*

$$(f, f_0) \in \mathcal{E}(M_0, q, r),$$

if a sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ *satisfies* $\lambda_n \geq \Lambda_0$ *and* $\lambda_i \lambda_{i+1}^{-1} \geq \Lambda_0$ *for each integer* i *such that* $1 \leq i < n$ *and if* $x \in X$ *satisfies*

$$f(x) + \sum_{i \in I_1} \lambda_i |g_i(x) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(x) - c_i, 0\}$$

$$\leq \inf\{f(z) + \sum_{i \in I_1} \lambda_i |g_i(z) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(z) - c_i, 0\} : \ z \in X\} + \delta,$$

then there exists $y \in A$ *such that*

$$\rho(x, y) \leq \epsilon \ \textit{and} \ f(y) \leq \inf(f; A) + \Lambda_1 \epsilon.$$

Theorem 2.35 implies the following results.

Theorem 2.36. *Assume that (A1) and (A2) hold and let* q *be a positive number. Then there exist numbers* $\Lambda_0, r > 0$ *such that for each* $f \in \mathcal{M}$ *satisfying*

$$(f, f_0) \in \mathcal{E}(M_0, q, r)$$

and each sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ *which satisfies* $\lambda_n \geq \Lambda_0$ *and* $\lambda_i \lambda_{i+1}^{-1} \geq \Lambda_0$ *for each integer* i *such that* $1 \leq i < n$ *the following assertion holds:*

If a sequence $\{x_k\}_{k=1}^\infty \in X$ *satisfies*

$$\lim_{k \to \infty} [f(x_k) + \sum_{i \in I_1} \lambda_i |g_i(x_k) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(x_k) - c_i, 0\}]$$

$$= \inf\{f(z) + \sum_{i \in I_1} \lambda_i |g_i(z) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(z) - c_i, 0\} : \ z \in X\},$$

then there exists a sequence $\{y_k\}_{k=1}^\infty \subset A$ *such that*

$$\lim_{k \to \infty} f(y_k) = \inf(f; A) \ \textit{and} \ \lim_{k \to \infty} \rho(x_k, y_k) = 0.$$

Theorem 2.36 implies the following result.

Theorem 2.37. *Assume that (A1) and (A2) hold and that there exists* $\bar{x} \in A$ *for which the following conditions hold:*

$f_0(\bar{x}) = \inf(f; A)$;

any sequence $\{x_k\}_{k=1}^\infty \subset A$ *which satisfies* $\lim_{k \to \infty} f_0(x_k) = \inf(f_0; A)$ *converges to* $\bar{x}$ *in* (X, ρ).

Then there exists a positive number Λ_0 such that for each sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ which satisfies $\lambda_n \geq \Lambda_0$ and $\lambda_i\lambda_{i+1}^{-1} \geq \Lambda_0$ for each integer i such that $1 \leq i < n$ the point $\bar{x}$ is a unique solution of the minimization problem

$$\text{minimize } f_0(z)+\sum_{i\in I_1}\lambda_i|g_i(z)-c_i|+\sum_{i\in I_2}\lambda_i\max\{g_i(z)-c_i,0\} \text{ subject to } z\in X.$$

Theorem 2.36 implies the following result.

Theorem 2.38. *Assume that any bounded subset of X is compact and that (A1) hold. Let $q > 0$. Then there exist numbers $\Lambda_0 > 0$, $r > 0$ such that for each $f \in \mathcal{M}$ satisfying*

$$(f, f_0) \in \mathcal{E}(M_0, q, r)$$

and each sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ which satisfies $\lambda_n \geq \Lambda_0$ and $\lambda_i\lambda_{i+1}^{-1} \geq \Lambda_0$ for each integer i such that $1 \leq i < n$ the following assertion holds:

If x is a solution of the minimization problem

$$\text{minimize } f(z)+\sum_{i\in I_1}|g_i(z)-c_i|+\sum_{i\in I_2}\max\{g_i(z)-c_i,0\} \text{ subject to } z\in X,$$

then $x \in A$ and $f(x) = \inf(f; A)$.

The results of this section have been obtained in [134].

2.12 Proof of Theorem 2.35

For each $f \in \mathcal{M}$ and each $\lambda = (\lambda_1, \dots, \lambda_n) \in (0,\infty)^n$ define

$$\psi_{f\lambda}(x) = f(x)+\sum_{i\in I_1}\lambda_i|g_i(x)-c_i|+\sum_{i\in I_2}\lambda_i\max\{g_i(x)-c_i,0\},\ x\in X. \quad (2.241)$$

It is easy to see that $\psi_{f\lambda} \in \mathcal{M}$ for each $f \in \mathcal{M}$ and each $\lambda = (\lambda_1, \dots, \lambda_n) \in (0,\infty)^n$.

We show that there exist numbers $\Lambda_0 > 0$, $r > 0$ such that the following property holds:

(P1) For each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that for each $f \in \mathcal{M}$ satisfying

$$(f, f_0) \in \mathcal{E}(M_0, q, r),$$

each sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ satisfying $\lambda_n \geq \Lambda_0$ and $\lambda_i\lambda_{i+1}^{-1} \geq \Lambda_0$ for all integers i such that $1 \leq i < n$ and for each $x \in X$ satisfying

$$\psi_{f\lambda}(x) \leq \inf(\psi_{f\lambda}) + \delta$$

the set $A \cap B(x, \epsilon)$ is nonempty.

Assume the contrary. Then for each integer $k \geq 1$ there exist

$$\epsilon_k \in (0,1),\ \lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_n^{(k)}) \in (0,\infty)^n,\ f_k \in \mathcal{M} \text{ and } x_k \in X \quad (2.242)$$

such that

$$(f_0, f_k) \in \mathcal{E}(M_0, q, 1/k), \quad (2.243)$$

$$\lambda_n^{(k)} \geq k \text{ and } \lambda_i^{(k)}(\lambda_{i+1}^{(k)})^{-1} \geq k \text{ for all integers } i \text{ satisfying } 1 \leq i < n, \quad (2.244)$$

$$\psi_{f_k \lambda^{(k)}}(x_k) \leq \inf(\psi_{f_k \lambda^{(k)}}) + 2^{-1}\epsilon_k k^{-2}, \quad (2.245)$$

$$\rho(x_k, A) \geq \epsilon_k. \quad (2.246)$$

Since the functions f_0 and g_i for integers i satisfying $2 \leq i \leq n$ are Lipschitzian on all bounded subsets of X there exists a number $L_0 > 1$ such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0 \rho(z_1, z_2) \text{ for each } z_1, z_2 \in B(\theta, M_0) \quad (2.247)$$

and that for each natural number i satisfying $2 \leq i \leq n$

$$|g_i(z_1) - g_i(z_2)| \leq L_0 \rho(z_1, z_2) \text{ for each } z_1, z_2 \in B(\theta, M_0). \quad (2.248)$$

Let $k \geq 1$ be an integer. By (2.245) and Ekeland's variational principle [37] there exists $y_k \in X$ such that

$$\psi_{f_k \lambda^{(k)}}(y_k) \leq \psi_{f_k \lambda^{(k)}}(x_k), \quad (2.249)$$

$$\rho(y_k, x_k) \leq (2k)^{-1}\epsilon_k, \quad (2.250)$$

$$\psi_{f_k \lambda^{(k)}}(y_k) \leq \psi_{f_k \lambda^{(k)}}(z) + k^{-1}\rho(z, y_k) \text{ for all } z \in X. \quad (2.251)$$

In view of (2.246)

$$y_k \notin A \text{ for all integers } k \geq 1. \quad (2.252)$$

By (2.252), (2.236) and (2.239) there exists an integer $p \in \{1, \dots, n\}$ such that the following conditions hold:

the set $\{k:\ k$ is a natural number for which $y_k \notin A_p\}$ is infinite;

if an integer j satisfies $1 \leq j < p$, then the set

$$\{k:\ k \text{ is a natural number for which } y_k \notin A_j\}$$

is finite.

Extracting a subsequence and reindexing we may assume without loss of generality that the following conditions hold:

$$y_k \notin A_p \text{ for all integers } k \geq 1; \quad (2.253)$$

if an integer j satisfies $1 \leq j < p$, then

$$y_k \in A_j \text{ for all integers } k \geq 1. \quad (2.254)$$

Relations (2.240) and (2.254) imply that

$$y_k \in B_p \text{ for each integer } k \geq 1. \tag{2.255}$$

We show that

$$y_k \in B(\theta, M_0 - 2) \text{ for all integers } k \geq 1. \tag{2.256}$$

By (2.242), (2.233), (2.241), (2.249), (2.245) and the inclusion $\theta_1 \in A$ for all integers $k \geq 1$

$$\begin{aligned} &\phi(\rho(y_k, \theta)) - \bar{a} \leq f_k(y_k) \leq \psi_{f_k \lambda^{(k)}}(y_k) \leq \psi_{f_k \lambda^{(k)}}(x_k) \\ &\leq \inf(\psi_{f_k \lambda^{(k)}}) + 2^{-1}k^{-2} \leq \inf(\psi_{f_k \lambda^{(k)}}; A) + (2k^2)^{-1} \\ &= \inf(f_k; A) + (2k^2)^{-1} \leq f_k(\theta_1) + (2k^2)^{-1}. \end{aligned} \tag{2.257}$$

Together with (2.243), (2.237) and (2.234) this relation implies that for all integers $k \geq 1$

$$\begin{aligned} &\phi(\rho(y_k, \theta)) \leq (2k^2)^{-1} + \bar{a} + f_k(\theta_1) \\ &\leq 2^{-1} + \bar{a} + f_0(\theta_1) + k^{-1} \leq 2 + \bar{a} + f_0(\theta_1). \end{aligned} \tag{2.258}$$

It follows from this inequality and the choice of M_0 (see (2.237) and (2.238)) that

$$y_k \in B(\theta, M_0 - 2) \text{ for all integers } k \geq 1. \tag{2.259}$$

If $p \in I_2$, then in view of (2.253) and (2.239)

$$g_p(y_k) > c_p \text{ for all integers } k \geq 1. \tag{2.260}$$

If $p \in I_1$, then (2.253) implies that for each integer $k \geq 1$ either $g_p(y_k) > c_p$ or $g_p(y_k) < c_p$. Extracting a subsequence and reindexing we may assume without loss of generality that either

$$g_p(y_k) > c_p \text{ for all natural numbers } k \tag{2.261}$$

or

$$g_p(y_k) < c_p \text{ for all natural numbers } k \geq 1. \tag{2.262}$$

Put

$$\bar{g}_p = g_p \text{ and } \bar{c}_p = c_p \text{ if } p \in I_2, \tag{2.263}$$

$$\bar{g}_p = g_p \text{ and } \bar{c}_p = c_p \text{ if } p \in I_1 \text{ and (2.261) is valid}, \tag{2.264}$$

$$\bar{g}_p = -g_p, \ \bar{c}_p = -c_p \text{ if } p \in I_1 \text{ and (2.262) is valid}. \tag{2.265}$$

Clearly, in all these cases

$$\bar{g}_p(y_k) > \bar{c}_p \text{ for all natural numbers } k. \tag{2.266}$$

Let $k \geq 1$ be an integer. By (2.239), (2.240), (2.251), (2.241), (2.235) and (2.255) for each $z \in B_p$,

$$-k^{-1}\rho(z,y_k) \le \psi_{f_k\lambda^{(k)}}(z) - \psi_{f_k\lambda^{(k)}}(y_k)$$

$$= f_k(z) - f_k(y_k) + \sum\{\lambda_i^{(k)}|g_i(z) - c_i| : \ i \in I_1 \text{ and } i \ge p\}$$

$$- \sum\{\lambda_i^{(k)}|g_i(y_k) - c_i| : \ i \in I_1 \text{ and } i \ge p\}$$

$$+ \sum\{\lambda_i^{(k)} \max\{g_i(z) - c_i, 0\} : \ i \in I_2 \text{ and } i \ge p\}$$

$$- \sum\{\lambda_i^{(k)} \max\{g_i(y_k) - c_i, 0\} : \ i \in I_2 \text{ and } i \ge p\}. \tag{2.267}$$

It follows from (2.266) that there exists $\Delta \in (0,1)$ such that

$$\bar{g}_p(z) > \bar{c}_p \text{ for all } z \in B(y_k, \Delta). \tag{2.268}$$

By (2.267), (2.268), (2.263)–(2.265) and (2.266) for each $z \in B_p \cap B(y_k, \Delta)$

$$\lambda_p^{(k)}(\bar{g}_p(z) - \bar{g}_p(y_k)) \ge -k^{-1}\rho(z,y_k) + \sum\{\lambda_i^{(k)}|g_i(y_k) - c_i| : \ i \in I_1 \text{ and } i > p\}$$

$$- \sum\{\lambda_i^{(k)}|g_i(z) - c_i| : \ i \in I_1 \text{ and } i > p\}$$

$$+ \sum\{\lambda_i^{(k)} \max\{g_i(y_k) - c_i, 0\} : \ i \in I_2 \text{ and } i > p\}$$

$$- \sum\{\lambda_i^{(k)} \max\{g_i(z) - c_i, 0\} : \ i \in I_2 \text{ and } i > p\} + f_k(y_k) - f_k(z)$$

$$\ge -k^{-1}\rho(z,y_k) - \sum\{\lambda_i^{(k)}|g_i(y_k) - g_i(z)| : \ i \in \{1, \dots, n\} \text{ and } i > p\}$$

$$+ f_k(y_k) - f_k(z). \tag{2.269}$$

By (2.247), (2.259), the relation $\Delta \in (0,1)$, (2.256), (2.243) and (2.234) for each $z \in B_p \cap B(y_k, \Delta)$,

$$|f_k(y_k) - f_k(z)| \le |f_0(y_k) - f_0(z)| + |(f_k - f_0)(y_k) - (f_k - f_0)(z)|$$

$$\le L_0\rho(z,y_k) + q\rho(z,y_k). \tag{2.270}$$

In view of (2.248), (2.256) and the relation $\Delta \in (0,1)$ for each $z \in B_p \cap B(y_k, \Delta)$ and each natural number i satisfying $2 \le i \le n$

$$|g_i(y_k) - g_i(z)| \le L_0\rho(y_k, z). \tag{2.271}$$

Relations (2.269), (2.270) and (2.271) imply that for each $z \in B_p \cap B(y_k, \Delta)$

$$\lambda_p^{(k)}(\bar{g}_p(z) - \bar{g}_p(y_k)) \ge -k^{-1}\rho(z,y_k) - (L_0 + q)\rho(z,y_k)$$

$$- \sum\{\lambda_i^{(k)} : \ i \in \{1, \dots, n\} \text{ and } i > p\} L_0\rho(y_k, z).$$

Combined with (2.244) this inequality implies that for each $z \in B_p \cap B(y_k, \Delta)$

$$\bar{g}_p(z) - \bar{g}_p(y_k) \ge -\rho(z,y_k)[k^{-1}(\lambda_p^{(k)})^{-1} + (L_0 + q)(\lambda_p^{(k)})^{-1}$$

$$+L_0 \sum\{\lambda_i^{(k)}(\lambda_p^{(k)})^{-1} : i \in \{1,\dots,n\} \text{ and } i > p\}]$$

$$\geq -\rho(z, y_k)[k^{-1} + (L_0 + q)k^{-1} + L_0 2k^{-1}].$$

Together with (2.231) this relation implies that

$$\Xi_{\bar{g}_p|_{B_p}}(y_k) \geq -k^{-1}(1 + 3L_0 + q) \text{ for all natural numbers } k$$

and that

$$\liminf_{k\to\infty} \Xi_{\bar{g}_p|_{B_p}}(y_k) \geq 0. \tag{2.272}$$

It follows from (2.241), (2.257), (2.243) and (2.237) that for each integer $j_1 \in [p, n] \cap I_1$, each $j_2 \in [p, n] \cap I_2$ and each natural number k

$$\lambda_{j_1}^{(k)}|g_{j_1}(y_k) - c_{j_1}|,\ \lambda_{j_2}^{(k)} \max\{g_{j_2}(y_k) - c_{j_2}, 0\}$$

$$\leq \psi_{f_k \lambda^{(k)}}(y_k) \leq f_k(\theta_1) + 1 \leq f_0(\theta_1) + 2.$$

Combined with (2.244) this inequality implies that

$$\lim_{k\to\infty} |g_j(y_k) - c_j| = 0 \text{ for all } j \in [p, n] \cap I_1, \tag{2.273}$$

$$\lim_{k\to\infty} \max\{g_j(y_k) - c_j, 0\} = 0 \text{ for all } j \in [p, n] \cap I_2.$$

It follows from (2.273) and (2.260) that if $p \in I_1$, then

$$|g_p(y_k) - c_p| \leq \gamma_0 \text{ for all sufficiently large natural numbers } k \tag{2.274}$$

and if $p \in I_2$, then

$$0 < g_p(y_k) - c_p \leq \gamma_0 \text{ for all sufficiently large natural numbers } k. \tag{2.275}$$

By (2.274), (2.275), (2.272), (2.255), (A2), (2.263)–(2.265) and (2.259) there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that $\{y_{k_j}\}_{j=1}^{\infty}$ converges in (X, ρ) to $y_* \in X$:

$$\lim_{j\to\infty} \rho(y_{k_j}, y_*) = 0. \tag{2.276}$$

Relations (2.276), (2.255), (2.239) and (2.240) imply that

$$y_* \in B_p. \tag{2.277}$$

In view of (2.273) and (2.276),

$$g_j(y_*) = c_j \text{ for all } j \in [p, n] \cap I_1 \text{ and } g_j(y_*) \leq c_j \text{ for all } j \in [p, n] \cap I_2.$$

Combined with (2.239), (2.240) and (2.277) this implies that

$$y_* \in A. \tag{2.278}$$

It follows from (2.277), (2.276), (2.272), (2.255) and Proposition 2.31 that

$$\Xi_{\bar{g}_p|_{B_p}}(y_*) \geq 0. \tag{2.279}$$

Relation (2.257) implies that for each natural number k

$$f_k(y_k) \leq \inf(\psi_{f_k\lambda^{(k)}}) + (2k^2)^{-1} \leq \inf(f_k; A) + (2k^2)^{-1}. \tag{2.280}$$

In view of (2.259) and (2.243)

$$\lim_{k\to\infty} |f_0(y_k) - f_k(y_k)| = 0. \tag{2.281}$$

We show that

$$\lim_{k\to\infty} (\inf(f_k; A)) = \inf(f_0; A). \tag{2.282}$$

It follows from (2.243) and (2.237) that for each integer $k \geq 0$

$$f_k(\theta_1) \leq f_0(\theta_1) + 1.$$

By this inequality, (2.242), (2.233), (2.238) and the inclusion $\theta_1 \in A$ for each integer $k \geq 0$ and each $z \in X$ satisfying $\rho(\theta, z) > M_0 - 2$ we have

$$f_k(z) \geq \phi(\rho(\theta, z)) - \bar{a} > f_0(\theta_1) + 4 > f_k(\theta_1) + 3 \geq \inf(f_k; A) + 3.$$

This relation implies that for each integer $k \geq 0$

$$\inf(f_k; A) = \inf\{f_k(z) : \ z \in A \text{ and } \rho(\theta, z) \leq M_0 - 2\}. \tag{2.283}$$

Let k be a natural number. In view of (2.243) for each $z \in B(\theta, M_0)$

$$|f_k(z) - f_0(z)| \leq 1/k.$$

Combined with (2.283) this implies that

$$|\inf(f_k; A) - \inf(f_0; A)| \leq 1/k \text{ for all integers } k \geq 1.$$

This relation implies that (2.282) holds.

It follows from (2.276), (2.281), (2.280) and (2.282) that

$$f_0(y_*) = \lim_{j\to\infty} f_0(y_{k_j}) = \lim_{j\to\infty} f_{k_j}(y_{k_j})$$

$$\leq \lim_{j\to\infty} [\inf(f_{k_j}; A) + (2k_j^2)^{-1}] = \inf(f_0; A).$$

In view of this relation, (2.279) and (2.278),

$$y_* \in A, \ f_0(y_*) = \inf(f_0; A), \ \Xi_{\bar{g}_p}|_{B_p}(y_*) \geq 0.$$

These relations contradict (A1). The contradiction we have reached proves that there exist numbers Λ_0, $r > 0$ such that property (P1) holds.

We may assume without loss of generality that $r < 1$. Since f_0 is Lipschitzian on bounded subsets of X there exists

$$\Lambda_1 > 4(q+1) \tag{2.284}$$

such that

$$|f_0(x) - f_0(y)| \le 2^{-1}\Lambda_1\rho(x,y) \text{ for each } x, y \in B(\theta, M_0). \tag{2.285}$$

Assume that $\epsilon \in (0,1)$. Let $\delta \in (0,\epsilon)$ be as guaranteed by property (P1). Now assume that $f \in \mathcal{M}$ satisfies

$$(f, f_0) \in \mathcal{E}(M_0, q, r), \tag{2.286}$$

a sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ satisfies

$$\lambda_n \ge \Lambda_0 \text{ and } \lambda_i\lambda_{i+1}^{-1} \ge \Lambda_0 \text{ for each integer } i \text{ satisfying } 1 \le i < n \tag{2.287}$$

and that $x \in X$ satisfies

$$\psi_{f\lambda}(x) \le \inf(\psi_{f\lambda}) + \delta. \tag{2.288}$$

It follows from (P1) and the choice of δ that there exists $y \in X$ such that

$$y \in A \cap B(x, \epsilon). \tag{2.289}$$

By (2.241), (2.288) and the inclusion $\theta_1 \in A$,

$$\begin{aligned} f(x) &\le \psi_{f\lambda}(x) \le \inf(\psi_{f\lambda}) + \delta \le \inf(\psi_{f\lambda}) + 1 \\ &\le \inf(\psi_{f\lambda}; A) + 1 = \inf(f; A) + 1 \le f(\theta_1) + 1. \end{aligned} \tag{2.290}$$

It follows from (2.233), (2.290), (2.286), (2.237) and the inequality $r < 1$ that

$$\phi(\rho(x,\theta)) - \bar{a} \le f(x) \le f(\theta_1) + 1 \le f_0(\theta_1) + 1 + r \le f_0(\theta_1) + 2. \tag{2.291}$$

In view of (2.291) and (2.238)

$$x \in B(\theta, M_0 - 2). \tag{2.292}$$

Relations (2.292) and (2.289) imply that

$$\rho(\theta, y) \le \rho(\theta, x) + \rho(x, y) \le M_0 - 1.$$

Combined with (2.292), (2.285), (2.286) and (2.289) this inequality implies that

$$\begin{aligned} |f(x) - f(y)| &\le |f_0(x) - f_0(y)| + |(f - f_0)(x) - (f - f_0)(y)| \\ &\le 2^{-1}\Lambda_1\rho(x,y) + q\rho(x,y) \le (2^{-1}\Lambda_1 + q)\epsilon. \end{aligned}$$

Together with (2.241), (2.288) and (2.284) this inequality implies that

$$\begin{aligned} f(y) &\le (2^{-1}\Lambda_1 + q)\epsilon + f(x) \le (2^{-1}\Lambda_1 + q)\epsilon + \psi_{f\lambda}(x) \\ &\le (2^{-1}\Lambda_1 + q)\epsilon + \inf(\psi_{f,\lambda}) + \delta \\ &\le (2^{-1}\Lambda_1 + q + 1)\epsilon + \inf(\psi_{f,\lambda}; A) \le \Lambda_1\epsilon + \inf(f; A). \end{aligned}$$

This completes the proof of Theorem 2.35.

2.13 An extension of Theorem 2.35

In this section we use the notation and the definitions introduced in Section 2.11.

Assume that n is a natural number, $f_0 \in \mathcal{M}$ is Lipschitzian on all bounded subsets of X, $g_i : X \to R^1$, $i = 1, \dots, n$ are locally Lipschitzian functions such that g_i is Lipschitzian on all bounded subsets of X for any integer i satisfying $2 \le i \le n$ and that $c = (c_1, \dots, c_n) \in R^n$.

Let I_1 and I_2 be subsets of $\{1, \dots, n\}$ satisfying (2.235). (Note that one of the sets I_1, I_2 may be empty.) Define the set A by (2.236). We assume that $A \ne \emptyset$. We consider the sets A_i, $i = 1, \dots, n$ defined by (2.239) and the sets B_j, $j = 1, \dots, n$ defined by (2.240).

In the sequel we use the following assumption.

(A3) For each $p \in I_1$, c_p is not a critical value of $g_p|_{B_p}$ and $-c_p$ is not a critical value of $-g_p|_{B_p}$. For each $p \in I_2$, c_p is not a critical value of $g_p|_{B_p}$.

We will establish the following result.

Theorem 2.39. *Assume that (A2) and (A3) hold and let M_0 be a positive number. Then there exists a positive number M_1 such that for each $q > 0$ there exist numbers $\Lambda_0 > 0$, $\Lambda_1 > 0$ such that the following assertion holds:*

For each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that if $f \in \mathcal{M}$ satisfies

$$(f, f_0) \in \mathcal{E}(M_1, q, M_0), \tag{2.293}$$

if a sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ satisfies

$$\lambda_n \ge \Lambda_0 \text{ and } \lambda_i \lambda_{i+1}^{-1} \ge \Lambda_0 \text{ for each integer } i \text{ such that } 1 \le i < n \tag{2.294}$$

and if $x \in X$ satisfies

$$f(x) + \sum_{i \in I_1} \lambda_i |g_i(x) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(x) - c_i, 0\}$$

$$\le \inf\{f(z) + \sum_{i \in I_1} \lambda_i |g_i(z) - c_i| + \sum_{i \in I_2} \lambda_i \max\{g_i(z) - c_i, 0\} : \ z \in X\} + \delta, \tag{2.295}$$

then there exists $y \in A$ which satisfies

$$\rho(x, y) \le \epsilon \text{ and } f(y) \le \inf(f; A) + \Lambda_1 \epsilon.$$

Proof: For each $f \in \mathcal{M}$ and each $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$ define $\psi_{f\lambda} : X \to R^1$ by (2.241). Fix $\theta_1 \in A$. In view of (2.232) there exists

$$M_1 > \rho(\theta, \theta_1) + 8 \tag{2.296}$$

such that

$$\phi(M_1 - 4) > f_0(\theta_1) + M_0 + \bar{a} + 8. \tag{2.297}$$

Let q be a positive number. We show that there exists a number $\Lambda_0 > 0$ such that the following property holds:

(P2) For each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that for each $f \in \mathcal{M}$ satisfying (2.293) each sequence of positive numbers $\{\lambda_i\}_{i=1}^n$ satisfying (2.294) and each $x \in X$ satisfying (2.295) the set $A \cap B(x, \epsilon)$ is nonempty.

Assume the contrary. Then for each integer $k \geq 1$ there exist

$$\epsilon_k \in (0,1),\ \lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_n^{(k)}) \in (0, \infty)^n,\ f_k \in \mathcal{M},\ x_k \in X \tag{2.298}$$

such that $(f_0, f_k) \in \mathcal{E}(M_1, q, M_0)$ and the relations (2.244) and (2.246) hold.

Arguing as in the proof of Theorem 2.35 we can show that there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset X \setminus A$ such that for each integer $k \geq 1$ relations (2.249)–(2.251) are valid. Arguing as in the proof of Theorem 2.35 we can show that (without loss of generality) there exists $p \in \{1, \ldots, n\}$ such that $y_k \in B_p$ for all integers $k \geq 1$, if $p \in I_2$, then (2.260) holds and if $p \in I_1$, then either (2.261) holds or (2.262) is valid. Define $\bar{y}_p$, $\bar{c}_p$ as in the proof of Theorem 2.35 (see (2.263)–(2.265)). Then we have that (2.266) holds in all the cases. Since the functions f_0 and g_i for natural numbers i satisfying $2 \leq i \leq n$ are Lipschitzian on all bounded subsets of X there exists a positive number $L_0 > 1$ such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0 \rho(z_1, z_2) \text{ for each } z_1, z_2 \in B(\theta, M_1)$$

and that for each natural number i satisfying $2 \leq i \leq n$

$$|g_i(z_1) - g_i(z_2)| \leq L_0 \rho(z_1, z_2) \text{ for each } z_1, z_2 \in B(\theta, M_1).$$

Arguing as in the proof of Theorem 2.35 we can show that

$$y_k \in B(\theta, M_1 - 4) \text{ for all integers } k \geq 1 \tag{2.299}$$

and that for any integer $k \geq 1$

$$\Xi_{\bar{g}_p}|_{B_p}(y_k) \geq -k^{-1}(1 + 3L_0 + q) \text{ and } \liminf_{k \to \infty} \Xi_{\bar{g}_p}|_{B_p}(y_k) \geq 0. \tag{2.300}$$

As in the proof of Theorem 2.35 we show that (2.274) and (2.275) hold. By (A2), (2.274), (2.275), (2.300) and the inclusion $y_k \in B_p$ for all integers $k \geq 1$ there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that $\{y_{k_j}\}_{j=1}^{\infty}$ converges in (X, ρ) to $y_* \in X$. Arguing as in the proof of Theorem 2.35 we show that

$$y_* \in B_p\ y_* \in A \text{ and } \Xi_{\bar{g}_p|_{B_p}}(y_*) \geq 0. \tag{2.301}$$

Relations (2.301) contradict (A1). The contradiction we have reached proves that there exists $\Lambda_0 > 0$ such that (P2) holds. Arguing as in the proof of Theorem 2.35 we show that Theorem 2.39 follows from property (P2).

Theorem 2.39 has been obtained in [134].

2.14 Exact penalty property and Mordukhovich basic subdifferential

In this section we study two constrained minimization problems in infinite-dimensional Asplund spaces and establish a sufficient condition for exact penalty property using the notion of the Mordukhovich basic subdifferential [71, 73]. Note that a Banach space is an Asplund space if and only if every separable subspace has a separable dual [73].

In Section 2.1 the existence of the exact penalty for the equality-constrained problem is established under the assumption that the set of admissible points does not contain critical points of the constraint function. The notion of critical points used in Section 2.1 is based on Clarke's generalized gradients [21]. It should be mentioned that there exists also the construction of the Mordukhovich basic subdifferential introduced in [71] which is intensively used in the literature. See, for example, [73, 85] and the references mentioned there. In this section we generalize the results of Section 2.1 for minimization problems on Asplund spaces using the (less restrictive) notion of critical points via the Mordukhovich basic subdifferential.

Let $(X, ||\cdot||)$ be an Asplund space and $(X^*, ||\cdot||_*)$ its dual equipped with the weak topology w^*.

If $F : X \to 2^{X^*}$ is a set-valued mapping between the Banach space X and its dual X^*, then the notation

$$\limsup_{x\to\bar{x}} F(x) := \{x^* \in X^* : \text{ there exist sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^*$$

$$\text{as } k \to \infty \text{ with } x_k^* \in F(x_k) \text{ for all integers } k \geq 1\} \tag{2.302}$$

signifies the sequential Painleve–Kuratowski upper limit with respect to the norm topology of X and the weak* topology of X^*.

For each $x \in X$, each $x^* \in X^*$ and each positive number r put

$$B(x, r) = \{y \in X : ||y - x|| \leq r\},$$

$$B_*(x^*, r) = \{l \in X^* : ||l - x^*||_* \leq r\}. \tag{2.303}$$

In this section in order to obtain a sufficient condition for the existence of an exact penalty we use the notion of Mordukhovich basic subdifferential introduced in [71] (see also [73]). In order to meet this goal we first present the notion of an analytic ϵ-subdifferential [73].

Let $\phi : X \to R^1$, ϵ be a positive number and let $\bar{x} \in X$. Then the set

$$\widehat{\partial}_{a\epsilon}\phi(\bar{x}) := \{x^* \in X^* : \liminf_{x\to\bar{x}} [(\phi(x) - \phi(\bar{x}) - < x^*, x - \bar{x} >)||x - \bar{x}||^{-1}] \geq -\epsilon\} \tag{2.304}$$

is the analytic ϵ-subdifferential of ϕ at $\bar{x}$. By Theorem 1.8.9 of [73], the set

$$\partial\phi(\bar{x}) = \limsup_{x\xrightarrow{\phi}\bar{x},\ \epsilon\to 0^+} \widehat{\partial}_{a\epsilon}\phi(x) \tag{2.305}$$

is the Mordukhovich basic (limiting) subdifferential of the function ϕ at the point $\bar{x}$.

Here we use the notation that $x \xrightarrow{\phi} \bar{x}$ if and only if $x \to \bar{x}$ with $\phi(x) \to \phi(\bar{x})$, where $\phi(x) \to \phi(\bar{x})$ is superfluous if ϕ is continuous at $\bar{x}$.

Note that in this section we do not provide a definition of the Mordukhovich basic (limiting) subdifferential as it appears in the literature (see page 82 of [73]). Instead of it we work with the formula (2.305) which is more convenient for our goals.

Let $f : X \to R^1$ be a locally Lipschitzian function. For each $x \in X$ denote by $\partial f(x)$ the Mordukhovich basic subdifferential of f at x and set

$$\Xi_f(x) = \inf\{||l||_* : \ l \in \partial f(x)\}. \tag{2.306}$$

(We suppose that the infimum of an empty set is ∞.)

A point $x \in X$ is a critical point of f if $0 \in \partial f(x)$.

A real number $c \in R^1$ is called a critical value of f if there exists a critical point x of f such that $f(x) = c$.

For each function $h : X \to R^1$ and each nonempty set $A \subset X$ set

$$\inf(h) = \{h(z) : \ z \in X\}, \tag{2.307}$$

$$\inf(h; A) = \inf\{h(z) : \ z \in A\}. \tag{2.308}$$

For each $x \in X$ and each $B \subset X$ put

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}. \tag{2.309}$$

Let $f : X \to R^1$ be a function which is Lipschitzian on all bounded subsets of X and which satisfies the growth condition

$$\lim_{||x|| \to \infty} f(x) = \infty. \tag{2.310}$$

Clearly, the function f is bounded from below.

Let $g : X \to R^1$ be a locally Lipschitzian function.

We say that the function g satisfies the (P-S) condition on a set $M \subset X$ if for each normed-bounded sequence $\{z_i\}_{i=1}^{\infty} \subset M$ such that the sequence $\{g(z_i)\}_{i=1}^{\infty}$ is bounded and $\liminf_{i \to \infty} \Xi_g(z_i) = 0$ there exists a norm-convergent subsequence of $\{z_i\}_{i=1}^{\infty}$ [8, 76, 100].

Let $c \in R^1$ be such that $g^{-1}(c) \neq \emptyset$.

We consider the following constrained minimization problems:

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}(c) \tag{P$_e$}$$

and

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}((-\infty, c]). \tag{P$_i$}$$

We associate with these two problems the corresponding families of unconstrained minimization problems

$$\text{minimize } f(x) + \lambda|g(x) - c| \text{ subject to } x \in X \tag{$P_{\lambda e}$}$$

and

$$\text{minimize } f(x) + \lambda \max\{g(x) - c, 0\} \text{ subject to } x \in X \tag{$P_{\lambda i}$}$$

where $\lambda > 0$.

The following two theorems are our main results in this section.

Theorem 2.40. *Assume that there exists a positive number γ_* such that the functions g and $-g$ satisfy the (P-S) condition on the set $g^{-1}([c-\gamma_*, c+\gamma_*])$ and the following property holds:*

If $x \in g^{-1}(c)$ is a critical point of the function g or a critical point of the function $-g$, then $f(x) > \inf(f; g^{-1}(c))$.

Then there exists a number $\bar{\lambda} > 0$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:

If $\lambda > \bar{\lambda}$ and if $x \in X$ satisfies

$$f(x) + \lambda|g(x) - c| \leq \inf\{f(z) + \lambda|g(z) - c| : \ z \in X\} + \delta,$$

then there is $y \in g^{-1}(c)$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; g^{-1}(c)) + \delta.$$

Theorem 2.41. *Assume that there exists a positive number γ_* such that the function g satisfies the (P-S) condition on the set $g^{-1}([c, c+\gamma_*])$ and the following property holds:*

If $x \in g^{-1}(c)$ is a critical point of the function g, then

$$f(x) > \inf(f; g^{-1}(-\infty, c]).$$

Then there exists a number $\bar{\lambda} > 0$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:

If $\lambda > \bar{\lambda}$ and if $x \in X$ satisfies

$$f(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{f(z) + \lambda \max\{g(z) - c, 0\} : \ z \in X\} + \delta,$$

then there is $y \in g^{-1}(-\infty, c])$ such that

$$||y - x|| \leq \epsilon \text{ and } f(y) \leq \inf(f; g^{-1}(-\infty, c]) + \delta.$$

Theorems 2.40 and 2.41 have been established in [142].

2.15 Proofs of Theorems 2.40 and 2.41

We prove Theorems 2.40 and 2.41 simultaneously. Set

$$A = g^{-1}(c) \text{ in the case of Theorem 2.40} \tag{2.311}$$

and

$$A = g^{-1}((-\infty, c]) \text{ in the case of Theorem 2.2.} \tag{2.312}$$

For each positive number λ define a function $\psi_\lambda : X \to R^1$ by

$$\psi_\lambda(z) = f(z) + \lambda|g(z) - c|, \ z \in X \tag{2.313}$$

in the case of Theorem 2.40 and by

$$\psi_\lambda(z) = f(z) + \lambda \max\{g(z) - c, 0\}, \ z \in X \tag{2.314}$$

in the case of Theorem 2.41.

Evidently, the function ψ_λ is locally Lipschitzian for all positive numbers λ. We show that there exists $\bar{\lambda} > 0$ such that the following property holds:

(P) For each $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon)$ such that for each $\lambda > \bar{\lambda}$ and each $x \in X$ which satisfies

$$\psi_\lambda(x) \le \inf(\psi_\lambda) + \delta$$

the set

$$\{y \in A \cap B(x, \epsilon) : \ \psi_\lambda(y) \le \psi_\lambda(x)\}$$

is nonempty.

It is easy to see that the existence of $\bar{\lambda} > 0$ for which the property (P) holds implies the validity of Theorems 2.40 and 2.41.

Assume the contrary. Then for each integer $k \ge 1$ there exist

$$\epsilon_k \in (0, 1), \ \lambda_k > k, \ x_k \in X \tag{2.315}$$

such that

$$\psi_{\lambda_k}(x_k) \le \inf(\psi_{\lambda_k}) + 2^{-1}\epsilon_k k^{-2} \tag{2.316}$$

and

$$\{z \in A : \ ||z - x_k|| \le \epsilon_k \text{ and } \psi_{\lambda_k}(y_k) \le \psi_{\lambda_k}(x_k)\} = \emptyset. \tag{2.317}$$

Let $k \ge 1$ be an integer. By (2.316) and Ekeland's variational principle [37] there exists $y_k \in X$ such that

$$\psi_{\lambda_k}(y_k) \le \psi_{\lambda_k}(x_k), \tag{2.318}$$

$$||y_k - x_k|| \le (2k)^{-1}\epsilon_k, \tag{2.319}$$

$$\psi_{\lambda_k}(y_k) \le \psi_{\lambda_k)}(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{2.320}$$

It follows from (2.318), (2.319) and (2.317) that

$$y_k \notin A \text{ for all integers } k \geq 1. \tag{2.321}$$

In the case of Theorem 2.41 we obtain that

$$g(y_k) > c \text{ for all integers } k \geq 1. \tag{2.322}$$

In the case of Theorem 2.40 we obtain that for each integer $k \geq 1$ either $g(y_k) > c$ or $g(y_k) < c$.

In the case of Theorem 2.40 by extracting a subsequence and reindexing we may assume that either $g(y_k) > c$ for all integers $k \geq 1$ or $g(y_k) < c$ for all integers $k \geq 1$. Replacing g with $-g$ and c with $-c$ if necessary we may assume without loss of generality that (2.322) holds in the case of Theorem 2.40 too. Now (2.322) is valid in both cases.

Let $k \geq 1$ be an integer. Then by (2.322) there is an open neighborhood V_k of y_k in X such that

$$g(z) > c \text{ for all } z \in V_k.$$

Combined with (2.313), (2.314) and (2.320) this implies that for all $z \in V_k$

$$f(y_k)+\lambda_k(g(y_k)-c) = \psi_{\lambda_k}(y_k) \leq f(z)+\lambda_k(g(z)-c)+k^{-1}||z-y_k||. \tag{2.323}$$

Set

$$\phi_k(z) = f(z) + \lambda_k g(z) + k^{-1}||z - y_k||, \; z \in V_k. \tag{2.324}$$

It follows from (2.305), (2.304), (2.324) and (2.323) that

$$0 \in \partial(\phi_k)(y_k). \tag{2.325}$$

By (2.325), (2.324) and Theorem 3.36 of [73],

$$0 \in \partial f(y_k) + \lambda_k \partial g(y_k) + k^{-1}\partial(|| \cdot -y_k||)(y_k). \tag{2.326}$$

Relation (2.326) and Corollary 1.8.1 of [73] imply that

$$\begin{aligned} 0 &\in \partial g(y_k) + \lambda_k^{-1}\partial f(y_k) + \lambda_k^{-1}k^{-1}\partial(|| \cdot -y_k||)(y_k) \\ &\subset \partial g(y_k) + \lambda_k^{-1}\partial f(y_k) + \lambda_k^{-1}k^{-1}B_*(0,1). \end{aligned} \tag{2.327}$$

By (2.313)–(2.316) and (2.318) for all integers $k \geq 1$,

$$f(y_k) \leq \psi_{\lambda_k}(y_k) \leq \inf(\psi_{\lambda_k}) + 1 \leq \inf(\psi_{\lambda_k}; A) + 1 = \inf(f; A) + 1. \tag{2.328}$$

It follows from this inequality and the growth condition (2.310) that the sequence $\{y_k\}_{k=1}^{\infty}$ is bounded. Since the function f is Lipschitzian on bounded subsets of X it follows from the boundedness of the sequence $\{y_k\}_{k=0}^{\infty}$ and Corollary 1.8.1 of [73] that there exists a positive number L such that

$$\partial f(y_k) \subset B_*(0, L) \text{ for all integers } k \geq 1. \tag{2.329}$$

In view of (2.315), (2.327) and (2.329) for all integers $k \geq 1$,

$$0 \in \partial g(y_k) + \lambda_k^{-1} B_*(0, L) + k^{-1} B_*(0, 1)$$

and by (2.306),

$$\lim_{k \to \infty} \Xi_g(y_k) = 0. \tag{2.330}$$

It follows from (2.313)–(2.316), (2.318) and (2.322) that for all natural numbers k,

$$\inf(f) + \lambda_k(g(y_k) - c) \leq f(y_k) + \lambda_k(g(y_k) - c) = \psi_{\lambda_k}(y_k)$$

$$\leq \inf(f; A) + 1$$

and

$$0 < g(y_k) - c \leq \lambda_k^{-1}[\inf(f; A) + 1 - \inf(f)] \to 0 \text{ as } k \to \infty. \tag{2.331}$$

Thus there exists an integer $k_0 \geq 1$ such that for all integers $k \geq k_0$

$$g(y_k) \in (c, c + \gamma_*]. \tag{2.332}$$

It follows from (2.332), the boundedness of the sequence $\{y_k\}_{k=1}^{\infty}$ in the norm topology and the (P-S) condition that there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that $\{y_{k_j}\}_{j=1}^{\infty}$ converges in the norm topology to $y_* \in X$. Relation (2.331) implies that

$$g(y_*) = c. \tag{2.333}$$

It follows from (2.313), (2.314), (2.318) and (2.326) that

$$f(y_*) = \lim_{j \to \infty} f(y_{k_j}) \leq \limsup_{j \to \infty} \psi_{\lambda_{k_j}}(y_{k_j})$$

$$\leq \limsup_{j \to \infty} \inf(\psi_{\lambda_{k_j}}) \leq \limsup_{j \to \infty} \inf(\psi_{\lambda_{k_j}}; A) = \inf(f; A).$$

Combined with (2.333), (2.311) and (2.312) this implies that

$$f(y_*) = \inf(f; A). \tag{2.334}$$

We have already mentioned that the sequence $\{y_k\}_{k=1}^{\infty}$ is bounded. Fix a number $M > 4$ for which

$$\{y_k\}_{k=1}^{\infty} \subset B(0, M - 2). \tag{2.335}$$

Since the function f is Lipschitz on bounded subsets of X there exists $L_0 \geq 1$ such that

$$|f(z_1) - f(z_2)| \leq L_0 ||z_1 - z_2|| \text{ for all } z_1, z_2 \in B(0, M). \tag{2.336}$$

Let $k \geq 1$ be an integer. We may assume that

$$V_k \subset B(y_k, 1) \subset B(0, M). \tag{2.337}$$

It follows from (2.323), (2.337), (2.336) and (2.325) that for all $z \in V_k \setminus \{y_k\}$,

$$(g(z) - g(y_k))||z - y_k||^{-1} \geq ||z - y_k||^{-1}[f(y_k) - f(z)]\lambda_k^{-1} - k^{-1}\lambda_k^{-1}$$

$$\geq -L_0\lambda_k^{-1} - k^{-1} \geq -k^{-1}(1 + L_0).$$

In view of the relation above and the definition (2.304),

$$0 \in \widehat{\partial}_{a\gamma_k} g_k(y_k)$$

with

$$\gamma_k = k^{-1}(1 + L_0).$$

Combined with (2.305) and the equality $y_* = \lim_{j\to\infty} y_{k_j}$ in the norm topology this implies that $0 \in \partial g(y_*)$. This contradicts the relation (2.333) and (2.334). The contradiction we have reached proves the existence of $\bar{\lambda} > 0$ for which the property (P) holds.

This completes the proofs of Theorems 2.40 and 2.41.

2.16 Comments

In this chapter we use the penalty approach in order to study various classes of constrained minimization problems on Banach spaces and on metric spaces. Our goal is to find simple sufficient conditions for the existence of an exact penalty for which any solution of a penalized unconstrained problem is a solution of the original unconstrained problem. Since in this chapter we consider minimization problems on a general Banach space and on a general metric space the existence of their solutions is not guaranteed. Therefore we are interested in approximate solutions of the unconstrained penalized problem and in approximate solutions of the corresponding constrained problem. Under mild assumptions we establish the existence of a penalty coefficient $\Lambda_0 > 0$ such that the following property holds:

For each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ which depends only on ϵ such that if x is a $\delta(\epsilon)$-approximate solution of the unconstrained penalized problem whose penalty coefficient is larger than Λ_0, then there exists an ϵ-approximate solution y of the corresponding constrained problem such that $||y - x|| \leq \epsilon$.

In our study we use tools and methods of variational analysis and, in particular, the nonsmooth version of the (P–S) condition introduced in [100] for Lipschitzian functions. Note that recently a subdifferential (P–S) condition for set-valued mappings was introduced and applied in [9] for multiobjective optimization problems.

3

Stability of the Exact Penalty

3.1 Minimization problems with one constraint

In this section we study two constrained nonconvex minimization problems with Lipschitzian (on bounded sets) objective functions. The first problem is an equality-constrained problem in a Banach space with a locally Lipschitzian constraint function and the second problem is an inequality-constrained problem in a Banach space with a locally Lipschitzian constraint function.

Let $(X, ||\cdot||)$ be a Banach space, $(X^*, ||\cdot||_*)$ its dual space and let $f: X \to R^1$ be a locally Lipschitzian function.

For each $x \in X$ let

$$f^0(x,h) = \limsup_{t \to 0+,\ y \to x} [f(y+th) - f(y)]/t, \ h \in X,$$

be the Clarke generalized directional derivative of f at the point x [21], let

$$\partial f(x) = \{l \in X^* : \ f^0(x,h) \geq l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of f at x, [21] and set

$$\Xi_f(x) = \inf\{f^0(x,h) : \ h \in X \text{ and } ||h|| \leq 1\} \tag{3.1}$$

[100].

Recall that a point $x \in X$ is called a critical point of f if $0 \in \partial f(x)$. It is not difficult to see that $x \in X$ is a critical point of f if and only if $\Xi_f(x) = 0$.

A real number $c \in R^1$ is called a critical value of f if there is a critical point x of f such that $f(x) = c$.

It is known that $\partial(-f)(x) = -\partial f(x)$ for any $x \in X$ (see Chapter 2, Section 2.3 of [21]). This equality implies that $x \in X$ is a critical point of f if and only if x is a critical point of $-f$ and $c \in R^1$ is a critical value of f if and only if $-c$ is a critical value of $-f$.

For each function $h : X \to R^1$ set $\inf(h) = \inf\{h(z) : \ z \in X\}$.

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For each $x \in X$ and each nonempty set $B \subset X$ put

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

We say that a locally Lipschitz function $f : X \to R^1$ satisfies the Palais–Smale (P-S) condition on a set $A \subset X$ if for any sequence $\{x_i\}_{i=1}^{\infty} \subset A$ for which the sequence $\{f(x_i)\}_{i=1}^{\infty}$ is bounded and $\lim_{i\to\infty} \Xi_f(x_i) = 0$ there exists a norm convergent subsequence in X [8, 76, 100].

Remark 3.1. In Section 2.1 instead of the function Ξ_f we introduced a function $\tilde{\Xi}_f : X \to R^1$ defined by

$$\tilde{\Xi}_f(x) = \inf\{f^0(x, h) : \ h \in X \text{ and } ||h|| = 1\}, \ x \in X.$$

Clearly for all $x \in X$ we have $\Xi_f(x) \le \tilde{\Xi}_f(x)$ and $\tilde{\Xi}_f(x) \ge 0$ if and only if $\Xi_f(x) = 0$. It is not difficult to see that for each sequence $\{x_i\}_{i=1}^{\infty} \subset X$

$$\lim_{i\to\infty} \Xi_f(x_i) = 0 \text{ if and only if } \liminf_{i\to\infty} \tilde{\Xi}_f(x_i) \ge 0.$$

For each $x \in X$, each $x^* \in X^*$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||y - x|| \le r\}, \ B_*(x^*, r) = \{l \in X^* : \ ||l - x^*||_* \le r\},$$

$$B^0(x, r) = \{y \in X : \ ||y - x|| < r\}, \ B^0_*(x^*, r) = \{l \in X^* : \ ||l - x^*||_* < r\}.$$

We assume that the infimum over an empty set is infinity.

For each $c \in R^1$ and each pair of functions $f, g : X \to R^1$ set

$$\inf(f, c, g) = \inf\{f(z) : \ z \in g^{-1}(c)\}, \tag{3.2}$$

$$\inf(f, (-\infty, c], g) = \inf\{f(z) : \ z \in g^{-1}((-\infty, c])\}.$$

Let $\phi : [0, \infty) \to R^1$ be an increasing function such that

$$\lim_{t\to\infty} \phi(t) = \infty. \tag{3.3}$$

Let M_0, M_1, M_2 be positive numbers and let $h : X \to R^1$ be a continuous function. Now we define a family of functions which are close to the function h. This family is determined by the parameters M_0, M_1, M_2.

Denote by $\mathcal{U}(h, M_0, M_1, M_2)$ the set of all continuous functions $g : X \to R^1$ such that

$$|g(x) - h(x)| \le M_1 \text{ for all } x \in B(0, M_0), \tag{3.4}$$

$$|h(x) - g(x) - (h(y) - g(y))| \le M_2||x - y|| \text{ for all } x, y \in B(0, M_0) \tag{3.5}$$

and denote by $\mathcal{U}_\phi(h, M_0, M_1, M_2)$ the set of all $g \in \mathcal{U}(h, M_0, M_1, M_2)$ such that

$$g(x) \ge \phi(||x||) \text{ for all } x \in X. \tag{3.6}$$

Let $f_0 : X \to R^1$ be a locally Lipschitzian function which satisfies

$$f_0(x) \geq \phi(||x||) \text{ for all } x \in X, \tag{3.7}$$

$g_0 : X \to R^1$ be a locally Lipschitzian function and let

$$a_0 < c_0 < b_0. \tag{3.8}$$

We assume that

$$g_0^{-1}(c_0) \neq \emptyset \tag{3.9}$$

and consider the following constrained minimization problems:

$$\text{minimize } f_0(x) \text{ subject to } x \in g_0^{-1}(c_0) \tag{P$_\text{e}$}$$

and

$$\text{minimize } f_0(x) \text{ subject to } x \in g_0^{-1}((-\infty, c_0]). \tag{P$_\text{i}$}$$

We suppose that there exists $\theta \in X$ such that the following assumption holds:

(A0) In the case of the problem (P_e)

$$\theta \in g_0^{-1}(c_0) \text{ and } 0 \notin \partial g_0(\theta);$$

in the case of the problem (P_i)

$$\theta \in g_0^{-1}((-\infty, c_0]) \text{ and if } \theta \in g_0^{-1}(c_0), \text{ then } 0 \notin \partial g_0(\theta).$$

Assume that

$$f : X \to R^1, \; g : X \to R^1, \; c \in R^1.$$

If $g^{-1}(c) \neq \emptyset$, then we consider the equality-constraint problem

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}(c) \tag{P$_\text{c,e}^\text{(f,g)}$}$$

and if $g^{-1}((-\infty, c]) \neq \emptyset$, then we consider the inequality-constraint problem

$$\text{minimize } f(x) \text{ subject to } x \in g^{-1}((-\infty, c]). \tag{P$_\text{c,i}^\text{(f,g)}$}$$

We associate with these two problems the corresponding families of unconstrained minimization problems

$$\text{minimize } f(x) + \lambda|g(x) - c| \text{ subject to } x \in X, \tag{P$_{\lambda,\text{c,e}}^\text{(f,g)}$}$$

$$\text{minimize } f(x) + \lambda \max\{g(x) - c, 0\} \text{ subject to } x \in X \tag{P$_{\lambda,\text{c,i}}^\text{(f,g)}$}$$

where λ is a positive number.

In view of (3.3) there exists a number $\bar{M} > 0$ such that

$$\bar{M} > ||\theta|| + 4 \text{ and } \phi(\bar{M} - 4) > f_0(\theta) + 4. \tag{3.10}$$

In this paper we use the following assumptions:

(A1) f_0 is Lipschitz on $B(0, \bar{M})$;

(A2) for each positive number ϵ there exists $x_\epsilon \in g_0^{-1}(c_0)$ such that $f_0(x_\epsilon) \le \inf(f, c_0, g_0) + \epsilon$ and $0 \notin \partial g_0(x_\epsilon)$;

(A3) for each positive number ϵ there exists $x_\epsilon \in g_0^{-1}((-\infty, c_0])$ such that

$$f_0(x_\epsilon) \le \inf(f_0, (-\infty, c_0], g_0) + \epsilon$$

and if $x_\epsilon \in g_0^{-1}(c_0)$, then $0 \notin \partial g_0(x_\epsilon)$;

(A4) if $y \in g_0^{-1}(c_0)$ satisfies $f_0(y) = \inf(f_0, c_0, g_0)$, then $0 \notin \partial g_0(y)$;

(A5) if $y \in g_0^{-1}(c_0)$ satisfies $f_0(y) = \inf(f_0; (-\infty, c_0]; g_0)$, then $0 \notin \partial g_0(y)$;

(A6) each sequence $\{y_k\}_{k=1}^{\infty} \subset \{z \in B^0(0, \bar{M}) :\ g_0(z) \in [a_0, b_0]\}$ which satisfies $\liminf_{k \to \infty} \Xi_{g_0}(y_k) = 0$ possesses a norm-convergent subsequence;

(A7) each sequence

$$\{y_k\}_{k=1}^{\infty} \subset \{z \in B^0(0, \bar{M}) :\ g_0(z) \in [c_0, b_0]\}$$

which satisfies $\liminf_{k \to \infty} \Xi_{g_0}(y_k) = 0$ possesses a norm-convergent subsequence.

Remark 3.2. If (A1) holds, then it follows from (3.7) and (3.10) that f_0 is bounded from below on X.

Remark 3.3. Assumption (A6) ((A7) respectively) means that g_0 satisfies the (P-S) condition on the set $g_0^{-1}([a_0, b_0]) \cap B^0(0, \bar{M})$ ($g_0^{-1}([c_0, b_0]) \cap B^0(0, \bar{M})$ respectively).

We will prove the following two results.

Theorem 3.4. *Suppose that (A1), (A2), (A4) and (A6) hold. Let M be a positive number. Then there exist positive numbers α, λ_0 such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

For each $\lambda \ge \lambda_0$, each $c \in R^1$ satisfying $|c - c_0| \le \alpha$, each pair of functions $f : X \to R^1$ and $g : X \to R^1$ which satisfy

$$f \in \mathcal{U}_\phi(f_0, \bar{M}, \alpha, M),\ g \in \mathcal{U}(g_0, \bar{M}, \alpha, \alpha)$$

and each $x \in X$ satisfying

$$f(x) + \lambda|g(x) - c| \le \inf\{f(z) + \lambda|g(z) - c| :\ z \in X\} + \delta$$

there exists $y \in g^{-1}(c)$ such that $||y - x|| \le \epsilon$ and

$$f(y) \le f(x) + \lambda|g(x) - c| \le \inf\{f(z) + \lambda|g(z) - c| :\ z \in X\} + \delta.$$

Theorem 3.5. *Suppose that (A1), (A3), (A5) and (A7) hold. Let M be a positive number. Then there exist positive numbers α, λ_0 such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

For each $\lambda \geq \lambda_0$, each $c \in R^1$ satisfying $|c - c_0| \leq \alpha$, each pair of functions $f : X \to R^1$ and $g : X \to R^1$ which satisfy

$$f \in \mathcal{U}_\phi(f_0, \bar{M}, \alpha, M), \ g \in \mathcal{U}(g_0, \bar{M}, \alpha, \alpha)$$

and each $x \in X$ which satisfies

$$f(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{f(z) + \lambda \max\{g(z) - c, 0\} : \ z \in X\} + \delta$$

there is $y \in g^{-1}((-\infty, c])$ such that $||y - x|| \leq \epsilon$ and

$$f(y) \leq f(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{f(z) + \lambda \max\{g(z) - c, 0\} : \ z \in X\} + \epsilon.$$

Theorems 3.4 and 3.5 will be proved in Section 3.3. They imply the following result.

Theorem 3.6. *1. Suppose that (A1), (A2), (A4) and (A6) hold. Let M be a positive number. Then there exist positive numbers α, λ_0 such that for each $f \in \mathcal{U}_\phi(f_0, \bar{M}, \alpha, M)$, each $g \in \mathcal{U}(g_0, \bar{M}, \alpha, \alpha)$, each $c \in [c_0 - \alpha, c_0 + \alpha]$, each $\lambda \geq \lambda_0$ and for each sequence $\{x_i\}_{i=1}^\infty \subset X$ which satisfies*

$$\lim_{i \to \infty} [f(x_i) + \lambda |g(x_i) - c|] = \inf\{f(z) + \lambda |g(z) - c| : \ z \in X\}$$

there exists a sequence $\{y_i\}_{i=1}^\infty \subset g^{-1}(c)$ such that

$$\lim_{i \to \infty} f(y_i) = \inf(f, c, g), \ \lim_{i \to \infty} ||y_i - x_i|| = 0.$$

2. Suppose that (A1), (A3), (A5) and (A7) hold. Let M be a positive number. Then there exist positive numbers α, λ_0 such that for each $f \in \mathcal{U}_\phi(f_0, \bar{M}, \alpha, M)$, each $g \in \mathcal{U}(g_0, \bar{M}, \alpha, \alpha)$, each $c \in [c_0 - \alpha, c_0 + \alpha]$, each $\lambda \geq \lambda_0$ and for each sequence $\{x_i\}_{i=1}^\infty \subset X$ which satisfies

$$\lim_{i \to \infty} [f(x_i) + \lambda \max\{g(x_i) - c, 0\}] = \inf\{f(z) + \lambda \max\{g(z) - c, 0\} : \ z \in X\}$$

there exists a sequence $\{y_i\}_{i=1}^\infty \subset g^{-1}((-\infty, c])$ such that

$$\lim_{i \to \infty} f(y_i) = \inf(f; (-\infty, c], g), \ \lim_{i \to \infty} ||y_i - x_i|| = 0.$$

Note that the results of this section have been obtained in [140].

3.2 Auxiliary results

In this section we use the notation and definitions introduced in Section 3.1.

Let $(Y, ||\cdot||)$ and $(Z, ||\cdot||)$ be Banach spaces, $A \subset Y$, $B \subset Z$. We say that $h : A \to B$ is an $\mathcal{L}$-mapping if for each $x \in A$ there is a positive number r such that the restriction $h : A \cap B(x, r) \to B$ is Lipschitzian.

Assume that $g : X \to R^1$ is a locally Lipschitz function. We use the following auxiliary result of [100].

Lemma 3.7. *Let $\delta > 0$ and let $A \subset X$ be a nonempty closed subset of X such that $\Xi_g(x) < -\delta$ for all $x \in A$. Then there exists an $\mathcal{L}$-mapping $V : X \to X$ such that*

$$Vx \in B(0, 2) \text{ for all } x \in X, \ g^0(x, Vx) \le 0 \text{ for all } x \in X,$$

$$g^0(x, Vx) \le -\delta \text{ for all } x \in A.$$

Lemma 3.8. *Let $x_0 \in X$, δ be a positive number and let $\Xi_g(x_0) < -\delta$. Then there exist a positive number r and an $\mathcal{L}$-mapping $V : X \to X$ such that*

$$Vx \in B(0, 2) \text{ for all } x \in X, \ g^0(x, Vx) \le 0 \text{ for all } x \in X,$$

$$g^0(x, Vx) \le -\delta \text{ for all } x \in B(x_0, r).$$

Proof: It follows from upper semicontinuity of the Clarke generalized directional derivative $g^0(\xi, \eta)$ with respect to ξ that there exists a positive number r such that $\Xi_g(x) < -\delta$ for all $x \in B(x_0, r)$. Now Lemma 3.8 follows from Lemma 3.7.

Proposition 3.9. *Let*

$$x_0 \in B^0(0, \bar{M}) \cap g_0^{-1}(c_0), \tag{3.11}$$

$$0 \notin \partial g_0(x_0) \tag{3.12}$$

and let $\epsilon_0 > 0$. Then there exists $\delta_0 \in (0, \epsilon_0)$ such that for each function $g : X \to R^1$ which satisfies

$$|g(z) - g_0(z)| \le \delta_0, \ z \in B^0(0, \bar{M}), \tag{3.13}$$

$$|(g - g_0)(z_1) - (g - g_0)(z_2)| \le \delta_0 ||z_1 - z_2|| \text{ for all } z_1, z_2 \in B^0(0, \bar{M}) \tag{3.14}$$

and each $c \in R^1$ satisfying

$$|c - c_0| \le \delta_0 \tag{3.15}$$

there exists $x \in B^0(0, \bar{M})$ such that

$$x \in B^0(x_0, \epsilon_0) \text{ and } g(x) = c. \tag{3.16}$$

Proof: It follows from (3.1) and (3.12) that

$$\Xi_{g_0}(x_0) < 0.$$

Choose a positive number δ such that

$$\Xi_{g_0}(x_0) < -\delta. \tag{3.17}$$

Lemma 3.8 implies that there exist a positive number r and an $\mathcal{L}$-mapping $V : X \to X$ such that

$$r < \epsilon_0,\ B(x_0, r) \subset B^0(0, \bar{M}), \tag{3.18}$$

$$Vx \in B(0, 2) \text{ for all } x \in X,\ g_0^0(x, Vx) \le 0 \text{ for all } x \in X, \tag{3.19}$$

$$g_0^0(x, Vx) \le -\delta \text{ for all } x \in B(x_0, r). \tag{3.20}$$

It is easy to see that there exist a positive number t_0 and a differentiable function $\phi : [-t_0, t_0] \to X$ such that

$$\phi'(t) = V\phi(t),\ t \in [-t_0, t_0], \tag{3.21}$$

$$\phi(0) = x_0, \tag{3.22}$$

$$\phi(t) \in B^0(x_0, r),\ t \in [-t_0, t_0]. \tag{3.23}$$

By the properties of the Clarke generalized directional derivative [21] for each $t_1, t_2 \in [-t_0, t_0]$ satisfying $t_1 < t_2$,

$$g_0(\phi(t_2)) - g_0(\phi(t_1)) = l(\phi'(s))(t_2 - t_1),$$

where $s \in [t_1, t_2]$ and $l \in \partial g_0(\phi(s))$, and by (3.20), (3.21), (3.23) and the definition of Clarke's generalized gradient

$$g_0(\phi(t_2)) - g_0(\phi(t_1)) = (t_2 - t_1)l(V(\phi(s)))$$

$$\le (t_2 - t_1)g_0^0(\phi(s), V(\phi(s))) \le -\delta(t_2 - t_1).$$

Therefore

$$g_0(\phi(t_2)) - g_0(\phi(t_1)) \le -\delta(t_2 - t_1) \text{ for each } t_1, t_2 \in [-t_0, t_0] \text{ such that } t_1 < t_2. \tag{3.24}$$

Fix a number $\delta_0 > 0$ such that

$$\delta_0 < \min\{\epsilon_0, \delta t_0/8\}. \tag{3.25}$$

Assume that $g : X \to R^1$ satisfies (3.14) and $c \in R^1$ satisfies (3.15). It follows from (3.18) and (3.23) that for all $t \in [-t_0, t_0]$,

$$\phi(t) \in B^0(x_0, r) \subset B^0(0, \bar{M}) \cap B^0(x_0, \epsilon_0). \tag{3.26}$$

By (3.11), (3.22) and (3.24),

$$g_0(\phi(t_0/2)) \le g_0(\phi(0)) - \delta t_0/2 = g_0(x_0) - \delta t_0/2 = c_0 - \delta t_0/2, \tag{3.27}$$

$$g_0(\phi(-t_0/2)) \ge g_0(\phi(0)) + \delta t_0/2 = g_0(x_0) + \delta t_0/2 = c_0 + \delta t_0/2. \tag{3.28}$$

In view of (3.14), (3.18) and (3.23),

$$|(g - g_0)(\phi(t_0/2))|,\ |(g - g_0)(\phi(-t_0/2))| \le \delta_0. \tag{3.29}$$

It follows from (3.29), (3.27), (3.15), (3.25) and (3.28) that

$$g(\phi(t_0/2)) \le g_0(\phi(t_0/2)) + \delta_0 \le c_0 - \delta t_0/2 + \delta_0 \le c - \delta t_0/2 + 2\delta_0 \le c - 2\delta_0, \tag{3.30}$$

$$g(\phi(-t_0/2)) \ge g_0(\phi(-t_0/2)) - \delta_0 = c_0 + \delta t_0/2 - \delta_0 \ge c + \delta_0/2 - 2\delta_0 \ge c + 2\delta_0.$$

By (3.30) there exists $s \in (-t_0/2, t_0 2)$ such that

$$g(\phi(s)) = c.$$

Relations (3.23) and (3.26) imply that

$$\phi(s) \in B^0(0, \bar{M}) \cap B^0(x_0, \epsilon_0).$$

Proposition 3.9 is proved.

3.3 Proof of Theorems 3.4 and 3.5

We prove Theorems 3.4 and 3.5 simultaneously. For each real number c and each function $g : X \to R^1$ put

$$A_{g,c} = g^{-1}(c) \text{ in the case of Theorem 3.4,} \tag{3.31}$$

$$A_{g,c} = g^{-1}((-\infty, c]) \text{ in the case of Theorem 3.5.}$$

For each positive number λ, each real number c and each pair of functions $f, g : X \to R^1$ define a function $\psi_{\lambda,c}^{(f,g)} : X \to R^1$ by

$$\psi_{\lambda,c}^{(f,g)}(z) = f(z) + \lambda|g(z) - c|,\ z \in X \tag{3.32}$$

in the case of Theorem 3.4 and by

$$\psi_{\lambda,c}^{(f,g)}(z) = f(z) + \lambda \max\{g(z) - c, 0\},\ z \in X \tag{3.33}$$

in the case of Theorem 3.5.

Let M be a positive number. We assume that Theorem 3.4 (Theorem 3.5 respectively) does not hold. Then for each integer $k \ge 1$ there exist

$$\epsilon_k \in (0,1),\ \lambda_k \geq k,\ c_k \in R^1$$

satisfying

$$|c_k - c_0| \leq k^{-1}, \tag{3.34}$$

a pair of functions $f_k, g_k : X \to R^1$ which satisfy

$$f_k \in \mathcal{U}_\phi(f_0, \bar{M}, k^{-1}, M),\ g_k \in \mathcal{U}(g_0, \bar{M}, k^{-1}, k^{-1}) \tag{3.35}$$

and $x_k \in X$ satisfying

$$\psi^{(f_k,g_k)}_{\lambda_k,c_k}(x_k) \leq \inf(\psi^{(f_k,g_k)}_{\lambda_k,c_k}) + \epsilon_k 2^{-1} k^{-2}, \tag{3.36}$$

$$\{y \in A_{g_k,c_k} \cap B(x_k, \epsilon_k) :\ \psi^{(f_k,g_k)}_{\lambda_k,c_k}(y) \leq \psi^{(f_k,g_k)}_{\lambda_k,c_k}(x_k)\} = \emptyset. \tag{3.37}$$

Set

$$\phi^{(k)} = \psi^{(f_k,g_k)}_{\lambda_k,c_k},\ k = 1, 2, \dots. \tag{3.38}$$

Let $k \geq 1$ be an integer. It follows from Ekeland's variational principle [37], (3.32), (3.33), (3.35), (3.36) and (3.38) that there exists $y_k \in X$ such that

$$\phi^{(k)}(y_k) \leq \phi^{(k)}(x_k), \tag{3.39}$$

$$||y_k - x_k|| \leq (2k)^{-1}\epsilon_k, \tag{3.40}$$

$$\phi^{(k)}(y_k) \leq \phi^{(k)}(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{3.41}$$

Relations (3.37), (3.38), (3.39) and (3.40) imply that

$$y_k \notin A_{g_k,c_k}. \tag{3.42}$$

In the case of Theorem 3.5 we obtain that

$$g_k(y_k) > c_k. \tag{3.43}$$

In the case of Theorem 3.4 we obtain that either $g_k(y_k) > c_k$ or $g_k(y_k) < c_k$. In the case of Theorem 3.4 extracting a subsequence, reindexing and replacing g_0 by $-g_0$, g_k by $-g_k$, c_0 by $-c_0$ and c_k by $-c_k$, we may assume without loss of generality that in both cases

$$g_k(y_k) > c_k \text{ for all integers } k \geq 1. \tag{3.44}$$

By (A1) there exists a positive number L_0 such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0||z_1 - z_2|| \text{ for all } z_1, z_2 \in B(0, \bar{M}). \tag{3.45}$$

Evidently, for all integers $k \geq 1$, $\phi^{(k)}$ is locally Lipschitz on $B(0, \bar{M})$.

We continue the proof with four steps.

Step 1. We show that $y_k \in B(0, \bar{M} - 4)$ for all sufficiently large natural numbers k. By (3.10), (A0), Proposition 3.9, (3.31), (3.34) and (3.35), there exists an integer $k_0 \geq 1$ such that for each natural number $k \geq k_0$ there exists

$$\theta_k \in B^0(0, \bar{M}) \cap A_{g_k c_k} \tag{3.46}$$

such that

$$\lim_{k \to \infty} ||\theta_k - \theta|| = 0. \tag{3.47}$$

By (3.10), (3.32), (3.33), (3.35), (3.36), (3.38), (3.39), (3.45) and (3.46) for each natural number $k \geq k_0$

$$f_k(y_k) \leq \phi^{(k)}(y_k) \leq \phi^{(k)}(x_k) \leq \inf(\phi^{(k)}) + 2^{-1}k^{-2}$$

$$\leq \phi^{(k)}(\theta_k) + 2^{-1}k^{-2} = f_k(\theta_k) + 2^{-1}k^{-2} = f_0(\theta_k) + [f_k(\theta_k) - f_0(\theta_k)] + 2^{-1}k^{-2}$$

$$\leq f_0(\theta_k) + k^{-1} + k^{-2} = f_0(\theta) + [f_0(\theta_k) - f_0(\theta)] + k^{-1} + k^{-2}$$

$$\leq f_0(\theta) + L_0||\theta_k - \theta|| + 2k^{-1}. \tag{3.48}$$

Since $\lim_{k \to \infty} ||\theta_k - \theta|| = 0$ it follows from (4.48) that there exists an integer $k_1 \geq k_0$ such that for all natural numbers $k \geq k_1$,

$$f_k(y_k) \leq f_0(\theta) + 2^{-1}.$$

Combined with (3.6), (3.10) and (3.35) this implies that

$$y_k \in B(0, \bar{M} - 4) \text{ for all natural numbers } k \geq k_1. \tag{3.49}$$

Step 2. We show that $\lim_{k \to \infty} \Xi_{g_0}(y_k) = 0$. Let $k \geq k_1$ be an integer. It follows from (3.35) and (3.49) that for each $z \in B(y_k, 3/2)$,

$$||[g_k(z) - g_0(z)] - [g_k(y_k) - g_0(y_k)]|| \leq k^{-1}||z - y_k||, \tag{3.50}$$

$$||[f_k(z) - f_0(z)] - [f_k(y_k) - f_0(y_k)]|| \leq M||z - y_k||. \tag{3.51}$$

In view of (3.44) there exists $r_k \in (0, 3/2]$ such that

$$g_k(z) > c_k \text{ for all } z \in B(y_k, r_k). \tag{3.52}$$

It follows from (3.52), (3.41), (3.38), (3.32) and (3.33) that for all $z \in B(y_k, r_k)$,

$$f_k(y_k) + \lambda_k(g_k(y_k) - c_k) \leq f_k(z) + \lambda_k(g_k(z) - c_k) + k^{-1}||z - y_k||. \tag{3.53}$$

Relations (3.50), (3.51), (3.52) and (3.53) imply that for all $z \in B(y_k, r_k)$,

$$f_0(y_k) + \lambda_k g_0(y_k) \leq f_k(y_k) + \lambda_k g_k(y_k) + [f_0(y_k) - f_k(y_k)] + \lambda_k[g_0(y_k) - g_k(y_k)]$$

$$\leq f_k(z) + \lambda_k g_k(z) + k^{-1}||z - y_k|| + [f_0(y_k) - f_k(y_k)] + \lambda_k[g_0(y_k) - g_k(y_k)]$$

$$\leq f_0(z) + \lambda_k g_0(z) + (f_k(z) - f_0(z))$$

$$+\lambda_k[g_k(z) - g_0(z)] + k^{-1}||z - y_k|| + [f_0(y_k) - f_k(y_k)] + \lambda_k[(g_0(y_k) - g_k(y_k)]$$

$$= f_0(z) + \lambda_k g_0(z) + k^{-1}||z - y_k||$$

$$+[(f_k - f_0)(z) - (f_k - f_0)(y_k)] + \lambda_k[(g_k - g_0)(z) - (g_k - g_0)(y_k)]$$
$$\le f_0(z) + \lambda_k g_0(z) + k^{-1}||z - y_k|| + M||z - y_k|| + \lambda_k k^{-1}||z - y_k||. \quad (3.54)$$

By (3.54) for all $z \in B(y_k, z_k)$

$$g_0(y_k) + \lambda_k^{-1} f_0(y_k) \le g_0(z) + \lambda_k^{-1} f_0(z) + \lambda_k^{-1}||z - y_k||(k^{-1} + M + k^{-1}).$$

By the inequality above and the properties of Clarke's generalized gradient (see Chapter 2, Sect. 2.3 of [21])

$$0 \in \partial g_0(y_k) + \lambda_k^{-1}\partial f_0(y_k) + \lambda_k^{-1}(2/k + M)B_*(0,1). \quad (3.55)$$

Relations (3.45) and (3.49) imply that for each integer $k \ge k_1$,

$$\partial f_0(y_k) \subset B_*(0, L_0).$$

Combined with (3.55) this implies that for each integer $k \ge k_1$

$$0 \in \partial g_0(y_k) + \lambda_k^{-1}(L_0 + 2/k + M)B_*(0,1)$$

and

$$\lim_{k\to\infty} \Xi_{g_0}(y_k) = 0. \quad (3.56)$$

Step 3. Let us show that $\lim_{k\to\infty} g_0(y_k) = c_0$ and that $\{y_j\}_{j=1}^{\infty}$ possesses a convergent subsequence. It follows from (3.49), (3.35), (3.44), (3.38), (3.32), (3.33), (3.39), (3.36) and (3.46) that for all integers $k \ge k_1$,

$$-M + \lambda_k(g_k(y_k) - c_k) + f_0(y_k) \le f_k(y_k) + \lambda_k(g_k(y_k) - c_k) = \phi^{(k)}(y_k) \le \phi^{(k)}(x_k)$$
$$\le \inf(\phi^{(k)}) + 1 \le \phi^{(k)}(\theta_k) + 1 \le f_k(\theta_k) + 1 = f_0(\theta_k) + 1 + f_k(\theta_k) - f_0(\theta_k)$$
$$\le f_0(\theta_k) + 2$$

and

$$0 < g_k(y_k) - c_k \le \lambda_k^{-1}[\sup\{f_0(z) : \ z \in B(0, \bar{M})\} - \inf(f_0) + 2 + M].$$

Hence $\lim_{k\to\infty}(g_k(y_k) - c_k) = 0$. Combined with (3.34), (3.35) and (3.49) this implies that

$$\lim_{k\to\infty} g_0(y_k) = c_0. \quad (3.57)$$

It follows from (3.56), (3.57), (3.49), (A6), (A7), (3.8) and (3.43) that there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that $\{y_{k_j}\}_{j=1}^{\infty}$ converges in the norm topology of X to $y_* \in X$. By (3.57)

$$g_0(y_*) = c_0. \quad (3.58)$$

Step 4. Let us show that $f_0(y_*) = \inf(f_0; A_{g_0,c_0})$.

By (3.56) and and upper semicontinuity of the Clarke generalized directional derivative $g^0(\xi, \eta)$ with respect to ξ [21], $\Xi_{g_0}(y_*) = 0$ and

$$0 \in \partial g_0(y_*). \tag{3.59}$$

It follows from (3.45) and (3.49) that

$$f_0(y_*) = \lim_{j \to \infty} f_0(y_{k_j}). \tag{3.60}$$

Let $\Delta \in (0, 1)$. (A2) and (A3) imply that there exists

$$x(\Delta) \in A_{g_0, c_0} \tag{3.61}$$

such that

$$f_0(x(\Delta)) \le \inf(f_0; A(g_0, c_0)) + \Delta, \tag{3.62}$$

$$0 \notin \partial g_0(x(\Delta)) \text{ if } g(x(\Delta)) = c_0. \tag{3.63}$$

It follows from (3.61) and (A0) that

$$f_0(x(\Delta)) \le f_0(\theta) + 1. \tag{3.64}$$

By (3.64), (3.7) and (3.10),

$$x(\Delta) \in B^0(0, \bar{M} - 4). \tag{3.65}$$

In view of (3.65), (3.61), (3.63), (3.34), (3.35) and Proposition 3.9 there exists an integer $k_2 \ge k_1$ such that for each integer $k \ge k_2$ there exists

$$x_k(\Delta) \in B^0(0, \bar{M}) \cap A_{g_k, c_k} \tag{3.66}$$

such that

$$\lim_{k \to \infty} ||x_k(\Delta) - x(\Delta)|| = 0. \tag{3.67}$$

It follows from (3.49), (3.35), (3.38), (3.32), (3.33), (3.39), (3.36), (3.66), (3.49) and (3.45) that for all natural numbers $k \ge k_2$,

$$f_0(y_k) - k^{-1} \le f_k(y_k) \le \phi^{(k)}(y_k) \le \phi^{(k)}(x_k) \le \inf(\phi^{(k)}) + 2^{-1}k^{-2}$$
$$\le \phi^{(k)}(x_k(\Delta)) + 2^{-1}k^{-2}$$
$$= f_k(x_k(\Delta)) + 2^{-1}k^{-2} = f_0(x_k(\Delta)) + [f_k(x_k(\Delta)) - f_0(x_k(\Delta))] + 2^{-1}k^{-2}$$
$$\le f_0(x_k(\Delta)) + k^{-1} + 2^{-1}k^{-2}$$
$$= f_0(x(\Delta)) + [f_0(x_k(\Delta)) - f_0(x(\Delta))] + 1/k + (2k^2)^{-1}$$
$$\le f_0(x(\Delta)) + L_0||x_k(\Delta) - x(\Delta)|| + 2k^{-1}$$

and together with (3.62) this implies that

$$f_0(y_k) \le 3k^{-1} + L_0||x_k(\Delta) - x(\Delta)|| + \inf(f_0; A_{g_0, c_0}) + \Delta.$$

Combined with (3.67) this implies that

$$\limsup_{k\to\infty} f_0(y_k) \le \inf(f_0; A_{g_0,c_0}) + \Delta.$$

Since Δ is an arbitrary number from the interval $(0,1)$ we conclude that

$$\limsup_{k\to\infty} f_0(y_k) \le \inf(f_0; A_{g_0,c_0}).$$

Together with (3.60) this implies that

$$f_0(y_*) = \lim_{j\to\infty} f_0(y_{k_j}) \le \inf(f_0; A_{g_0,c_0})$$

and by (3.58),

$$f_0(y_*) = \inf(f_0; A_{g_0,c_0}).$$

The relation above, (3.58) and (3.59) contradict (A4) in the case of Theorem 3.4 and (A5) in the case of Theorem 3.5. The contradiction we have reached proves Theorems 3.4 and 3.5.

3.4 Problems with convex constraint functions

In this section we study a large class of inequality-constrained minimization problems

$$\text{minimize } f(x) \text{ subject to } x \in A \tag{P}$$

where

$$A = \{x \in X : \ g_i(x) \le c_i \text{ for } i = 1, \dots, n\}.$$

Here X is a Banach space, c_i, $i = 1, \dots, n$ are real numbers, and the constraint functions g_i, $i = 1, \dots, n$ and the objective function f are lower semicontinuous and satisfy certain assumptions.

We associate with the inequality-constrained minimization problem above the corresponding family of unconstrained minimization problems

$$\text{minimize } f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \text{ subject to } z \in X$$

where $\gamma > 0$ is a penalty. This problem is studied in Sections 2.4–2.6 where we establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem. In this chapter we show that the exact penalty property is stable under perturbations of cost functions, constraint functions and the right-hand side of constraints. More precisely, we consider a family of constrained minimization problems of type (P) with an objective function close to f and with constraint functions close to $g_1, \dots, g_n$ in a certain natural sense. We show that all the constrained minimization problems belonging to this family possess the exact penalty property with the same penalty coefficient which depends only on f, $g_1, \dots, g_n$, $c_1, \dots, c_n$. It should be mentioned that for a general natural number n we suppose that constraint functions of any problem from our family are convex while for $n = 1$ constraint functions are not assumed to be necessarily convex.

In this section we use the convention that $\lambda \cdot \infty = \infty$ for all $\lambda \in (0, \infty)$, $\lambda + \infty = \infty$ and $\max\{\lambda, \infty\} = \infty$ for any real number λ and that supremum over empty set is $-\infty$.

Let $(X, ||\cdot||)$ be a Banach space. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||x - y|| \le r\}, \ B^o(x, r) = \{y \in X : \ ||x - y|| < r\}.$$

For each function $f : X \to R^1 \cup \{\infty\}$ and each nonempty set $A \subset X$ put

$$\mathrm{dom}(f) = \{x \in X : \ f(x) < \infty\}, \ \inf(f) = \inf\{f(z) : \ z \in X\},$$

$$\inf(f; A) = \inf\{f(z) : \ z \in A\}.$$

For each $x \in X$ and each $B \subset X$ set

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}. \tag{3.68}$$

Let $n \ge 1$ be an integer. For each $\kappa \in (0, 1)$ denote by Ω_κ the set of all $\gamma = (\gamma_1, \dots, \gamma_n) \in R^n$ such that

$$\kappa \le \min\{\gamma_i : \ i = 1, \dots, n\} \text{ and } \max\{\gamma_i : \ i = 1, \dots, n\} = 1. \tag{3.69}$$

Assume that $\phi : X \to R^1$ satisfies

$$\lim_{||x|| \to \infty} \phi(x) = \infty \text{ and } \inf(\phi) > -\infty. \tag{3.70}$$

We consider problems of type (P) with objective functions f which satisfy $f(x) \ge \phi(x)$ for all $x \in X$.

Let $\bar{c}_0 \in R^1$, $\bar{c} = (\bar{c}_1, \dots, \bar{c}_n) \in R^n$ and let $f_i : X \to R^1 \cup \{\infty\}$, $i = 1, \dots, n$ be convex lower semicontinuous functions and put

$$A = \{x \in X : \ f_i(x) \le \bar{c}_i \text{ for all } i = 1, \dots, n\}. \tag{3.71}$$

We consider a family of problems of type (P) with constraint functions close to $f_1, \dots, f_n$ in a certain natural sense. We assume that $\theta \in X$ satisfies

$$f_i(\theta) < \bar{c}_i, \ i = 1, \dots, n. \tag{3.72}$$

Let us now describe the collection of objective functions (denoted by $\mathcal{A}$) which corresponds to our family of constrained minimization problems.

It follows from (3.70) there exists a positive number M_0 such that

$$||\theta|| + 4 < M_0,$$

$$\phi(z) > \bar{c}_0 + 4 \text{ for all } z \in X \text{ satisfying } ||z|| \ge M_0 - 4. \tag{3.73}$$

Let X_0 be a nonempty convex subset of $B^o(0, M_0)$ such that $\theta \in X_0$.

Assume that a function $h : B^o(0, M_0) \times X_0 \to R^1 \cup \{\infty\}$ satisfies the following assumptions:

(A1) $h(z, y)$ is finite for all $y, z \in X_0$ and $h(y, y) = 0$ for each $y \in X_0$;

(A2) for each $y \in X_0$ the function $h(\cdot, y) : B^o(0, M_0) \to R^1 \cup \{\infty\}$ is convex;

(A3) for each $z \in X_0$

$$\sup\{h(z, y) : \ y \in X_0\} < \infty.$$

Let M_1 be a positive number. We denote by $\mathcal{A}$ a set of all lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ such that

$$f(x) \geq \phi(x) \text{ for all } x \in X, \tag{3.74}$$

$$f(\theta) \leq \bar{c}_0,$$

$$B^o(0, M_0) \cap \mathrm{dom}(f) \subset X_0$$

and that the following assumption holds:

(A4) for each $y \in \mathrm{dom}(f) \cap B^o(0, M_0)$ there exists a neighborhood V of y in X such that $V \subset B^o(0, M_0)$ and that

$$f(z) - f(y) \leq M_1 h(z, y) \text{ for all } z \in V.$$

Below we consider two examples of the function h and the set $\mathcal{A}$.

Example 3.10. Assume that a function $f_0 : X \to R^1$ is Lipschitz on bounded subsets of X, $f_0(\theta) \leq \bar{c}_0$ and that $f_0(x) \geq \phi(x)$ for all $x \in X$. Then there exists a positive number L_0 such that

$$|f_0(z_1) - f_0(z_2)| \leq L_0||z_1 - z_2|| \text{ for all } z_1, z_2 \in B(0, M_0).$$

Let L_1 be a positive number and put

$$M_1 = L_0 + L_1, \ h(z, y) = ||z - y||, \ z, y \in X,$$

$$X_0 = B^o(0, M_0).$$

Clearly, the set $\mathcal{A}$ contains all lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ such that $f(x) \geq \phi(x)$ for all $x \in X$, $f(\theta) \leq \bar{c}_0$, $f(z)$ is finite for all $z \in B^o(0, M_0)$ and that

$$|(f - f_0)(z_1) - (f - f_0)(z_2)| \leq L_1||z_1 - z_2|| \text{ for all } z_1, z_2 \in B^o(M_0).$$

Example 3.11. Assume that $f_0 : X \to R^1 \cup \{\infty\}$ is a lower semicontinuous convex function such that $f_0(x) \geq \phi(x)$ for all $x \in X$ and $f_0(\theta) \leq \bar{c}_0$. Let L_1 be a positive number and put

$$X_0 = \mathrm{dom}(f_0) \cap B^o(0, M_0).$$

It is easy to see that $\theta \in X_0$. Set $M_1 = 1$.

For each $z \in B^o(0, M_0)$ and each $y \in X_0$ define

$$h(z, y) = \sup\{\xi(z) - \xi(y) : \ \xi \in \mathcal{B}\} + L_1||z - y||, \tag{3.75}$$

where $\mathcal{B}$ is the set of all convex functions $\xi : B^o(0, M_0) \to R^1 \cup \{\infty\}$ such that

$$|\xi(v) - f_0(v)| \le L_1 \text{ for all } v \in \ \mathrm{dom}(f_0) \cap B^o(0, M_0).$$

It is not difficult to see that the function $h(\cdot, \cdot)$ is well defined and that the assumptions (A1), (A2) and (A3) hold.

Assume that $f : X \to R^1 \cup \{\infty\}$ is a lower semicontinuous function such that

$$f(\theta) \le \bar{c}_0, \ f(x) \ge \phi(x) \text{ for all } x \in X, \tag{3.76}$$

$$\mathrm{dom}(f) \cap B^o(0, M_0) = \mathrm{dom}(f_0) \cap B^o(0, M_0)$$

and that there exists a convex function $g : B^o(0, M_0) \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(f) \cap B^0(0, M_0) = \mathrm{dom}(g) \cap B^o(0, M_0), \tag{3.77}$$

$$|f_0(z) - g(z)| \le L_1 \text{ for all } z \in \ \mathrm{dom}(f_0) \cap B^o(0, M_0), \tag{3.78}$$

$$|(f - g)(z_1) - (f - g)(z_2)| \le L||z_1 - z_2|| \text{ for all } z_1, z_2 \in \mathrm{dom}(f_0) \cap B^o(0, M_0).$$

We show that $f \in \mathcal{A}$. In order to meet this goal it is sufficient to show that (A4) holds.

Let $y \in B^o(0, M_0) \cap$ dom(f) and $z \in B^o(0, M_0)$. It is easy to see that if $f(z) = \infty$, then $f_0(z) = \infty$ and

$$h(z, y) \ge f_0(z) - f_0(y) = \infty = f(z) - f(y).$$

If $f(z) < \infty$, then by (3.76) and (3.77) $f_0(z) < \infty$, $g(z) < \infty$, $f_0(y) < \infty$ and $g(y) < \infty$. Combined with (3.75)–(3.78) this implies that

$$f(z) - f(y) = g(z) - g(y) + [(f - g)(z) - (f - g)(y)]$$

$$\le g(z) - g(y) + L_1||z - y|| \le h(z, y).$$

Therefore $f(z) - f(y) \le h(z, y)$ in both cases, (A4) holds and $f \in \mathcal{A}$.

Let us now describe the collections of constraint functions which correspond to our family of constrained minimization problems.

For each $i \in \{1, \ldots, n\}$ and each positive number ϵ denote by $\mathcal{V}(i, \epsilon)$ the set of all convex lower semicontinuous functions $g : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(g) \cap B^o(0, M_0) = \mathrm{dom}(f_i) \cap B^o(0, M_0), \tag{3.79}$$

$$|f_i(x) - g(x)| \le \epsilon \text{ for all } x \in \mathrm{dom}(f_i) \cap B^o(0, M_0). \tag{3.80}$$

We prove the following result.

Theorem 3.12. *Let $\kappa \in (0,1)$. Then there exist a positive number Λ_0 and $\Delta_0 \geq 1$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ for which the following assertion holds:*

If $g_0 \in \mathcal{A}$, $g_i \in \mathcal{V}(i, \Delta_0^{-1})$, $i = 1, \dots, n$, $\gamma \in \Omega_\kappa$, $\lambda \geq \Lambda_0$, $c = (c_1, \dots, c_n) \in R^n$ satisfies

$$|\bar{c}_i - c_i| \leq \Delta_0^{-1},\ i = 1, \dots, n$$

and if $x \in X$ satisfies

$$g_0(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}$$

$$\leq \inf\{g_0(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} :\ z \in X\} + \delta,$$

then there is $y \in X$ such that

$$||y - x|| \leq \epsilon,\ g_i(y) \leq c_i,\ i = 1, \dots, n,$$

$$g_0(y) \leq \inf\{g_0(z) :\ z \in X \text{ and } g_i(z) \leq c_i,\ i = 1, \dots, n\} + \epsilon.$$

Theorem 3.12 implies the following result.

Corollary 3.13. *Let $\kappa \in (0,1)$ and let $\Lambda_0 > 0$ and $\Delta_0 \geq 1$ be as guaranteed by Theorem 3.12. Then for each $g_0 \in \mathcal{A}$, each $g_i \in \mathcal{V}(i, \Delta_0^{-1})$, $i = 1, \dots, n$, each $\gamma \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$, each $c = (c_1, \dots, c_n) \in R^n$ which satisfies $|\bar{c}_i - c_i| \leq \Delta_0^{-1}$, $i = 1, \dots, n$ and each sequence $\{x_i\}_{i=1}^\infty \subset X$ which satisfies*

$$\lim_{j \to \infty} [g_0(x_j) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x_j) - c_i, 0\}]$$

$$= \inf\{g_0(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} :\ z \in X\}$$

there is a sequence $\{y_i\}_{i=1}^\infty \subset \{z \in X :\ g_i(z) \leq c_i,\ i = 1, \dots, n\}$ such that

$$\lim_{j \to \infty} g_0(y_j) = \inf\{g_0(z) :\ z \in X \text{ and } g_i(z) \leq c_i,\ i = 1, \dots, n\},$$

$$\lim_{i \to \infty} ||x_i - y_i|| = 0.$$

The results of this section have been obtained in [143].

3.5 Proof of Theorem 3.12

We show that there is $\Lambda_0 \geq 1$ such that the following property holds:

(P1) For each $\epsilon \in (0,1)$ there is $\delta \in (0,\epsilon)$ such that for each $g_0 \in \mathcal{A}$, each $g_i \in \mathcal{V}(i, \Lambda_0^{-1})$, $i = 1, \dots, n$, each $c = (c_1, \dots, c_n) \in R^n$ satisfying

$$|\bar{c}_i - c_i| \leq \Lambda_0^{-1},\ i = 1, \dots, n,$$

each $\gamma \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$, each $x \in X$ which satisfies

$$g_0(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}$$

$$\leq \inf\{g_0(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} :\ z \in X\} + \delta$$

there exists $y \in B(x,\epsilon)$ such that

$$g_i(y_i) \leq c_i \text{ for all } i = 1, \dots, n,$$

$$g_0(y) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(y) - c_i, 0\} \leq g_0(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}.$$

Evidently, Theorem 3.12 easily follows from the property (P1).

Assume that there is no $\Lambda_0 \geq 1$ such that (P1) holds. Then for each integer $k \geq 1$ there exist

$$\epsilon_k \in (0,1),\ g_0^{(k)} \in \mathcal{A},\ g_i^{(k)} \in \mathcal{V}(i, k^{-1}),\ i = 1, \dots, n, \tag{3.81}$$

$c^{(k)} = (c_1^{(k)}, \dots, c_n^{(k)}) \in R^n$ satisfying

$$|c_i^{(k)} - \bar{c}_i| \leq 1/k,\ i = 1, \dots, n, \tag{3.82}$$

$$\gamma^{(k)} = (\gamma_1^{(k)}, \dots, \gamma_n^{(k)}) \in \Omega_\kappa,\ \lambda_k \geq k \tag{3.83}$$

and $x_k \in X$ which satisfies

$$g_0^{(k)}(x_k) + \sum_{i=1}^{n} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(x_k) - c_i^{(k)}, 0\} \tag{3.84}$$

$$\leq \inf\{g_0^{(k)}(z) + \lambda_k \sum_{i=1}^{n} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\} :\ z \in X\} + 2^{-1}\epsilon_k k^{-2}$$

and

$$\{y \in B(x_k, \epsilon_k) :\ g_i^{(k)}(y) \leq c_i \text{ for all } i = 1, \dots, n \text{ and}$$

$$g_0^{(k)}(y) + \sum_{i=1}^{n} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(y) - c_i^{(k)}, 0\} \tag{3.85}$$

$$\leq g_0^{(k)}(x_k) + \sum_{i=1}^{n} \lambda_k \gamma_i^{(k)} \max\{g_i(x_k) - c_i^{(k)}, 0\}\} = \emptyset.$$

For any natural number k put

$$\psi_k(z) = g_0^{(k)}(z) + \sum_{i=1}^{n} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\},\ z \in X. \tag{3.86}$$

It is easy to see that for any integer $k \geq 1$ the function ψ_k is lower semicontinuous and

$$\psi_k(z) \geq \phi(z) \text{ for all } z \in X. \tag{3.87}$$

Let $k \geq 1$ be an integer. By (3.84), (3.86) and Ekeland's variational principle [37], there exists $y_k \in X$ such that

$$\psi_k(y_k) \leq \psi_k(x_k), \tag{3.88}$$

$$||y_k - x_k|| \leq (2k)^{-1}\epsilon_k, \tag{3.89}$$

$$\psi_k(y_k) \leq \psi_k(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{3.90}$$

It follows from (3.85), (3.86), (3.88) and (3.89) that for all integers $k \geq 1$

$$y_k \notin \{z \in X :\ g_i^{(k)}(z) \leq c_i^{(k)} \text{ for all } i = 1, \dots, n\}. \tag{3.91}$$

For each integer $k \geq 1$ put

$$I_{1k} = \{i \in \{1, \dots, n\} :\ g_i^{(k)}(y_k) > c_i^{(k)}\}, \tag{3.92}$$

$$I_{2k} = \{i \in \{1, \dots, n\} :\ g_i^{(k)}(y_k) = c_i^{(k)}\},$$

$$I_{3k} = \{i \in \{1, \dots, n\} :\ g_i^{(k)}(y_k) < c_i^{(k)}\}.$$

By (3.91) and (3.92),

$$I_{1k} \neq \emptyset \text{ for all natural numbers } k. \tag{3.93}$$

Extracting a subsequence and reindexing we may assume without loss of generality that

$$I_{1k} = I_{11},\ I_{2k} = I_{21},\ I_{3k} = I_{31} \text{ for all integers } k \geq 1. \tag{3.94}$$

Relation (3.72) implies that there exists an integer $k_0 \geq 1$ such that

$$8k_0^{-1} < \min\{\bar{c}_i - f_i(\theta) :\ i = 1, \dots, n\}. \tag{3.95}$$

Assume that a natural number $k \geq k_0$. It follows from (3.81), (3.82), (3.72), (3.73) and (3.95) that for all $i = 1, \dots, n$

$$c_i^{(k)} - g_i^{(k)}(\theta) \geq -k^{-1} + \bar{c}_i - f_i(\theta) - k^{-1} \geq 8k_0^{-1} - 3k^{-1} > 0. \tag{3.96}$$

It follows from (3.81), the definition of $\mathcal{A}$, (3.96), (3.86), (3.84), (3.81), (3.88) and (3.92)–(3.94) that

$$\begin{aligned}
\bar{c}_0 \geq g_0^{(k)}(\theta) \geq \inf\{g_0^{(k)}(z) :& \\
z \in X \text{ and } g_i^{(k)}(z) \leq c_i^{(k)} &\text{ for all } i = 1, \dots, n\} \\
= \inf\{\psi_k(z) :\ z \in X \text{ and } g_i^{(k)}(z) &\leq c_i^{(k)} \text{ for all } i = 1, \dots, n\} \\
\geq \inf(\psi_k) \geq \psi_k(x_k) - 1 &\geq \psi_k(y_k) - 1 \\
= g_0^{(k)}(y_k) + \sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)}&(g_i^{(k)}(y_k) - c_i^{(k)}) - 1.
\end{aligned} \tag{3.97}$$

In view of (3.92)–(3.94) and (3.97),

$$g_0^{(k)}(y_k) \leq \bar{c}_0 + 1 \text{ for all integers } k \geq k_0. \tag{3.98}$$

Combined with (3.81) and (3.74) this implies that

$$\phi(y_k) \leq \bar{c}_0 + 1 \text{ for all integers } k \geq k_0. \tag{3.99}$$

By (3.99) and (3.73),

$$y_k \in B(0, M_0 - 4) \text{ for all integers } k \geq k_0. \tag{3.100}$$

Let $k \geq k_0$ be a natural number. (A4), (3.81), (3.98) and (3.100) imply that there exists a neighborhood V_k of y_k in X such that

$$V_k \subset B^o(0, M_0),$$

$$g_0^{(k)}(z) - g_0^{(k)}(y_k) \leq M_1 h(z, y_k) \text{ for all } z \in V_k. \tag{3.101}$$

It follows from (3.97), (3.92)–(3.94), (3.83), (3.81), (3.74), (3.70) and (3.69) that for each $i \in I_{11}$ and each natural number $k \geq k_0$,

$$\begin{aligned}
0 < g_i^{(k)}(y_k) - c_i^{(k)} &\leq [1 + \bar{c}_0 - \inf(g_0^{(k)})](\gamma_i^{k)})^{-1} k^{-1} \\
&\leq [1 + \bar{c}_0 - \inf(\phi)] k^{-1} \kappa^{-1}.
\end{aligned} \tag{3.102}$$

Assume that $k \geq k_0$ is a natural number. Since the functions $g_i^{(k)}$, $i = 1, \dots, n$ are lower semicontinuous it follows from (3.92)–(3.94) that there exists $r_k \in (0, 1)$ such that for each $y \in B(y_k, r_k)$

$$g_i^{(k)}(y) > c_i^{(k)} \text{ for each } i \in I_{11}. \tag{3.103}$$

By (3.86), (3.92)–(3.94), (3.103), (3.88), (3.84), (3.97), (3.98) and (3.90) for each $z \in B(y_k, r_k) \cap \mathrm{dom}(g_0^{(k)})$,

$$\sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} (g_i^{(k)}(z) - c_i^{(k)}) + \sum_{i \in I_{21} \cup I_{31}} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\}$$

$$- \sum_{i \in I_{11}} \lambda_k \gamma_i^{(k)} (g_i^{(k)}(y_k) - c_i^{(k)}) - \sum_{i \in I_{21} \cup I_{31}} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}$$

$$= \psi_k(z) - \psi_k(y_k) - g_0^{(k)}(z) + g_0^{(k)}(y_k) \geq -k^{-1}||z - y_k|| + g_0^{(k)}(y_k) - g_0^{(k)}(z).$$

Therefore for each $z \in B(y_k, r_k)$

$$\sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\}$$

$$- \sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(y_k) - \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}$$

$$+\lambda_k^{-1}(g_0^{(k)}(z) - g_0^{(k)}(y_k)) \geq -\lambda_k^{-1} k^{-1} ||y_k - z||. \tag{3.104}$$

(3.104) and (3.101) imply that for each $z \in B(y_k, r_k) \cap V_k$

$$\sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\}$$

$$+\lambda_k^{-1} M_1 h(z, y_k) + \lambda_k^{-1} k^{-1} ||y_k - z||$$

$$\geq \sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(y_k) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}. \tag{3.105}$$

In view of (3.81) the functions $g_i^{(k)}$, $i = 1, \ldots, n$ are convex. By (A2), (3.98), (3.81), (3.100) and (3.74) the function

$$\sum_{i \in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i \in I_{21} \cup I_{31}} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\}$$

$$+\lambda_k^{-1} M_1 h(z, y_k) + \lambda_k^{-1} k^{-1} ||z - y_k||, \ z \in B^o(0, M_0)$$

is convex. Together with (A1) this implies that (3.105) holds for all $z \in B^o(0, M_0)$.

Extracting a subsequence and reindexing we may assume without loss of generality that for each $i \in \{1, \ldots, n\}$ there exists

$$\gamma_i = \lim_{k \to \infty} \gamma_i^{(k)} \in [0, 1]. \tag{3.106}$$

It is easy to see that

$$\gamma = (\gamma_1, \ldots, \gamma_n) \in \Omega_\kappa.$$

Assume that

$$z \in B^0(0, M_0) \cap [\cap_{i=1}^n \operatorname{dom}(f_i)] \cap X_0. \tag{3.107}$$

In view of (3.83) and (3.100),

$$\lim_{k\to\infty} \lambda_k^{-1} k^{-1} ||z - y_k|| = 0. \tag{3.108}$$

It follows from (3.83), (A3), (3.107), (3.100), (3.98), (3.81) and (3.74) that

$$\lim_{k\to\infty} \lambda_k^{-1} M_1 h(z, y_k) = 0. \tag{3.109}$$

It follows from (3.107), (3.81), (3.92), (3.80), (3.82), (3.108), (3.109), (3.105) which holds for z, (3.92), (3.94), (3.106) and (3.102) that

$$\begin{aligned}
&\sum_{i\in I_{11}} \gamma_i f_i(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i \max\{f_i(z) - \bar{c}_i, 0\} \\
&= \lim_{k\to\infty} [\sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\} \\
&\quad + \lambda_k^{-1} k^{-1} ||z - y_k|| + \lambda_k^{-1} M_1 h(z, y_k)] \\
&\ge \limsup_{k\to\infty} \sum_{i\in I_{11}} \gamma_i^{(k)} g_i^{(k)}(y_k) = \sum_{i\in I_{11}} \lim_{k\to\infty} (\gamma_i^{(k)} g_i^{(k)}(y_k)) = \sum_{i\in I_{11}} \gamma_i \bar{c}_i.
\end{aligned}$$

Hence we have shown that for each $z \in X$ satisfying (3.107)

$$\sum_{i\in I_{11}} \gamma_i f_i(z) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i \max\{f_i(z) - \bar{c}_i, 0\} \ge \sum_{i\in I_{11}} \gamma_i \bar{c}_i. \tag{3.110}$$

It follows from (3.110), (3.72), (3.73) the inclusions $\theta \in X_0$ and $\gamma \in \Omega_\kappa$ and (3.92)–(3.94) that

$$\begin{aligned}
\sum_{i\in I_{11}} \gamma_i \bar{c}_i &\le \sum_{i\in I_{11}} \gamma_i f_i(\theta) + \sum_{i\in I_{21}\cup I_{31}} \gamma_i \max\{f_i(\theta) - \bar{c}_i, 0\} \\
&= \sum_{i\in I_{11}} \gamma_i f_i(\theta) < \sum_{i\in I_{11}} \gamma_i \bar{c}_i.
\end{aligned}$$

The contradiction we have reached proves that there exists $\Lambda_0 \ge 1$ such that the property (P1) holds. Theorem 3.12 is proved.

3.6 An extension of Theorem 3.12 for problems with one constraint function

In this section we assume that $n = 1$ and use the notation and definitions from Section 3.4. In this case

$$\bar{c} \in R^1. \tag{3.111}$$

Put

$$f = f_1. \tag{3.112}$$

In this case we also have that

$$A = \{x \in X : f(x) \le \bar{c}\}, \; f(\theta) < \bar{c}. \tag{3.113}$$

For each positive number ϵ denote by $\mathcal{U}(\epsilon)$ the set of all lower semicontinuous functions $g : X \to R^1 \cup \{\infty\}$ for which there exists a convex function $h : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(g) \cap B^o(0, M_0) = \mathrm{dom}(f) \cap B^o(0, M_0) = \mathrm{dom}(h) \cap B^o(0, M_0), \tag{3.114}$$

$$|f(x) - h(x)| \le \epsilon \text{ for all } x \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{3.115}$$

$$|(h - g)(z_1) - (h - g)(z_2)| \le \epsilon ||z_1 - z_2|| \text{ for all } z_1, z_2 \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{3.116}$$

$$|h(z) - g(z)| \le \epsilon \text{ for all } z \in \mathrm{dom}(f) \cap B^o(0, M_0). \tag{3.117}$$

We will prove the following result.

Theorem 3.14. *There exist a positive number Λ_0 and $\Delta_0 \ge 1$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ for which the following assertion holds:*

If $g_0 \in \mathcal{A}$, $g \in \mathcal{U}(\Delta_0^{-1})$, $\lambda \ge \Lambda_0$, $c \in R^1$ satisfies $|\bar{c} - c| \le \Delta_0^{-1}$ and if $x \in X$ satisfies

$$g_0(x) + \lambda \max\{g(x) - c, 0\} \le \inf\{g_0(z) + \lambda \max\{g(z) - c, 0\} : z \in X\} + \delta,$$

then there exists $y \in X$ such that

$$||y - x|| \le \epsilon, \; g(y) \le c,$$

$$g_0(y) \le \inf\{g_0(z) : z \in X \text{ and } g(z) \le c\} + \epsilon.$$

Since the set $\mathcal{U}(\epsilon)$ is larger than the set $\mathcal{V}(1, \epsilon)$ Theorem 3.14 is a generalization of Theorem 3.12 in the case $n = 1$.

Theorem 3.14 implies the following result.

Corollary 3.15. *Let Λ_0 be a positive number and $\Delta_0 \ge 1$ be as guaranteed by Theorem 3.14. Then for each $g_0 \in \mathcal{A}$, each $g \in \mathcal{U}(\Delta_0^{-1})$, each $\lambda \ge \Lambda_0$, each $c \in R^1$ which satisfies $|\bar{c} - c| \le \Delta_0^{-1}$, and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies*

$$\lim_{j \to \infty} [g_0(x_j) + \lambda \max\{g(x_j) - c, 0\}] = \inf\{g_0(z) + \lambda \max\{g(z) - c, 0\} : z \in X\}$$

there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset \{z \in X : g(z) \le c\}$ such that

$$\lim_{j \to \infty} g_0(y_j) = \inf\{g_0(z) : z \in X \text{ and } g(z) \le c\} \text{ and } \lim_{i \to \infty} ||x_i - y_i|| = 0.$$

The results of this section have been obtained in [143].

3.7 Proof of Theorem 3.14

We show that there is $\Lambda_0 \geq 1$ such that the following property holds:

(P2) For each $\epsilon \in (0,1)$ there is $\delta \in (0,\epsilon)$ such that for each $g_0 \in \mathcal{A}$, each $g \in \mathcal{U}(\Lambda_0^{-1})$, each $c \in R^1$ satisfying $|\bar{c}-c| \leq \Lambda_0^{-1}$, each $\lambda \geq \Lambda_0$ and each $x \in X$ which satisfies

$$g_0(x) + \lambda \max\{g(x) - c, 0\} \leq \inf\{g_0(z) + \lambda \max\{g(z) - c, 0\} :\ z \in X\} + \delta \tag{3.118}$$

there exists $y \in X$ such that

$$||y - x|| \leq \epsilon,\ g(y) \leq c, \tag{3.119}$$

$$g_0(y) + \lambda \max\{g(y) - c, 0\} \leq g_0(x) + \lambda \max\{g(x) - c, 0\}. \tag{3.120}$$

Clearly, Theorem 3.14 easily follows from the property (P2).

Assume that there is no $\Lambda_0 \geq 1$ such that (P2) holds. Then for each integer $k \geq 1$ there exist

$$\epsilon_k \in (0,1),\ g_0^{(k)} \in \mathcal{A},\ g^{(k)} \in \mathcal{U}(k^{-1}), \tag{3.121}$$

$c^{(k)} \in R^1$ satisfying

$$|c_k - \bar{c}| \leq 1/k, \tag{3.122}$$

$$\lambda_k \geq k \tag{3.123}$$

and $x_k \in X$ which satisfies

$$g_0^{(k)}(x_k) + \lambda_k \max\{g^{(k)}(x_k) - c_k, 0\}$$

$$\leq \inf\{g_0^{(k)}(z) + \lambda_k \max\{g^{(k)}(z) - c_k, 0\} :\ z \in X\} + 2^{-1}\epsilon_k k^{-2} \tag{3.124}$$

and

$$\{y \in B(x_k, \epsilon_k) :\ g^{(k)}(y) \leq c_k \text{ and}$$

$$g_0^{(k)}(y) + \lambda_k \max\{g^{(k)}(y) - c_k, 0\} \leq g_0^{(k)}(x_k) + \lambda_k \max\{g^{(k)}(x_k) - c_k, 0\}\} = \emptyset. \tag{3.125}$$

For any natural number k put

$$\psi_k(z) = g_0^{(k)}(z) + \lambda_k \max\{g^{(k)}(z) - c_k, 0\},\ z \in X. \tag{3.126}$$

It is easy to see that for any integer $k \geq 1$ the function ψ_k is lower semicontinuous and

$$\psi_k(z) \geq \phi(z) \text{ for all } z \in X. \tag{3.127}$$

Assume that $k \geq 1$ is an integer. By (3.124), (3.126), (3.127) and Ekeland's variational principle [37] there exists $y_k \in X$ such that

$$\psi_k(y_k) \leq \psi_k(x_k), \tag{3.128}$$

$$||y_k - x_k|| \le (2k)^{-1}\epsilon_k, \tag{3.129}$$

$$\psi_k(y_k) \le \psi_k(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{3.130}$$

It follows from (3.125), (3.126), (3.128) and (3.129) that for all integers $k \ge 1$

$$g^{(k)}(y_k) > c_k. \tag{3.131}$$

In view of (3.123) there exists an integer $k_0 \ge 1$ such that

$$8k_0^{-1} < \bar{c} - f(\theta). \tag{3.132}$$

Assume that a natural number $k \ge k_0$. It follows from (3.122), (3.121), (3.125), (3.123), (3.126) and (3.132) that

$$c_k - g^{(k)}(\theta) \ge -k^{-1} + \bar{c} - f(\theta) - k^{-1} \ge 8k_0^{-1} - 2k^{-1} > 0. \tag{3.133}$$

In view of (3.121), the definition of $\mathcal{A}$, (3.126), (3.124), (3.121), (3.128) and (3.131),

$$\bar{c}_0 \ge g_0^{(k)}(\theta) \ge \inf\{g_0^{(k)}(z) : \; z \in X \text{ and } g^{(k)}(z) \le c_k\}$$

$$= \inf\{\psi_k(z) : \; z \in X \text{ and } g^{(k)}(z) \le c_k\}$$

$$\ge \inf(\psi_k) \ge \psi_k(x_k) - 1 \ge \psi_k(y_k) - 1 = g_0^{(k)}(y_k) + \lambda_k(g^{(k)}(y_k) - c_k) - 1. \tag{3.134}$$

It follows from (3.131) and (3.134) that

$$g_0^{(k)}(y_k) \le \bar{c}_0 + 1 \text{ for all integers } k \ge k_0. \tag{3.135}$$

Combined with (3.121) and (3.74) this implies that

$$\phi(y_k) \le \bar{c}_0 + 1 \text{ for all integers } k \ge k_0.$$

Together with (3.73) this implies that

$$y_k \in B(0, M_0 - 4) \text{ for all integers } k \ge k_0. \tag{3.136}$$

Assume that $k \ge k_0$ is a natural number. In view of (A4), (3.121), (3.136), (3.135) and (3.74) there exists a neighborhood V_k of y_k in X such that

$$V_k \subset B^o(0, M_0),$$

$$g_0^{(k)}(z) - g_0^{(k)}(y_k) \le M_1 h(z, y_k) \text{ for all } z \in V_k. \tag{3.137}$$

It follows from (3.131), (3.134), (3.121) and (3.74) that for each natural number $k \ge k_0$,

$$0 < g^{(k)}(y_k) - c_k \le [1 + \bar{c}_0 - \inf(g_0^{(k)})]\lambda_k^{-1} \le [1 + \bar{c}_0 - \inf(\phi)]k^{-1}. \tag{3.138}$$

Assume that $k \ge k_0$ is a natural number. Since the function $g^{(k)}$ is lower semicontinuous it follows from (3.131) that there exists $r_k \in (0, 1)$ such that

$$B(y_k, r_k) \subset V_k,$$

$$g^{(k)}(y) > c_k \text{ for each } y \in B(y_k, r_k). \tag{3.139}$$

By (3.139), (3.126) and (3.135) for each $z \in B(y_k, r_k) \cap \mathrm{dom}(g_0^{(k)})$

$$\lambda_k(g^{(k)}(z) - c_k) - \lambda_k(g^{(k)}(y_k) - c_k) = \psi_k(z) - \psi_k(y_k) - g_0^{(k)}(z) + g_0^{(k)}(y_k)$$
$$\geq -k^{-1}||z - y_k|| + g_0^{(k)}(y_k) - g_0^{(k)}(z)$$

and

$$g^{(k)}(z) - g^{(k)}(y_k) \geq \lambda_k^{-1}(g_0^{(k)}(y_k) - g_0^{(k)}(z)) - \lambda_k^{-1}k^{-1}||y_k - z||. \tag{3.140}$$

It follows from (3.121) and the definition of $\mathcal{U}(k^{-1})$ that there exists a convex function $h_k : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(g^{(k)}) \cap B^o(0, M_0) = \mathrm{dom}(f) \cap B^o(0, M_0) = \mathrm{dom}(h_k) \cap B^o(0, M_0), \tag{3.141}$$

$$|f(x) - h_k(x)| \leq 1/k \text{ for all } x \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{3.142}$$

$$|(h_k - g^{(k)})(z_1) - (h_k - g^{(k)})(z_2)| \leq ||z_1 - z_2||/k$$
$$\text{for all } z_1, z_2 \in \mathrm{dom}(f) \cap B^o(0, M_0), \tag{3.143}$$

$$|h_k(z) - g^{(k)}(z)| \leq 1/k \text{ for all } z \in \mathrm{dom}(f) \cap B^o(0, M_0). \tag{3.144}$$

By (3.139), (3.137), (3.141), (3.138), (3.143) and (3.136) for each

$$z \in B(y_k, r_k) \cap \mathrm{dom}(g_0^{(k)}) \cap \mathrm{dom}(f),$$

$$h_k(z) - h_k(y_k) = g^{(k)}(z) - g^{(k)}(y_k) + ((h_k - g^{(k)})(z) - (h_k - g^{(k)})(y_k))$$
$$\geq g^{(k)}(z) - g^{(k)}(y_k) - ||z - y_k||/k$$
$$\geq -||z - y_k||/k + \lambda_k^{-1}(g_0^{(k)}(y_k) - g_0^{(k)}(z)) - \lambda_k^{-1}k^{-1}||y_k - z||.$$

Together with (3.137) and (3.139) this implies that for each $z \in B(y_k, r_k) \cap \mathrm{dom}(f)$,

$$0 \leq h_k(z) - h_k(y_k) + ||z - y_k||(1/k + (\lambda_k k)^{-1}) + \lambda_k^{-1}(g_0^{(k)}(z) - g_0^{(k)}(y_k))$$
$$\leq h_k(z) - h_k(y_k) + ||z - y_k||(1/k + (\lambda_k k)^{-1}) + \lambda_k^{-1}M_1 h(z, y_k).$$

It follows from the relation above, (3.141), (3.137) and (3.139) that for all $z \in B(y_k, r_k)$,

$$h_k(y_k) \leq h_k(z) + ||z - y_k||(1/k + (\lambda_k k)^{-1}) + \lambda_k^{-1}M_1 h(z, y_k). \tag{3.145}$$

By (A2), (3.136), (3.135), (3.121) and (3.74) the function

$$z \to h_k(z) + \lambda_k^{-1}M_1 h(z, y_k) + (\lambda_k^{-1}k^{-1} + k^{-1})||z - y_k||, \; z \in B^o(0, M_0)$$

is convex. Together with (A1), (3.74), (3.135), (3.136), (3.121) and (3.138) this implies that (3.145) holds for all $z \in B^0(0, M_0)$.

In view of (3.136) and (3.123) for all $z \in B^0(0, M_0)$

$$\lim_{k\to\infty} (k^{-1} + \lambda_k^{-1} k^{-1})||z - y_k|| = 0. \tag{3.146}$$

By (3.123), (3.136), (A3), (3.135), (3.121) and (3.74) for all $z \in X_0$

$$\lim_{k\to\infty} \lambda_k^{-1} M_1 h(z, y_k) = 0. \tag{3.147}$$

It follows from (3.146), (3.147), (3.142), (3.145) which holds for all $z \in X_0 \cap \mathrm{dom}(f)$, (3.136), (3.138), (3.141), (3.144), (3.131) and (3.122) that

$$f(z) = \lim_{k\to\infty} [h_k(z) + \lambda_k^{-1} M_1 h(z, y_k) + (\lambda_k^{-1} k^{-1} + k^{-1})||z - y_k||]$$

$$\geq \limsup_{k\to\infty} h_k(y_k) = \limsup_{k\to\infty} g^{(k)}(y_k) \geq \lim_{k\to\infty} c_k = \bar{c}.$$

Since $\theta \in X_0 \cap \mathrm{dom}(f)$ (see (3.113)) we conclude that $f(\theta) \geq \bar{c}$. This contradicts (3.113). The contradiction we have reached proves that there exists $\Lambda_0 \geq 1$ such that the property (P2) holds. Theorem 3.14 is proved.

3.8 Nonconvex inequality-constrained minimization problems

In this section we study a large class of inequality-constrained minimization problems

$$\text{minimize } f(x) \text{ subject to } x \in A \tag{P$_\mathrm{i}$}$$

where

$$A = \{x \in X : \ g_i(x) \leq c_i \text{ for all } i = 1, \dots, n\}.$$

Here X is a Banach space, c_i, $i = 1, \dots, n$ are real numbers, and the constraint functions g_i, $i = 1, \dots, n$ and the objective function f are locally Lipschitz.

We associate with the inequality-constrained minimization problem above the corresponding family of unconstrained minimization problems

$$\text{minimize } f(z) + \gamma \sum_{i=1}^{n} \max\{g_i(z) - c_i, 0\} \text{ subject to } z \in X$$

where $\gamma > 0$ is a penalty. We establish the existence of a penalty coefficient for which approximate solutions of the unconstrained penalized problem are close enough to approximate solutions of the corresponding constrained problem and study the stability of this exact penalty property under perturbations of the functions f and $g_1, \dots, g_n$ and of the parameters $c_1, \dots, c_n$. More precisely, we consider a family of constrained minimization problems of type (P_i) with an objective function close to a given function f, with constraint functions close to given functions $g_1, \dots, g_n$ and with the right-hand side of constraints close to given constants $c_1, \dots, c_n$ in a certain natural sense. Under certain conditions on $f, g_1, \dots, g_n, c_1, \dots, c_n$ we show that all the constrained minimization problems belonging to this family possess the exact penalty property with the same penalty coefficient which depends only on f, $g_1, \dots, g_n$, $c_1, \dots, c_n$.

Let $(X, ||\cdot||)$ be a Banach space and let $(X^*, ||\cdot||_*)$ be its dual space. For each $x \in X$, each $x^* \in X^*$ and each $r > 0$ set

$$B(x,r) = \{y \in X : \ ||y-x|| \le r\}, \ B_*(x^*,r) = \{l \in X^* : \ ||l-x^*||_* \le r\}.$$

Assume that $f : U \to R^1$ be a Lipschitz function which is defined on a nonempty open set $U \subset X$. For each $x \in U$ let

$$f^0(x,h) = \limsup_{t \to 0^+, y \to x} [f(y+th) - f(y)]/t, \ h \in X$$

be the Clarke generalized directional derivative of f at the point x [21], let

$$\partial f(x) = \{l \in X^* : \ f^0(x,h) \ge l(h) \text{ for all } h \in X\}$$

be Clarke's generalized gradient of f at x [21] and set

$$\Xi_f(x) = \inf\{f^0(x,h) : \ h \in X \text{ and } ||h| = 1\}.$$

As usual a point $x \in X$ is called a critical point of f if $0 \in \partial f(x)$. A real number $c \in R^1$ is called a critical value of f if there is a critical point $x \in U$ of f such that $f(x) = c$.

In order to consider a constrained minimization problem with several constraints we use the notion of a critical point and the version of Palais–Smale condition for a Lipschitz mapping $F : X \to R^n$ introduced in Section 2.7.

Assume that $n \ge 1$ is an integer, U is a nonempty open subset of X and that $F = (f_1, \dots, f_n) : U \to R^n$ is a locally Lipschitz mapping.

Let $\kappa \in (0,1)$. For each $x \in U$ set

$$\Xi_{F,\kappa}(x) = \inf\{||\sum_{i=1}^{n}(\alpha_{i1}\eta_{i1} - \alpha_{i2}\eta_{i2})|| : \tag{3.148}$$

$$\eta_{i1}, \ \eta_{i2} \in \partial f_i(x), \ \alpha_{i1}, \ \alpha_{i2} \in [0,1], \ i = 1, \dots, n$$

$$\text{and there is } j \in \{1, \dots, n\} \text{ such that } \alpha_{j1}\alpha_{j2} = 0 \text{ and } |\alpha_{j1}| + |\alpha_{j2}| \ge \kappa\}.$$

It is known (see Chapter 2, Sect. 2.3 of [21]) that for each $x \in U$ and all $i = 1, \dots, n$,

$$\partial(-f_i)(x) = -\partial f_i(x). \tag{3.149}$$

This equality implies that

$$\Xi_{-F,\kappa}(x) = \Xi_{F,\kappa}(x) \text{ for each } x \in U. \tag{3.150}$$

In the sequel we assume that $U = X$.

A point $x \in X$ is called a critical point of F with respect to κ if $\Xi_{F,\kappa}(x) = 0$.

A vector $c = (c_1, \dots, c_n) \in R^n$ is called a critical value of F with respect to κ if there is a critical point $x \in X$ of F with respect to κ such that $F(x) = c$.

Let M be a nonempty subset of X. We say that the mapping $F : X \to R^n$ satisfies the (P-S) condition on M with respect to κ if for each bounded with respect to the norm topology sequence $\{x_i\}_{i=1}^\infty \subset M$ such that $\{F(x_i)\}_{i=1}^\infty$ is bounded and $\liminf_{i\to\infty} \Xi_{F,\kappa}(x_i) = 0$ there exists a convergent subsequence of $\{x_i\}_{i=1}^\infty$ in X with the norm topology.

For each function $h : X \to R^1$ and each nonempty set $A \subset X$ put

$$\inf(h) = \inf\{h(z) : \ z \in X\}, \ \inf(h; A) = \inf\{h(z) : \ z \in A\}.$$

For each $x \in X$ and each $B \subset X$ put

$$d(x, B) = \inf\{||x - y|| : \ y \in B\}.$$

We assume that the sum over empty set is zero.

Denote by $\mathcal{M}$ the set of all continuous functions $h : X \to R^1$. We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\begin{aligned}\mathcal{E}(M, q, \epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \ &|f(x) - g(x)| \le \epsilon \text{ for all } x \in B(0, M)\} \\ \cap\{(f, g) \in \mathcal{M} \times \mathcal{M} : \ &|(f - g)(x) - (f - g)(y)| \\ &\le q||x - y|| \text{ for each } x, y \in B(0, M)\},\end{aligned} \tag{3.151}$$

where M, q, ϵ are positive numbers. It is not difficult to see that this uniform space is metrizable and complete.

Let $n \ge 1$ be an integer, $f \in \mathcal{M}$, $G = (g_1, \dots, g_n)$ with $g_i \in \mathcal{M}$ for all $i = 1, \dots, n$ and let $c = (c_1, \dots, c_n) \in R^n$.

Put

$$A(G, c) = \{x \in X : \ g_i(x) \le c_i \text{ for all } i = 1, \dots, n\} \tag{3.152}$$

and consider the following constrained minimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in A(G, c). \tag{P}$$

We associate with the problem (P) the corresponding family of unconstrained minimization problems

$$\text{minimize } f(x) + \sum_{i=1}^n \lambda_i \max\{g_i(x) - c_i, 0\} \text{ subject to } x \in X, \tag{P$_\lambda$}$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$.

For each $\kappa \in (0,1)$ put

$$\Omega_\kappa = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \geq \kappa$$
$$\text{for all } i = 1, \dots, n \text{ and } \max_{i=1,\dots,n} x_i = 1\}. \tag{3.153}$$

Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that

$$\lim_{t \to \infty} \phi(t) = \infty \tag{3.154}$$

and $\bar{a}$ be a positive number. Denote by $\mathcal{M}_\phi$ the set of all functions $h \in \mathcal{M}$ such that

$$h(x) \geq \phi(||x||) - \bar{a} \text{ for all } \in X. \tag{3.155}$$

Assume that

$$\bar{f} \in \mathcal{M}_\phi \tag{3.156}$$

is Lipschitz on all bounded subsets of X, $\bar{G} = (\bar{g}_1, \dots, \bar{g}_n) : X \to R^1$ is a locally Lipschitz mapping and that $\bar{c} = (\bar{c}_1, \dots, \bar{c}_n) \in R^n$.

We assume that $A(\bar{G}, \bar{c}) \neq \emptyset$ and fix

$$\theta \in A(\bar{G}, \bar{c}). \tag{3.157}$$

It follows from (3.154) that there exists a positive number M_0 such that

$$M_0 > 2 + ||\theta|| \text{ and } \phi(M_0 - 2) > \bar{f}(\theta) + \bar{a} + 4. \tag{3.158}$$

For each $x \in A(\bar{G}, \bar{c})$ put

$$I(x) = \{i \in \{1, \dots, n\} : \ \bar{c}_i = \bar{g}_i(x)\}. \tag{3.159}$$

Fix $\kappa \in (0,1)$. In this paper we use the following assumptions.

(A1) If $x \in A(\bar{G}, \bar{c})$, $q \geq 1$ is the cardinality of a subset $\{i_1, \dots, i_q\}$ of $I(x)$ with $i_1 < i_2 < \dots < i_q$ and if x is a critical point of the mapping

$$(\bar{g}_{i_1}, \dots, \bar{g}_{i_q}) : X \to R^q$$

with respect to κ, then $\bar{f}(x) > \inf(\bar{f}; A(\bar{G}, \bar{c}))$.

(A2) There exists a positive number γ_* such that for each finite strictly increasing sequence of natural numbers $\{i_1, \dots, i_q\}$ which satisfies $\{i_1, \dots, i_q\} \subset \{1, \dots, n\}$ the mapping $(\bar{g}_{i_1}, \dots, \bar{g}_{i_q}) : \ X \to R^q$ satisfies the (P-S) condition on the set

$$\cap_{j \in \{i_1, \dots, i_q\}} (\bar{g}_j^{-1}([\bar{c}_j - \gamma_*, \bar{c}_j + \gamma_*]))$$

with respect to κ.

(A3) For each positive number ϵ there is $x_\epsilon \in A(\bar{G}, \bar{c})$ such that $\bar{f}(x_\epsilon) \leq \inf(\bar{f}; A(\bar{G}, \bar{c})) + \epsilon$ and if $I(x_\epsilon) \neq \emptyset$, then zero does not belong to the convex hull of the set

$$\cup_{i \in I(x_\epsilon)} \partial \bar{g}_i(x_\epsilon).$$

The following theorem is the main result of this section.

Theorem 3.16. *Let (A1), (A2) and (A3) hold and let q be a positive number. Then there exist numbers $\Lambda_0, r > 0$ such that for each positive number ϵ there exists $\delta \in (0, \epsilon)$ such that the following assertion holds:*

If $f \in \mathcal{M}_\phi$ satisfies

$$(f, \bar{f}) \in \mathcal{E}(M_0, q, r),$$

if $G = (g_1, \dots, g_n) : X \to R^n$ satisfies

$$g_i \in \mathcal{M} \text{ and } (g_i, \bar{g}_i) \in \mathcal{E}(M_0, r, r) \text{ for all } i = 1, \dots, n,$$

if $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$, $\lambda \geq \Lambda_0$, $c = (c_1, \dots, c_n) \in R^n$ satisfies

$$|\bar{c}_i - c_i| \leq r \text{ for all } i = 1, \dots, n$$

and if $x \in X$ satisfies

$$f(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\} \leq \inf\{f(z)$$

$$+ \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : z \in X\} + \delta,$$

then there exists $y \in A(G, c)$ such that

$$||x - y|| \leq \epsilon \text{ and } f(y) \leq \inf(f; A(G, c)) + \epsilon.$$

Theorem 3.16 easily implies the following result.

Theorem 3.17. *Let (A1), (A2) and (A3) hold and let q be a positive number. Then there exist numbers $\Lambda_0, r > 0$ such that for each $f \in \mathcal{M}_\phi$ satisfying $(f, \bar{f}) \in \mathcal{E}(M_0, q, r)$, each mapping $G = (g_1, \dots, g_n) : X \to R^n$ which satisfies*

$$g_i \in \mathcal{M} \text{ and } (g_i, \bar{g}_i) \in \mathcal{E}(M_0, r, r) \text{ for all } i = 1, \dots, n, \tag{3.160}$$

each $c = (c_1, \dots, c_n) \in R^n$ satisfying

$$|\bar{c}_i - c_i| \leq r \text{ for all } i = 1, \dots, n,$$

each $\gamma = (\gamma_1, \dots, \gamma_n) \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$ and each sequence $\{x_k\}_{k=1}^\infty \subset X$ which satisfies

$$\lim_{k \to \infty} [f(x_k) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x_k) - c_i, 0\}]$$

$$= \inf\{f(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : z \in X\}$$

there exists a sequence $\{y_k\}_{k=1}^\infty \subset A(G, c)$ such that

$$\lim_{k \to \infty} ||y_k - x_k|| = 0 \text{ and } \lim_{k \to \infty} f(y_k) = \inf(f; A(G, c)).$$

Corollary 3.18. *Let (A1), (A2) and (A3) hold and let q be a positive number. Then there exist $\Lambda_0, r > 0$ such that if $f \in \mathcal{M}_\phi$ satisfies $(f, \bar{f}) \in \mathcal{E}(M_0, q, r)$, if a mapping $G = (g_1, \ldots, g_n) : X \to R^n$ satisfies (3.160), if $c = (c_1, \ldots, c_n) \in R^n$ satisfies*

$$|\bar{c}_i - c_i| \leq r \text{ for all } i = 1, \ldots, n,$$

if $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Omega_\kappa,\ \lambda \geq \Lambda_0$ and if $x \in X$ satisfies

$$f(x) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(x) - c_i, 0\}$$

$$= \inf\{f(z) + \sum_{i=1}^{n} \lambda\gamma_i \max\{g_i(z) - c_i, 0\} : \ z \in X\},$$

then $x \in A(G, c)$ and $f(x) = \inf(f; A(G, c))$.

Note that Theorem 3.16 was obtained in [140].

3.9 Proof of Theorem 3.16

For each $f \in \mathcal{M}_\phi$, each $G = (g_1, \ldots, g_n) : X \to R^n$, each $c = (c_1, \ldots, c_n) \in R^n$ and each $\lambda = (\lambda_1, \ldots, \lambda_n) \in (0, \infty)^n$ define for all $z \in X$

$$\psi_{\lambda,c}^{(f,G)}(z) = f(z) + \sum_{i=1}^{n} \lambda_i \max\{g_i(z) - c_i, 0\}. \tag{3.161}$$

We show that there exist $\Lambda_0, r > 0$ such that the following property holds:

(P1) For each $\epsilon \in (0, 1)$ there exists $\delta \in (0, \epsilon)$ such that for each $f \in \mathcal{M}_\phi$ satisfying

$$(f, \bar{f}) \in \mathcal{E}(M_0, q, r),$$

each $G = (g_1, \ldots, g_n) : X \to R^n$ satisfying

$$g_i \in \mathcal{M} \text{ and } (g_i, \bar{g}_i) \in \mathcal{E}(M_0, r, r) \text{ for all } i = 1, \ldots, n,$$

each $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Omega_\kappa$, each $\lambda \geq \Lambda_0$, each $c = (c_1, \ldots, c_n) \in R^n$ satisfying

$$|c_i - \bar{c}_i| \leq r, \ i = 1, \ldots, n$$

and each $x \in X$ satisfying

$$\psi_{\lambda\gamma,c}^{(f,G)}(x) \leq \inf(\psi_{\lambda\gamma,c}^{(f,G)}) + \delta$$

the set

$$\{y \in B(x, \epsilon) \cap A(G, c) : \ \psi_{\lambda\gamma,c}^{(f,G)}(y) \leq \psi_{\lambda\gamma,c}^{(f,G)}(x)\}$$

is nonempty.

Clearly, (P1) implies the validity of Theorem 3.16.

Assume that there are no positive numbers Λ_0, r such that the property (P1) holds. Then for each integer $k \geq 1$ there exist $\epsilon_k \in (0,1)$, $f^{(k)} \in \mathcal{M}_\phi$ satisfying

$$(f_k, \bar{f}) \in \mathcal{E}(M_0, q, k^{-1}), \tag{3.162}$$

$G^{(k)} = (g_1^{(k)}, \dots, g_n^{(k)}) : X \to R^n$ satisfying

$$g_i^{(k)} \in \mathcal{M} \text{ and } (g_i^{(k)}, \bar{g}_i) \in \mathcal{E}(M_0, k^{-1}, k^{-1}) \text{ for all } \ i = 1, \dots, n, \tag{3.163}$$

$$\gamma^{(k)} = (\gamma_1^{(k)}, \dots, \gamma_n^{(k)}) \in \Omega_\kappa, \ \lambda_k \geq k, \tag{3.164}$$

$c^{(k)} = (c_1^{(k)}, \dots, c_n^{(k)}) \in R^n$ satisfying

$$|c_i^{(k)} - \bar{c}_i| \leq k^{-1}, \ i = 1, \dots, n, \tag{3.165}$$

and $x_k \in X$ such that

$$\psi_{\lambda_k \gamma^{(k)}, c^{(k)}}^{(f^{(k)}, G^{(k)})}(x_k) \leq \inf(\psi_{\lambda_k \gamma^{(k)}, c^{(k)}}^{(f^{(k)}, G^{(k)})}) + 2^{-1}\epsilon_k k^{-2}, \tag{3.166}$$

$$\{y \in B(x_k, \epsilon_k) \cap A(G^{(k)}, c^{(k)}) : \ \psi_{\lambda_k \gamma^{(k)}, c^{(k)}}^{(f^{(k)}, G^{(k)})}(y) \leq \psi_{\lambda_k \gamma^{k}, c^{(k)}}^{(f^{(k)}, G^{(k)})}(x_k)\} = \emptyset. \tag{3.167}$$

For each integer $k \geq 1$ set

$$\psi_k = \psi_{\lambda_k \gamma^{(k)}, c^{(k)}}^{(f^{(k)}, G^{(k)})}. \tag{3.168}$$

Put

$$\bar{\psi} = \psi_{\lambda_k \gamma^{(k)}, c^{(k)}}^{(\bar{f}, \bar{G})}. \tag{3.169}$$

Let $k \geq 1$ be an integer. By (3.166) and Ekeland's variational principle [37] there exists $y_k \in X$ such that

$$\psi_k(y_k) \leq \psi_k(x_k), \tag{3.170}$$

$$||y_k - x_k|| \leq (2k)^{-1}\epsilon_k, \tag{3.171}$$

$$\psi_k(y_k) \leq \psi_k(z) + k^{-1}||z - y_k|| \text{ for all } z \in X. \tag{3.172}$$

Relations (3.167), (3.168), (3.170) and (3.171) imply that

$$y_k \notin A(G^{(k)}, c^{(k)}) \text{ for all integers } k \geq 1. \tag{3.173}$$

For each integer $k \geq 1$ put

$$I_k = \{i \in \{1, \dots, n\} : \ g_i^{(k)}(y_k) = c_i^{(k)}\}, \tag{3.174}$$

$$I_{k+} = \{i \in \{1, \dots, n\} : \ g_i^{(k)}(y_k) > c_i^{(k)}\},$$

$$I_{k-} = \{i \in \{1, \dots, n\} : \ g_i^{(k)}(y_k) < c_i^{(k)}\}.$$

Relations (3.173), (3.174) and (3.152) imply that

$$I_{k+} \neq \emptyset \text{ for all natural numbers } k. \tag{3.175}$$

Extracting a subsequence and reindexing we may assume without loss of generality that for all integers $k \geq 1$,

$$I_k = I_1, \ I_{k+} = I_{1+}, \ I_{k-} = I_{1-}. \tag{3.176}$$

We continue the proof with two steps.

Step 1. We will show that for all sufficiently large natural numbers k

$$A(G^{(k)}, c^{(k)}) \neq \emptyset, \ y_k \in B(0, M_0 - 2)$$

and that

$$\limsup_{k\to\infty} f^{(k)}(y_k) \leq \limsup_{k\to\infty} \inf(f^{(k)}; A(G^{(k)}, c^{(k)})) \leq \inf(\bar{f}; A(\bar{G}, \bar{c})).$$

Let $\delta_0 \in (0, 2^{-1})$. (A3) implies that there exists

$$z_0 \in A(\bar{G}, \bar{c}) \tag{3.177}$$

such that

$$\bar{f}(z_0) \leq \inf(\bar{f}; A(\bar{G}, \bar{c})) + \delta_0; \tag{3.178}$$

$$\text{if } I(z_0) \neq \emptyset, \text{ then 0 does not belong}$$
$$\text{to the convex hull of the set } \cup_{i\in I(z_0)} \partial \bar{g}_i(z_0). \tag{3.179}$$

It follows from (3.177), (3.178) and (3.157) that

$$\bar{f}(z_0) \leq \bar{f}(\theta) + 1. \tag{3.180}$$

By (3.180), (3.156), (3.155) and (3.158),

$$z_0 \in B(0, M_0 - 2). \tag{3.181}$$

Define $z_1 \in X$ as follows:

$$\text{If } I(z_0) = \emptyset, \text{ then set } z_1 = z_0. \tag{3.182}$$

Assume that

$$I(z_0) \neq \emptyset. \tag{3.183}$$

Fix $\delta_1 \in (0, 1)$ such that

$$\bar{c}_i > \bar{g}_i(z_0) + 4\delta_1 \text{ for all natural numbers } i \in \{1, \ldots, n\} \setminus I(z_0). \tag{3.184}$$

It follows from (3.179) and (3.183) that there exists

$$\eta \in X \text{ such that } ||\eta|| = 1 \text{ and } \delta_2 \in (0, 1) \tag{3.185}$$

such that

$$l(\eta) \leq -2\delta_2 \text{ for all } l \in \cup_{i\in I(z_0)}\partial \bar{g}_i(z_0). \tag{3.186}$$

Inequality (3.186) implies that

$$\bar{g}_i^0(z_0, \eta) \leq -2\delta_2 \text{ for all } i \in I(z_0). \tag{3.187}$$

Since the function $\bar{f}$ is Lipschitz on bounded subsets of X and the functions $\bar{g}_i^0(\cdot, \eta)$, $i = 1, \dots, n$ are upper semicontinuous it follows from (3.184) and (3.187) that there exists a positive number $\delta_3 < \min\{1, \delta_1\}$ such that

$$\bar{g}_i^0(z, \eta) \leq -(3/2)\delta_2 \text{ for all } i \in I(z_0) \text{ and all } z \in B(z_0, \delta_3), \tag{3.188}$$

$$\bar{c}_i > \bar{g}_i(z) + 3\delta_1 \text{ for all } i \in \{1, \dots, n\} \setminus I(z_0) \text{ and all } z \in B(z_0, \delta_3), \tag{3.189}$$

$$|\bar{f}(z) - \bar{f}(z_0)| \leq \delta_0 \text{ for all } z \in B(z_0, \delta_3). \tag{3.190}$$

Set

$$z_1 = z_0 + \delta_3\eta. \tag{3.191}$$

It follows from (3.191), (3.189) and (3.185) that

$$\bar{c}_i > \bar{g}_i(z_1) + 3\delta_1 \text{ for all } i \in \{1, \dots, n\} \setminus I(z_0). \tag{3.192}$$

Assume that $j \in I(z_0)$. It follows from the mean value theorem (see Theorem 2.3.7 of [21]), (3.185) and (3.188) that there exist

$$s \in [0, \delta_3] \text{ and } l \in \partial \bar{g}_j(z_0 + s\eta)$$

such that

$$\bar{g}_j(z_0+\delta_3\eta)-\bar{g}_j(z_0) = l(\delta_3\eta) \leq \bar{g}_j^0(z+s\eta, \delta_3\eta) = \delta_3\bar{g}_j^0(z_0+s\eta, \eta) \leq \delta_3(-3/2)\delta_2.$$

Together with (3.159) and (3.191) this implies that

$$\bar{g}_j(z_1) \leq \bar{c}_j - (3/2)\delta_2\delta_3 \text{ for all } j \in I(z_0). \tag{3.193}$$

In view of (3.192) and (3.193),

$$\bar{g}_j(z_1) \leq \bar{c}_j - (3/2)\delta_2\delta_3 \text{ for all } j \in \{1, \dots, n\}. \tag{3.194}$$

It follows from (3.190), (3.191), (3.185) and (3.178) that

$$\bar{f}(z_1) \leq \bar{f}(z_0) + \delta_0 \leq \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0. \tag{3.195}$$

By (3.191), (3.185) and (3.181),

$$||z_1|| \leq ||z_0|| + \delta_3 \leq M_0 - 1. \tag{3.196}$$

Now we conclude that in both cases which were considered separately ($I(z_0) = \emptyset$; $I(z_0) \neq \emptyset$) we have defined $z_1 \in X$ such that

$$\bar{g}_j(z_1) < \bar{c}_j,\ j = 1, \dots, n, \tag{3.197}$$

$$\bar{f}(z_1) \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0, \tag{3.198}$$

$$z_1 \in B(0, M_0 - 1) \tag{3.199}$$

(see (3.194)-(3.196), (3.182), (3.181), (3.177) and (3.178)). By (3.197), (3.199), (3.163), (3.165), (3.152) and (3.151) there exists an integer $k_0 \ge 1$ such that

$$z_1 \in A(G^{(k)}, c^{(k)}) \text{ for all natural numbers } k \ge k_0. \tag{3.200}$$

It follows from (3.155), (3.168), (3.161), (3.170), (3.166), (3.200), (3.199), (3.162) and (3.198) that for any natural number $k \ge k_0$

$$\phi(||y_k||) - \bar{a} \le f^{(k)}(y_k) \le \psi_k(y_k) \le \psi_k(x_k) \le \inf(\psi_k) + (2k^2)^{-1}$$
$$\le \inf(\psi_k; A(G^{(k)}, c^{(k)})) + (2k^2)^{-1} = \inf(f^{(k)}; A(G^{(k)}, c^{(k)})) + 2^{-1}k^{-2}$$
$$\le f^{(k)}(z_1) + 2^{-1}k^{-2} \le \bar{f}(z_1) + k^{-1} + 2^{-1}k^{-2} \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0 + 2k^{-1}. \tag{3.201}$$

In view of (3.201), the inequality $\delta_0 < 1/2$, (3.157) and (3.158) for all natural numbers $k \ge k_0$

$$\phi(y_k) - \bar{a} \le \bar{f}(\theta) + 2,\ ||y_k|| \le M_0 - 2. \tag{3.202}$$

By (3.190) and (3.202) for all sufficiently large natural numbers k

$$A(G^{(k)}, c^{(k)}) \ne \emptyset,\ y_k \in B(0, M_0 - 2). \tag{3.203}$$

Inequality (3.201) implies that

$$\limsup_{k \to \infty} f^{(k)}(y_k) \le \limsup_{k \to \infty} \inf(f^{(k)}, A(G^{(k)}, c^{(k)})) \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2\delta_0.$$

Since δ_0 is an arbitrary element of the interval $(0, 1/2)$ we conclude that

$$\limsup_{k \to \infty} f^{(k)}(y_k) \le \limsup_{k \to \infty} \inf(f^{(k)}, A(G^{(k)}, c^{(k)})) \le \inf(\bar{f}; A(\bar{G}, \bar{c})). \tag{3.204}$$

Step 2. In this step we will complete the proof of the theorem. By (3.201), the inclusion $\delta_0 \in (0, 1/2)$, (3.168), (3.161), the inclusion $f^{(k)} \in \mathcal{M}_\phi$ and (3.155) for each natural number $k \ge k_0$ and each $i \in I_{1+}$

$$-\bar{a} + \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\} \le \inf(\bar{f}; A(\bar{G}, \bar{c})) + 2.$$

Combined with (3.164) and (3.153) this implies for each integer $k \ge k_0$ and each $i \in I_{1+}$

$$g_i^{(k)}(y_k) - c_i^{(k)} = \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\} \le k^{-1}\kappa^{-1}(\inf(\bar{f}; A(\bar{G}, \bar{c})) + 2 + \bar{a}). \tag{3.205}$$

Then for all sufficiently large natural numbers k

$$0 \le g_i^{(k)}(y_k) - c_i^{(k)} \le \gamma_*/2 \text{ for all } i \in I_{1+}.$$

Combined with (3.174), (3.176), (3.163), (3.165) and (3.203) this implies that for all sufficiently large natural numbers k

$$-\gamma_* \le \bar{g}_i(y_k) - \bar{c}_i \le \gamma_* \text{ for all } i \in I_{1+} \cup I_1. \tag{3.206}$$

Since $\bar{f}$ is Lipschitz on bounded subsets of X there is $L_0 > 1$ such that

$$|\bar{f}(u_1) - \bar{f}(u_2)| \le L_0 ||u_1 - u_2|| \text{ for each } u_1, u_2 \in B(0, M_0). \tag{3.207}$$

Assume that $k \ge k_0$ is an integer. By (3.202), (3.174) and (3.176) there exists an open neighborhood V of y_k in X such that for each $y \in V$

$$g_i^{(k)}(y) > c_i^{(k)} \text{ for all } i \in I_{1+},\ g_i^{k)}(y) < c_i^{(k)} \text{ for all } i \in I_{1-}, \tag{3.208}$$

$$V \subset B(0, M_0 - 1).$$

By (3.174), (3.176), (3.161), (3.168) and (3.208) for each $z \in V$

$$f^{(k)}(y_k) + \lambda_k \sum_{i \in I_{1+}} \gamma_i^{(k)}(g_i^{(k)}(y_k) - c_i^{(k)}) + \lambda_k \sum_{i \in I_1} \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}$$

$$= f^{(k)}(y_k) + \sum_{i=1}^{n} \lambda_k \gamma_i^{(k)} \max\{g_i^{(k)}(y_k) - c_i^{(k)}, 0\}$$

$$= \psi_{\lambda_k \gamma^{(k)}, c_k}^{(f^k, G^{(k)})} = \psi_k(y_k) \le \psi_k(z) + k^{-1}||z - y_k||$$

$$= \psi_{\lambda_k \gamma^{(k)}, c^{(k)}}^{(f^{(k)}, G^{(k)})}(z) + k^{-1}||z - y_k||$$

$$= f^{(k)}(z) + \lambda_k \sum_{i \in I_{1+}} \gamma_i^{(k)}(g_i^{(k)}(z) - c_i^{(k)})$$

$$+\lambda_k \sum_{i \in I_1} \gamma_i^{(k)} \max\{g_i^{(k)}(z) - c_i^{(k)}, 0\} + k^{-1}||z - y_k||.$$

It follows from the relation above, (3.208) and the properties of Clarke's generalized gradient (see Chapter 2, Sect. 2.3 of [21]) that

$$0 \in \partial f^{(k)}(y_k) + \lambda_k \sum_{i \in I_{1+}} \gamma_i^{(k)} \partial g_i^{(k)}(y_k)$$

$$+\lambda_k \sum_{k, i \in I_1} \gamma_i^{(k)}(\cup\{\alpha \partial g_i^{(k)}(y_k) :\ \alpha \in [0,1]\}) + k^{-1} B_*(0,1). \tag{3.209}$$

By the properties of Clarke's generalized gradient [21], (3.202), (3.162) and (3.151),

$$\partial f^{(k)}(y_k) = \partial(\bar{f} + (f^{(k)} - \bar{f}))(y_k) \subset \partial \bar{f}(y_k) + \partial(f^{(k)} - \bar{f})(y_k)$$

$$\subset \partial \bar{f}(y_k) + qB_*(0,1). \tag{3.210}$$

It follows from the properties of Clarke's generalized gradient [21], (3.202), (3.163) and (3.151) that for all $i \in I_1 \cup I_{1+}$,

$$\partial g_i^{(k)}(y_k) = \partial(\bar{g}_i + (g_i^{(k)} - \bar{g}_i))(y_k) \subset \partial \bar{g}_i(y_k) + \partial(g_i^{(k)} - \bar{g}_i)(y_k)$$
$$\subset \partial \bar{g}_i(y_k) + k^{-1}B_*(0,1). \tag{3.211}$$

In view of (3.209), (3.164), (3.210), (3.211), (3.153) and (3.164),

$$0 \in \lambda_k^{-1}\partial f^{(k)}(y_k) + \sum_{i \in I_{1+}} \gamma_i^{(k)} \partial g_i^{(k)}(y_k) + \sum_{i \in I_1} \gamma_i^{(k)} (\cup\{\alpha \partial g_i^{(k)}(y_k) :\ \alpha \in [0,1]\})$$
$$+k^{-2}B_*(0,1) \subset \lambda_k^{-1}\partial \bar{f}(y_k) + k^{-1}qB_*(0,1)$$
$$+ \sum_{i \in I_{1+}} \gamma_i^{(k)}[\partial \bar{g}_i(y_k) + k^{-1}B_*(0,1)]$$
$$+\sum_{i \in I_1} \gamma_i^{(k)}(\cup\{\alpha \partial \bar{g}_i(y_k) + \alpha k^{-1}B_*(0,1) :\ \alpha \in [0,1]\}) + k^{-2}B_*(0,1)$$
$$\subset \lambda_k^{-1}\partial \bar{f}(y_k) + \sum_{i \in I_{1+}} \gamma_i^{(k)} \partial \bar{g}_i(y_k) + \sum_{i \in I_1} \gamma_i^{(k)} (\cup\{\alpha \partial \bar{g}_i(y_k) :\ \alpha \in [0,1]\})$$
$$+(q/k + n/k + n/k + k^{-2})B_*(0,1). \tag{3.212}$$

Inclusion (3.212) implies that there exists $l_* \in X^*$ satisfying

$$l_* \in B_*(0,1),$$

$$l_0 \in \partial \bar{f}(y_k),\ l_i \in \partial \bar{g}_i(y_k),\ i \in I_{1+} \cup I_1,\ \alpha_i \in [0,1],\ i \in I_1 \tag{3.213}$$

such that

$$0 = k^{-1}(q + 2n + k^{-1})l_* + \lambda_k^{-1} l_0 + \sum_{i \in I_{1+}} \gamma_i^{(k)} l_i + \sum_{i \in I_1} \alpha_i \gamma_i^{(k)} l_i.$$

Together with (3.213), (3.202) and (3.207) this implies that

$$|| \sum_{i \in I_{1+}} \gamma_i^{(k)} l_i + \sum_{i \in I_1} \alpha_i \gamma_i^{(k)} l_i|| \le k^{-1}(q + 2n + 1) + k^{-1}L_0. \tag{3.214}$$

It follows from (3.165) and (3.176) that there exists a finite strictly increasing sequence of natural numbers $i_1 < \cdots < i_q$, where q is a natural number, such that

$$\{i_1, \dots, i_q\} = I_{1+} \cup I_1.$$

Consider a mapping $G = (\bar{g}_{i_1}, \dots, \bar{g}_{i_q}) : X \to R^q$. In view of (3.214), (3.213), (3.148), (3.175), (3.176), (3.153) and (3.164),

$$\Xi_{G,\kappa}(y_k) \le k^{-1}(q + 2n + 1 + L_0) \text{ for each integer } k \ge k_0. \tag{3.215}$$

By (3.215), (A2), (3.206) and (3.202) there exists a subsequence $\{y_{k_p}\}_{p=1}^{\infty}$ of the sequence $\{y_k\}_{k=1}^{\infty}$ which converges to $y_* \in X$ in the norm topology:

$$\lim_{p\to\infty} ||y_{k_p} - y_*|| = 0. \tag{3.216}$$

Relations (3.215), (3.216) and Proposition 2.21 imply that

$$\Xi_{G,\kappa}(y_*) = 0. \tag{3.217}$$

By (3.216), (3.165), (3.163), (3.202), (3.205), (3.174) and (3.176) for $s \in \{1, \dots, q\}$

$$\bar{g}_{i_s}(y_*) - \bar{c}_{i_s} = \lim_{p\to\infty} (\bar{g}_{i_s}(y_{k_p}) - c_{i_s}^{(k_p)}) = \lim_{p\to\infty} (g_{i_s}^{(k_p}(y_{k_p}) - c_{i_s}^{(k_p)}) = 0. \tag{3.218}$$

For any

$$j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_q\},$$

we have $j \in I_{1-}$ and in view of (3.174), (3.176), (3.216), (3.202) and (3.163),

$$\bar{g}_j(y_*) = \lim_{p\to\infty} \bar{g}_j(y_{k_p)} = \lim_{p\to\infty} g_j^{(k_p})(y_{k_p}) \le \lim_{p\to\infty} c_j^{(k_p)} = \bar{c}_j.$$

Combined with (3.218) this implies that

$$y_* \in A(\bar{G}, \bar{c}). \tag{3.219}$$

Relation (3.218) implies that

$$\{i_1, \dots, i_q\} \subset I(y_*). \tag{3.220}$$

It follows from (3.216), (3.202), (3.162) and (3.204) that

$$\bar{f}(y_*) = \lim_{p\to\infty} \bar{f}(y_{k_p}) = \lim_{p\to\infty} f^{k_p}(y_{k_p}) \le \inf(\bar{f}; A(\bar{G}, \bar{c})).$$

Together with (3.219), (3.217) and (3.220) this implies that

$$y_* \in A(\bar{G}, \bar{c}),\ \bar{f}(y_*) = \inf(\bar{f}; A(\bar{G}, \bar{c})),\ \Xi_{G,\kappa}(y_*) = 0.$$

Combined with (3.220) this contradicts (A1). The contradiction we have reached proves that there exist positive numbers Λ_0, r such that the property (P1) holds. Theorem 3.16 is proved.

3.10 Comments

In this chapter we study the stability of the exact penalty. We show that exact penalty properties established in Chapter 2 are stable under perturbations of objective functions, constraint functions and the right-hand side of constraints. The stability results are established for equality-constrained problems and inequality-constrained problems with one constraint and with locally Lipschitz objective and constrained functions in Sections 3.1–3.3, for inequality-constrained problems with a lower semicontinuous objective function and convex lower semicontinuous constraint functions in Sections 3.4–3.7, and for inequality-constrained problems with locally Lipschitz objective and constraint functions in Sections 3.8 and 3.9. The stability of the exact penalty property is crucial in practice. One reason is that in practice we deal with a problem which consists of a perturbation of the problem we wish to consider. Another reason is that the computations introduce numerical errors.

4

Generic Well-Posedness of Minimization Problems

4.1 A generic variational principle

We will obtain many results of this chapter as a realization of a variational principle which will be considered in this section. This variational principle is a modification of the variational principle of [52] which has its prototype in the variational principle of Deville, Godefroy and Zizler [31].

We consider a metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology. (Note that these topologies can coincide.)

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$. In our study we use the following basic hypotheses about the functions.

(H1) For any $a \in \mathcal{A}$, any $\epsilon > 0$ and any $\gamma > 0$ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A} : \ d(a, b) < \epsilon\} \neq \emptyset$$

and for any $b \in \mathcal{W}$

(i) $\inf(f_b)$ is finite;

(ii) if $z \in X$ is such that $f_b(z) \leq \inf(f_b) + \eta$, then $\rho(z, x) \leq \gamma$ and $|f_b(z) - \alpha| \leq \gamma$.

(H2) if $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in X.

We show (see Theorem 4.1) that if (H1) and (H2) hold, then for a generic $a \in \mathcal{A}$ the problem minimize $f_a(x)$ subject to $x \in X$, has a unique solution. This result generalizes the variational principle in [52] which was obtained for

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the complete domain space (X, ρ). Note that if (X, ρ) is complete, the weak and strong topologies on $\mathcal{A}$ coincide and for any $a \in \mathcal{A}$ the function f_a is not identically ∞, then the variational principles in [52] and in this section are equivalent.

The variational principle which is proved in this section was suggested in [105]. For the classes of optimal control problems considered in [105] the domain space is not complete. Since the variational principle in [52] was established only for complete domain spaces it cannot be applied to these classes of optimal control problems. Fortunately, instead of the completeness assumption we can use (H2) and this hypothesis holds for spaces of integrands (integrand-map pairs) which satisfy the Cesari growth condition [105].

Given $a \in \mathcal{A}$ we say that the problem of minimization of f_a on X is well-posed with respect to data in $\mathcal{A}$ (or just with respect to $\mathcal{A}$) if the following assertions hold:

(1) $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$.

(2) For each $\epsilon > 0$ there are a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(x_a, z) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

(This property was introduced in [52]. In a slightly different setting a similar property was introduced in [146].)

Theorem 4.1. *Assume that (H1) and (H2) hold. Then there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem of f_a on X is well-posed with respect to $\mathcal{A}$.*

Following the tradition, we can summarize the theorem by saying that under the assumptions (H1) and (H2) the minimization problem for f_a on (X, ρ) is generically strongly well-posed with respect to $\mathcal{A}$ or that the minimization problem for f_a is well-posed with respect to $\mathcal{A}$ for a generic $a \in \mathcal{A}$.

Proof: Let $a \in \mathcal{A}$. By (H1) for any natural $n = 1, 2, \dots$ there are a nonempty open set $\mathcal{U}(a, n)$ in $\mathcal{A}$ with the weak topology, $x(a, n) \in X$, $\alpha(a, n) \in R^1$ and $\eta(a, n) > 0$ such that

$$\mathcal{U}(a, n) \cap \{b \in \mathcal{A} : \ d(a, b) < 1/n\} \ne \emptyset$$

and for any $b \in \mathcal{U}(a, n)$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \eta(a, n)$, then

$$\rho(z, x(a, n)) \le 1/n, \ |f_b(z) - \alpha(a, n)| \le 1/n.$$

Define $\mathcal{B}_n = \cup\{\mathcal{U}(a, m) : \ a \in \mathcal{A}, \ m \ge n\}$ for $n = 1, 2, \dots$. Clearly for each integer $n \ge 1$ the set $\mathcal{B}_n$ is open in the weak topology and everywhere dense in the strong topology. Set $\mathcal{B} = \cap_{n=1}^{\infty} \mathcal{B}_n$. Since for each integer $n \ge 1$ the set $\mathcal{B}_n$ is also open in the strong topology generated by the complete metric d we conclude that $\mathcal{B}$ is everywhere dense in the strong topology.

Let $b \in \mathcal{B}$. Evidently $\inf(f_b)$ is finite. There are a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$ and a strictly increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that $b \in \mathcal{U}(a_n, k_n)$, $n = 1, 2, \dots$. Assume that $\{z_n\}_{n=1}^{\infty} \subset X$ and $\lim_{n\to\infty} f_b(z_n) = \inf(f_b)$.

Let $m \geq 1$ be an integer. Clearly for all large enough n the inequality $f_b(z_n) < \inf(f_b) + \eta(a_m, k_m)$ is true and it follows from the definition of $\mathcal{U}(a_m, k_m)$ that

$$\rho(z_n, x(a_m, k_m)) \leq k_m^{-1},\ |f_b(z_n) - \alpha(a_m, k_m)| \leq k_m^{-1} \tag{4.1}$$

for all large enough n. Since m is an arbitrary natural number we conclude that $\{z_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence. By (H2) there is $\bar{x} = \lim_{n\to\infty} z_n$. As f_b is lower semicontinuous, we have $f_b(\bar{x}) = \inf(f_b)$. Clearly f_b does not have another minimizer for otherwise we would be able to construct a nonconvergent sequence $\{z_n\}_{n=1}^{\infty}$. This proves the first part of the theorem. We further note that by (4.1)

$$\rho(\bar{x}, x(a_m, k_m)) \leq k_m^{-1},\ |f_b(\bar{x}) - \alpha(a_m, k_m)| \leq k_m^{-1},\ m = 1, 2, \dots. \tag{4.2}$$

We turn now to the second assertion. Let $\epsilon > 0$. Choose a natural number m for which $4k_m^{-1} < \epsilon$. Let $a \in \mathcal{U}(a_m, k_m)$. Clearly $\inf(f_a)$ is finite. Let $z \in X$ and $f_a(z) \leq \inf(f_a) + \eta(a_m, k_m)$. By the definition of $\mathcal{U}(a_m, k_m)$,

$$\rho(z, x(a_m, k_m)) \leq k_m^{-1},\ |f_a(z) - \alpha(a_m, k_m)| \leq k_m^{-1}.$$

Together with (4.2) this implies that

$$\rho(z, \bar{x}) \leq 2k_m^{-1},\ |f_b(\bar{x}) - f_a(z)| \leq 2k_m^{-1} < \epsilon.$$

The second assertion is proved.

4.2 Two classes of minimization problems

Let (X, ρ) be a complete metric space. We consider two classes of minimization problems. Fix $\theta \in X$. Denote by $\mathcal{M}$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ which are not identically ∞ and satisfy

$$f(x) \to \infty \text{ as } \rho(x, \theta) \to \infty.$$

We study the existence of a solution of the minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in X \tag{P1}$$

with $f \in \mathcal{M}$. This is our first class of minimization problems. We endow the set $\mathcal{M}$ with an appropriate uniformity and show that for a generic $f \in \mathcal{M}$ the minimization problem (P1) has a unique solution.

Now we describe the second class of minimization problems. Denote by $\mathcal{L}$ the set of all functions $f \in \mathcal{M}$ such that the epigraph of f is the closure of its interior. Let $S(X)$ be the set of all nonempty closed subsets of X. The spaces $\mathcal{L}$ and $S(X)$ will be equipped with appropriate complete uniformities. We consider the following minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in A, \tag{P2}$$

where $A \in S(X)$ and $f \in \mathcal{L}$. We show that for a generic pair $(f, A) \in \mathcal{L} \times S(X)$ the minimization problem (P2) has a unique solution.

We equip the space of functions (respectively, function-set pairs) with weak and strong topologies. We construct a subset of this space which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets such that for each function (respectively, function-set pair) the corresponding minimization problem has a unique solution.

In the sequel we use the following notation.

For each function $f : Y \to R^1 \cup \{\infty\}$ where Y is a nonempty set we define

$$\mathrm{dom}(f) = \{y \in Y : f(y) < \infty\}, \ \inf(f) = \inf\{f(y) : y \in Y\}$$

and

$$\mathrm{epi}(f) = \{(y, \alpha) \in Y \times R^1 : \alpha \geq f(y)\}.$$

4.3 The generic existence result for problem (P1)

Let (X, ρ) be a complete metric space. Fix $\theta \in X$. Denote by $\mathcal{M}$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ such that

$$\mathrm{dom}(f) \neq \emptyset \text{ and } f(x) \to \infty \text{ as } \rho(x, \theta) \to \infty. \tag{4.3}$$

We equip the set $\mathcal{M}$ with strong and weak topologies.

For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E_s(n) = \{(f, g) \in \mathcal{M} \times \mathcal{M} :$$
$$f(x) \leq g(x) + n^{-1} \text{ and } g(x) \leq f(x) + n^{-1} \text{ for all } x \in X\}, \tag{4.4}$$

where n is a natural number. Clearly this uniform space $\mathcal{M}$ is metrizable and complete. Denote by τ_s the topology in $\mathcal{M}$ induced by this uniformity. The topology τ_s is called the strong topology.

Now we equip the set $\mathcal{M}$ with a weak topology. We consider the complete metric space $X \times R^1$ with the metric $\Delta(\cdot, \cdot)$ defined by

$$\Delta((x_1, \alpha_1), (x_2, \alpha_2)) = \rho(x_1, x_2) + |\alpha_1 - \alpha_2|, \ x_1, x_2 \in X \ , \alpha_1, \alpha_2 \in R^1. \tag{4.5}$$

For each lower semicontinuous bounded from below function $f : X \to R^1 \cup \{\infty\}$ with a nonempty epigraph define a function $\Delta_f : X \times R^1 \to R^1$ by

$$\Delta_f(x,\alpha) = \inf\{\Delta((x,\alpha),(y,\beta)) : (y,\beta) \in \text{epi}(f)\},\ (x,\alpha) \in X \times R^1. \quad (4.6)$$

For each natural number n denote by $E_w(n)$ the set of all pairs $(f,g) \in \mathcal{M} \times \mathcal{M}$ which have the following property:

C(i) For each $(x,\alpha) \in X \times R^1$ satisfying $\rho(x,\theta) + |\alpha| \le n$,

$$|\Delta_f(x,\alpha) - \Delta_g(x,\alpha)| \le n^{-1}; \quad (4.7)$$

C(ii) For each $(x,\alpha) \in X \times R^1$ satisfying

$$\alpha \le n,\ \min\{\Delta_f(x,\alpha), \Delta_g(x,\alpha)\} \le n \quad (4.8)$$

the inequality (4.7) is valid.

We will show (see Section 4.4, Lemma 4.7) that for each natural number n and each $(f,g),(g,h) \in E_w(2n)$ the relation $(f,h) \in E_w(n)$ is true. Therefore for the set $\mathcal{M}$ there exists the uniformity generated by the base $E_w(n)$, $n = 1,2,\dots$. This uniformity is metrizable (by a metric $\widehat{\Delta}_w$) and it induces in $\mathcal{M}$ a topology τ_w which is weaker than τ_s. The topology τ_w is called the weak topology. The following auxiliary result establishes an important property of the weak topology τ_w.

Proposition 4.2. *Let $f \in \mathcal{M}$ and M be a positive number. Then there exists a neighborhood U of f in $\mathcal{M}$ with the weak topology τ_w and a positive number r such that for each $g \in U$ and each $x \in X$ satisfying $\rho(\theta,x) \ge r$ the relation $g(x) > M$ holds.*

Proof: There exists $r > 4$ such that

$$f(x) \ge 2M + 4 \text{ for all } x \in X \text{ satisfying } \rho(x,\theta) \ge r/2 - 1.$$

Fix an integer $n_0 > 4M + 4$ and put

$$U = \{g \in \mathcal{M} : (f,g) \in E_w(n_0)\}.$$

Assume that

$$g \in U,\ x \in X \text{ and } \rho(x,\theta) \ge r. \quad (4.9)$$

Assume that $g(x) \le M$. Then it follows from the definition of U, (4.7) and (4.8) that the inequality $\Delta_f(x, g(x)) < 1$ holds and there exists $(y,\alpha) \in \text{epi}(f)$ such that $\rho(y,x) \le 1$ and $|\alpha - g(x)| \le 1$. Combined with (4.9) this implies that $\rho(\theta,y) \ge 2^{-1}r$ and $f(y) \le \alpha \le g(x) + 1 \le M + 1$. This is contradictory to the definition of r. Proposition 4.2 is proved.

For the set $\mathcal{M}$ we consider the metrizable uniformity determined by the following base:

$$\mathcal{E}(n) = \{(f,g) \in \mathcal{M} \times \mathcal{M} : (4.7) \text{ is valid for all } (x,\alpha) \in X \times R^1$$
$$\text{satisfying } \rho(x,\theta) + |\alpha| \le n\} \quad (4.10)$$

where $n = 1, 2, \ldots$. Denote by τ_* the topology induced by this uniformity. The topology τ_* is called the epi-distance topology [2]. Clearly the topology τ_* is weaker than τ_w.

Let $\phi \in \mathcal{M}$. Denote by $\mathcal{M}(\phi)$ the set of all $f \in \mathcal{M}$ satisfying $f(x) \geq \phi(x)$ for all $x \in X$. It is easy to verify that $\mathcal{M}(\phi)$ is a closed subset of $\mathcal{M}$ with the topology τ_w. We consider the topological subspace $\mathcal{M}(\phi) \subset \mathcal{M}$ with the relative weak and strong topologies. Clearly the topologies τ_w and τ_* induce the same relative topology on $\mathcal{M}(\phi)$. In [52] the generic existence result was established for the minimization problem (P1) with a cost function f belonging to the space $\mathcal{M}(\phi)$ endowed with the epi-distance topology (the strong and the weak topologies in $\mathcal{M}(\phi)$ coincide). Since the epi-distance topology τ_* in $\mathcal{M}$ does not have the property established in Proposition 4.2 we cannot prove our generic existence result for the space $\mathcal{M}$ when its weak topology is τ_*. The following proposition shows that τ_w is the weakest topology among all topologies which are stronger than the epi-distance topology and have the property established in Proposition 4.2.

Proposition 4.3. *Assume that a topology τ in $\mathcal{M}$ is stronger than τ_* and has the following property:*

For each $f \in \mathcal{M}$ and each positive number M there exist a neighborhood U of f in $\mathcal{M}$ with the topology τ and a positive number r such that

$$\inf\{\inf(g) : g \in U\} > -\infty \text{ and}$$

$$g(x) > M \text{ for all } g \in U \text{ and all } x \in X \text{ satisfying } \rho(x, \theta) \geq r.$$

Then the topology τ is stronger than τ_w.

Proof: Assume that $f \in \mathcal{M}$ and n is a natural number. Fix an integer $n_0 \geq 4n + 4$. There exist a neighborhood U_1 of f in $\mathcal{M}$ with the topology τ and a positive number r such that $g(x) > 2n_0 + 2$ for all $g \in U_1$ and all $x \in X$ satisfying $\rho(x, \theta) \geq r$ and $\inf_{g \in U_1} \inf(g) > -\infty$. Fix a natural number $n_1 > 2r + 2n_0 + 2 + 4|\inf_{g \in U_1} \inf(g)|$ and put

$$U = \{g \in \mathcal{M} : \ (f, g) \in \mathcal{E}(n_1)\} \cap U_1.$$

Clearly,

$$U \subset \{g \in \mathcal{M} : \ (f, g) \in E_w(n)\}.$$

This completes the proof of the proposition.

Set $\mathcal{A} = \mathcal{M}$ and $f_a = a$ for each $a \in \mathcal{A}$. In this chapter we prove the following two results.

Theorem 4.4. *The minimization problem for a generic $f \in \mathcal{M}$ is well-posed with respect to $\mathcal{M}$.*

Theorem 4.5. *Let $\phi \in \mathcal{M}$. The minimization problem for a generic $f \in \mathcal{M}(\phi)$ is well-posed with respect to $\mathcal{M}(\phi)$.*

Note that Theorems 4.4 and 4.5 have been obtained in [110].

4.4 The weak topology on the space $\mathcal{M}$

In this section we study the weak topology τ_w.

Lemma 4.6. *Let n be a natural number and let $f \in \mathcal{M}$. Then there exists a positive number a such that for each $g \in \mathcal{M}$ satisfying $(f, g) \in G(n)$ the inequality $g(x) \geq -a$ holds for all $x \in X$.*

Proof: There exists a positive number a_0 such that $f(x) \geq -a_0$ for all $x \in X$. Fix a number $a > 2a_0 + 4$. Assume that $g \in \mathcal{M}$, $(f, g) \in G(n)$ and $x \in X$. We will show that $g(x) \geq -a$. Assume the contrary. Then $g(x) < -a$ and $(x, -a) \in \mathrm{epi}(g)$. By the definition of $E_w(n)$, $\Delta_f(x, -a) \leq n^{-1}$. This implies that there exists $(y, \beta) \in \mathrm{epi}(f)$ such that $\Delta((x, -a), (y, \beta)) \leq 2$. Thus $\beta \geq f(y)$, $\beta \leq 2 - a$ and $f(y) \leq 2 - a < -2a_0 - 2$. This is contradictory to the definition of a_0. The contradiction we have reached proves the lemma.

The following lemma implies that the collection of the sets $E_w(n)$, $n = 1, 2, \ldots$ is the base of a uniformity.

Lemma 4.7. *For each natural number n and each $(f, g), (g, h) \in E_w(2n)$ the inclusion $(f, h) \in E_w(n)$ holds.*

Proof: Let n be a natural number,

$$(f, g) \text{ and } (g, h) \in E_w(2n). \tag{4.11}$$

It follows from (4.11) and property C(i) that for each $(x, \alpha) \in X \times R^1$ satisfying

$$\rho(x, \theta) + |\alpha| \leq n \tag{4.12}$$

the following inequalities hold:

$$|\Delta_f(x, \alpha) - \Delta_g(x, \alpha)| \leq (2n)^{-1}, \ |\Delta_g(x, \alpha) - \Delta_h(x, \alpha)| \leq (2n)^{-1}$$

and

$$|\Delta_f(x, \alpha) - \Delta_h(x, \alpha)| \leq n^{-1}. \tag{4.13}$$

Assume that $(x, \alpha) \in X \times R^1$,

$$\alpha \leq n \text{ and } \min\{\Delta_f(x, \alpha), \Delta_h(x, \alpha)\} \leq n. \tag{4.14}$$

We show that (4.13) holds. We may assume that $\Delta_f(x, \alpha) \leq n$. By this inequality, (4.14), (4.11) and property C(ii)

$$|\Delta_f(x, \alpha) - \Delta_g(x, \alpha)| \leq (2n)^{-1}, \ \Delta_g(x, \alpha) \leq 2n. \tag{4.15}$$

It follows from these inequalities, (4.14), (4.11) and property C(ii) that

$$|\Delta_g(x, \alpha) - \Delta_h(x, \alpha)| \leq (2n)^{-1}. \tag{4.16}$$

Combined with (4.15) this implies (4.13). Lemma 4.7 is proved.

Proposition 4.8. *The metric space* $(\mathcal{M}, \widehat{\Delta}_w)$ *is complete.*

Proof: Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}$ be a Cauchy sequence. We show that it converges in $(\mathcal{M}, \widehat{\Delta}_w)$. For each natural number n there exists a natural number $q(n)$ such that

$$(f_i, f_j) \in E_w(n) \text{ for all integers } i, j \geq q(n). \tag{4.17}$$

We may assume that

$$q(n+1) > q(n) + 4, \; n = 1, 2, \dots. \tag{4.18}$$

We consider a collection of all closed nonempty subsets of $X \times R^1$ denoted by $S(X \times R^1)$. For the set $S(X \times R^1)$ we consider the uniformity determined by the following base:

$$\tilde{E}(n) = \{(A, B) \in S(X \times R^1) \times S(X \times R^1) : \; \sup\{|\inf_{y \in A} \Delta((x, \alpha), y) - \tag{4.19}$$

$$\inf_{y \in B} \Delta((x, \alpha), y)| : \; x \in X, \; \alpha \in R^1 \text{ and } \rho(x, \theta) + |\alpha| \leq n\} < n^{-1}\},$$

where $n = 1, 2 \dots$. It is well-known [2] that the uniform space $S(X \times R^1)$ is metrizable and complete. It is easy to see that the sequence $\{\text{epi}(f_j)\}_{j=1}^{\infty}$ is a Cauchy sequence in the uniform space $S(X \times R^1)$. There exists a set $\Omega \in S(X \times R^1)$ such that

$$\Omega = \lim_{j \to \infty} \text{epi}(f_j) \tag{4.20}$$

in the uniform space $S(X \times R^1)$. Clearly,

$$\Omega = \{(x, \alpha) \in X \times R^1 : \; \text{there exists a strictly increasing sequence of natural} \tag{4.21}$$

$$\text{numbers } \{j_k\}_{k=1}^{\infty} \text{ and a sequence} \{(x_k, \alpha_k)\}_{k=1}^{\infty} \subset X \times R^1 \text{ such that}$$

$$(x_k, \alpha_k) \in \text{ epi}(f_{j_k}), \; k = 1, 2, \dots \text{ and } \lim_{k \to \infty} (x_k, \alpha_k) = (x, \alpha)\}.$$

Lemma 4.6 implies that there exists a positive number $\bar{a}$ such that for $j = 1, 2, \dots$

$$f_j(x) \geq -\bar{a} \text{ for all } x \in X. \tag{4.22}$$

Clearly, there exists a lower semicontinuous function $f : X \to R^1 \cup \{\infty\}$ such that

$$f(x) \geq -\bar{a} \text{ for all } x \in X \tag{4.23}$$

and

$$\Omega = \{(x, \alpha) \in X \times R^1 : \; \alpha \geq f(x)\} = \text{ epi}(f). \tag{4.24}$$

In view of (4.20), (4.19), (4.24) and (4.6) for each natural number n there is an integer $q_0(n) \geq q(n)$ such that the following property holds:

(a) for each natural number $j \geq q_0(n)$ and each $(x, \alpha) \in X \times R^1$ satisfying $\rho(x, \theta) + |\alpha| \leq n$ the inequality $|\Delta_f(x, \alpha) - \Delta_{f_j}(x, \alpha)| \leq n^{-1}$ holds.

Let $n \geq 1$ and $j \geq q(2n+1)$ be natural numbers and let $(x, \alpha) \in X \times R^1$. We show that if

$$\alpha \leq n \text{ and } \min\{\Delta_f(x,\alpha), \Delta_{f_j}(x,\alpha)\} \leq n, \tag{4.25}$$

then

$$|\Delta_f(x,\alpha) - \Delta_{f_j}(x,\alpha)| \leq n^{-1}. \tag{4.26}$$

Assume that (4.25) holds. There are two cases:

$$(1)\ \Delta_f(x,\alpha) \leq \Delta_{f_j}(x,\alpha); \tag{4.27}$$

$$(2)\ \Delta_f(x,\alpha) > \Delta_{f_j}(x,\alpha). \tag{4.28}$$

Consider the case (1). Then

$$\Delta_f(x,\alpha) \leq n \tag{4.29}$$

and there exists

$$(y,\beta) \in \text{ epi}(f) \tag{4.30}$$

such that

$$\rho(y,x) + |\alpha - \beta| \leq \Delta_f(x,\alpha) + (16n)^{-1} \leq n + (16n)^{-1}. \tag{4.31}$$

In view of (4.31) and (4.25)

$$\beta \leq 2n + 1/2. \tag{4.32}$$

It follows from (4.30), (4.24) and (4.21) that there exists a natural number $i > j + 4$ and

$$(z,\gamma) \in \text{ epi}(f_i) \tag{4.33}$$

such that

$$\rho(z,y) + |\gamma - \beta| \leq (16n)^{-1}. \tag{4.34}$$

It follows from (4.17) that

$$(f_i, f_j) \in E_w(2n+1).$$

By the inclusion above, the definition of $E_w(2n+1)$ (see property C(ii)), (4.33), (4.32) and (4.34),

$$\Delta_{f_j}(z,\gamma) \leq (2n+1)^{-1}.$$

Thus there exists $(u,\xi) \in$ epi(f_j) such that

$$\rho(z,u) + |\gamma - \xi| < (2n)^{-1}.$$

In view of the inequality above and (4.34),

$$\rho(u, y) + |\beta - \xi| < (2n)^{-1} + (16n)^{-1}.$$

Combined with (4.31) this implies that

$$\rho(x, u) + |\xi - \alpha| < (2n)^{-1} + (16n)^{-1} + \rho(y, x) + |\alpha - \beta| < \Delta_f(x, \alpha) + n^{-1},$$

$$\Delta_{f_j}(x, \alpha) < n^{-1} + \Delta_f(x, \alpha).$$

Hence in case (1) the inequality (4.26) holds.

Consider the case (2). We have

$$\Delta_{f_j}(x, \alpha) \leq n. \tag{4.35}$$

Assume that an integer $i \geq j$. In view of (4.17),

$$(f_i, f_j) \in E_w(2n + 1).$$

By the inclusion above, the definition of $E_w(2n + 1)$ (see property C(i)) and (4.25),

$$|\Delta_{f_i}(x, \alpha) - \Delta_{f_j}(x, \alpha)| \leq (2n + 1)^{-1}. \tag{4.36}$$

There exists $(x_i, \alpha_i) \in \text{epi}(f_i)$ for which

$$\rho(x, x_i) + |\alpha_i - \alpha| \leq \Delta_{f_i}(x, \alpha) + (16n)^{-1}. \tag{4.37}$$

Consider the sequence $\{(x_i, \alpha_i)\}_{i=j}^{\infty} \subset X \times R^1$. It follows from (4.37), (4.36) and (4.35) that for all integers $i \geq j$

$$|\alpha_i| + \rho(x_i, \theta) \leq |\alpha_i| + \rho(x, x_i) + \rho(x, \theta) \leq |\alpha| + 2 + n + \rho(x, \theta). \tag{4.38}$$

Fix a natural number

$$p_0 > |\alpha| + 4n + 2 + \rho(x, \theta) + 4.$$

In view of the choice of p_0 and (4.38),

$$|\alpha_i| + \rho(x_i, \theta) < p_0 \text{ for all integers } i \geq j. \tag{4.39}$$

Choose a natural number

$$p > j + q_0(p_0).$$

Property (a), (4.39) and the choice of p imply that

$$\Delta_f(x_p, \alpha_p) \leq p_0^{-1}.$$

Since (4.36) and (4.37) hold with $i = p$ it follows from the inequality above and the choice of p_0 that

$$\Delta_f(x, \alpha) \leq \rho(x, x_p) + |\alpha_p - \alpha| + \Delta_f(x_p, \alpha_p) \leq p_0^{-1} + \Delta_{f_p}(x, \alpha) + (16n)^{-1}$$

$$\leq \Delta_{f_j}(x, \alpha) + (2n + 1)^{-1} + (16n)^{-1} + p_0^{-1} \leq \Delta_{f_j}(x, \alpha) + n^{-1}.$$

Therefore in case (2) relation (4.26) is also valid.

We have shown that the following property holds:

(b) For each natural number n, each integer $j \geq q(2n+1)$ and each $(x, \alpha) \times X \times R^1$ satisfying (4.25) the inequality (4.26) is valid.

To complete the proof of the proposition we need to verify that $f \in \mathcal{M}$. It is sufficient to show that $f(x) \to \infty$ as $\rho(x, \theta) \to \infty$. This follows from property (b). Proposition 4.8 is proved.

4.5 Proofs of Theorems 4.4 and 4.5

We can easily prove the following auxiliary result.

Lemma 4.9. *Let $f \in \mathcal{M}$ and let δ be a positive number. Then there exists a neighborhood U of f in $\mathcal{M}$ with the weak topology such that for each $g \in U$*

$$\inf\{g(x) :\ x \in X\} \leq \inf\{f(x) :\ x \in X\} + \delta.$$

Lemma 4.10. *Let $f \in \mathcal{M}$ and let γ be a positive number. Then there exists a neighborhood U of f in $\mathcal{M}$ with the weak topology such that for each $g \in U$*

$$\inf_{x \in X} f(x) < \inf_{x \in X} g(x) + \gamma. \tag{4.40}$$

Proof: Fix an integer n such that

$$n > |\inf_{x \in X} f(x)| + 4 + 8\gamma^{-1} \tag{4.41}$$

and put

$$U = \{g \in \mathcal{M} :\ (g, f) \in E_w(n)\}. \tag{4.42}$$

Let $g \in U$. We show that (4.40) holds. There exists $x_0 \in X$ such that

$$g(x_0) \leq \inf_{x \in X} g(x) + \gamma/4. \tag{4.43}$$

We may assume that

$$g(x_0) \leq n. \tag{4.44}$$

In view of the definition of $E_w(n)$ (see property C(ii), (4.8)), (4.42) and (4.44),

$$\Delta_f(x_0, g(x_0)) \leq n^{-1}.$$

There exists $(x_1, \alpha_1) \in \text{epi}(f)$ such that

$$\rho(x_0, x_1) + |g(x_0) - \alpha_1| \leq 2n^{-1}. \tag{4.45}$$

(4.45), (4.43) and (4.41) imply that

$$f(x_1) \le \alpha_1 \le g(x_0) + 2n^{-1} \le \inf_{x \in X} g(x) + \gamma/4 + 2n^{-1}$$

$$\le \inf_{x \in X} g(x) + \gamma/2.$$

This implies (4.40). Lemma 4.10 is proved.

Denote by $\mathcal{E}$ the set of all $f \in \mathcal{M}$ for which there exists $x_f \in X$ such that $f(x_f) = \inf\{f(x) : x \in X\}$.

Lemma 4.11. *Let $f \in \mathcal{M}$. Then there exists a sequence $f_n \in \mathcal{E}$, $n = 1,,2\dots$ such that $f_n(x) \ge f(x)$ for all $x \in X$ and $n = 1, 2, \dots$ and $f_n \to f$ as $n \to \infty$ in the strong topology.*

Proof: For each natural number n there exists $x_n \in X$ such that $f(x_n) \le \inf\{f(x) : x \in X\} + 1/n$. For $n = 1, 2, \dots$ define

$$f_n(x) = \max\{f(x), f(x_n)\} \text{ for all } x \in X.$$

It is easy to see that $f_n \in \mathcal{E}$, $n = 1, 2\dots$ and $\lim_{n\to\infty} f_n = f$ in the strong topology. Lemma 4.11 is proved.

Lemma 4.12. *Let $f \in \mathcal{E}$, $x_f \in X$,*

$$f(x_f) = \inf\{f(x) : x \in X\},\ \epsilon \in (0,1),\ \delta \in (0, 16^{-1}\epsilon^2). \tag{4.46}$$

Define a function $\bar{f} \in \mathcal{E}$ by

$$\bar{f}(x) = f(x) + \epsilon/2 \min\{\rho(x, x_f), 1\} \text{ for all } x \in X.$$

Then there exists a neighborhood U of $\bar{f}$ in $\mathcal{M}$ with the weak topology such that for each $g \in U$ and each $x \in X$ satisfying $g(x) \le \inf(g) + \delta$,

$$|g(x) - f(x_f)| \le \epsilon \text{ and } \rho(x, x_f) \le \epsilon. \tag{4.47}$$

Proof: Lemmas 4.9 and 4.10 imply that there exists a neighborhood U_0 of $\bar{f}$ in $\mathcal{M}$ with the weak topology such that

$$|\inf(g) - \inf(\bar{f})| < \delta/2 \text{ for all } g \in U_0. \tag{4.48}$$

Fix an integer

$$n_0 > |\inf(f)| + 4 + 4\delta^{-1} \tag{4.49}$$

and put

$$U = U_0 \cap \{g \in \mathcal{M} :\ (\bar{f}, g) \in E_w(n_0)\}. \tag{4.50}$$

Let $g \in U$ and let $x \in X$ satisfy

$$g(x) \le \inf(g) + \delta. \tag{4.51}$$

In view of (4.51) and (4.48),

$$g(x) \le \inf(\bar{f}) + 2\delta. \tag{4.52}$$

It follows from (4.49), (4.50), (4.52), (4.7) and (4.8) that there exist $(y, \alpha) \in \text{epi}(\bar{f})$ such that

$$\rho(y, x) \le 2n_0^{-1} \text{ and } |\alpha - (\inf(\bar{f}) + 2\delta)| \le 2n_0^{-1}. \tag{4.53}$$

By (4.53), (4.49), (4.46) and the definition of $\bar{f}$,

$$f(y) + \epsilon/2 \min\{\rho(y, x_f), 1\} = \bar{f}(y) \le \inf(\bar{f}) + 2\delta + 2n_0^{-1} \le \inf(\bar{f}) + 3\delta \tag{4.54}$$

$$= f(x_f) + 3\delta \le f(y) + 3\delta, \ \rho(y, x_f) \le \epsilon/2.$$

It is easy to see that (4.47) follows from (4.54), (4.53), (4.52) and (4.48). Lemma 4.12 is proved.

To complete the proof of Theorems 4.4 and 4.5 it is sufficient to note that (H1) follows from Lemmas 4.11 and 4.12.

4.6 An extension of Theorem 4.4

Denote by $\mathcal{M}_b$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ which are not identically infinity. We equip the set $\mathcal{M}_b$ with a weak and a strong topology.

For each natural number n denote by $G_w(n)$ the set of all $(f, g) \in \mathcal{M}_b \times \mathcal{M}_b$ such that

$$\sup\{\Delta_f(x, \alpha) : (x, \alpha) \in \text{epi}(g)\}, \ \sup\{\Delta_g(x, \alpha) : (x, \alpha) \in \text{epi}(f)\} \le n^{-1}.$$

For the set $\mathcal{M}_b$ we consider the complete metrizable uniformity determined by the base $G_w(n)$, $n = 1, 2, \dots$. We equip the space $\mathcal{M}_b$ with a topology τ_w induced by this uniformity. The topology τ_w is called a weak topology.

Also for the set $\mathcal{M}_b$ we consider the complete metrizable uniformity determined by the following base:

$$U(n) = \{(f, g) \in \mathcal{M}_b \times \mathcal{M}_b :$$

$$f(x) \le g(x) + n^{-1} \text{ and } g(x) \le f(x) + n^{-1}, \ x \in X\}.$$

The space $\mathcal{M}_b$ is endowed with the topology τ_s induced by this uniformity. The topology τ_s is called a strong topology.

Set $\mathcal{A} = \mathcal{M}_b$ and $f_a = a$ for each $a \in \mathcal{A}$. We will establish the following result.

Theorem 4.13. *The minimization problem for a generic $f \in \mathcal{M}_b$ is well-posed with respect to $\mathcal{M}_b$.*

In order to prove Theorem 4.13 we need the following two lemmas. The first lemma is proved in a straightforward manner.

Lemma 4.14. *Assume that $f \in \mathcal{M}_b$ and that ϵ is a positive number. Then there exists a neighborhood U_0 of f in $\mathcal{M}_b$ with the weak topology such that*

$$|\inf(g) - \inf(f)| \le \epsilon \text{ for all } g \in U.$$

Denote by $\mathcal{E}$ the set of all $f \in \mathcal{M}_b$ such that there exists $x_f \in X$ satisfying $f(x_f) = \inf\{f(x) : x \in X\}$. Analogously to Lemma 4.11 we can show that $\mathcal{E}$ is everywhere dense in $\mathcal{M}_b$ with the strong topology.

Lemma 4.15. *Let $f \in \mathcal{E}$, $x_f \in X$ and let*

$$f(x_f) = \inf\{f(x) : x \in X\}, \ \epsilon \in (0,1), \ \delta \in (0, 16^{-1}\epsilon^2). \tag{4.55}$$

Define a function $\bar{f} \in \mathcal{E}$ by

$$\bar{f}(x) = f(x) + 2^{-1}\epsilon \min\{\rho(x, x_f), 1\} \text{ for all } x \in X. \tag{4.56}$$

Then there exists a neighborhood U of $\bar{f}$ in $\mathcal{M}$ with the weak topology such that for each $g \in U$ and each $x \in X$ satisfying $g(x) \le \inf(g) + \delta$ the following relations hold:

$$|g(x) - f(x_f)| \le \epsilon \text{ and } \rho(x, x_f) \le \epsilon. \tag{4.57}$$

Proof: There exist a neighborhood U of $\bar{f}$ in $\mathcal{M}_b$ with the weak topology such that for each $g \in U$

$$\Delta_g(x,\alpha) \le 16^{-1}\delta, \ (x,\alpha) \in \text{ epi}(\bar{f}) \text{ and } \Delta_{\bar{f}}(x,\alpha) \le 16^{-1}\delta, \ (x,\alpha) \in \text{ epi}(g). \tag{4.58}$$

Let $g \in U$, $x \in X$ and $g(x) \le \inf(g) + \delta$. Combined with (4.58) this implies that

$$g(x) \le \inf(\bar{f}) + 2\delta \text{ and } g(x) \ge \inf(\bar{f}) - 2\delta. \tag{4.59}$$

In view of (4.58) there exists $(y, \alpha) \in \text{epi}(\bar{f})$ such that

$$\rho(y, x) \le 8^{-1}\delta \text{ and } |\alpha - (\inf(\bar{f}) + 2\delta)| \le 8^{-1}\delta. \tag{4.60}$$

By (4.56) and (4.59),

$$f(y) + 2^{-1}\epsilon \min\{\rho(y, x_f), 1\} = \bar{f}(y) \le \inf(\bar{f}) + 2\delta + 8^{-1}\delta$$

$$\le f(x_f) + 3\delta \le f(y) + 3\delta, \ \rho(y, x_f) \le 2^{-1}\epsilon.$$

It is easy to see that (4.57) follows from the relation above, (4.60) and (4.59). This completes the proof of Lemma 4.15.

To complete the proof of Theorem 4.13 it is sufficient to note that (H1) follows from Lemma 4.15.

Note that Theorem 4.13 has been obtained in [110].

4.7 The generic existence result for problem (P2)

We consider a complete metric space (X, ρ). Fix $\theta \in X$ and set $||x|| = \rho(x, \theta)$ for all $x \in X$ and define $B(r) = \{x \in X : \ ||x|| \le r\}$ for all $r > 0$. Denote by $\mathcal{L}$ the set of all lower semicontinuous bounded from below functions $f : X \to R^1$ which satisfy the following assumptions:

A(i) $\inf\{f(x) : \ x \in X \setminus B(r)\} \to \infty$ as $r \to \infty$.

A(ii) For each $x \in X$, each neighborhood U of x in X and each positive number δ there exists an open nonempty set $V \subset U$ such that $f(y) \le f(x) + \delta$ for all $y \in V$.

Denote by $\mathcal{L}_c$ the set of all finite-valued continuous functions $f \in \mathcal{L}$.

Remark 4.16. Let $f : X \to R^1$ be a lower semicontinuous bounded from below function. Then f satisfies assumption A(ii) if and only if epi(f) is the closure of its interior.

For $x \in X$ and $A \subset X$ set $\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}$. Denote by $S(X)$ the collection of all closed nonempty sets in X. We equip the set $S(X)$ with a weak and a strong topology.

For the set $S(X)$ we consider the uniformity determined by the following base:

$$\mathcal{G}_w(n) = \{(A, B) \in S(X) \times S(X) : \ |\rho(x, A) - \rho(x, B)| \le n^{-1} \text{ for all } x \in B(n)\},$$

where $n = 1, 2, \ldots$. It is well known that the set $S(X)$ with this uniformity is metrizable (by a metric H_w) and complete. The bounded Hausdorff (Attouch–Wets) topology induced by this uniformity is a weak topology in the space $S(X)$.

Also for the set $S(X)$ we consider the uniformity determined by the following base:

$$\mathcal{G}_s(n) = \{(A, B) \in S(X) \times S(X) : \ \rho(x, B) \le n^{-1} \text{ for all } x \in A$$

$$\text{and } \rho(y, A) \le n^{-1} \text{ for all } y \in B\}, \ n = 1, 2, \ldots.$$

It is well known that the space $S(X)$ with this uniformity is metrizable (by a metric H_s) and complete. The Hausdorff topology induced by this uniformity is a strong topology in the space $S(X)$.

For the space $\mathcal{L}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_0(n) = \{(f, g) \in \mathcal{L} \times \mathcal{L} : \ |f(x) - g(x)| \le n^{-1} \text{ for all } x \in B(n)\},$$

where $n = 1, 2, \ldots$. Evidently this uniform space is metrizable (by a metric h_0) and $\mathcal{L}_c$ is a closed subset of $\mathcal{L}$ with the topology induced by the metric h_0.

For each natural number n denote by $\mathcal{E}_w(n)$ the set of all pairs $(f, g) \in \mathcal{L} \times \mathcal{L}$ which have the following property:

B(i) For each $x \in B(n)$ the inequality $|f(x) - g(x)| \le n^{-1}$ is valid;

B(ii) For each $x \in X$ satisfying $\inf\{f(x), g(x)\} \le n$ the inequality $|f(x) - g(x)| \le n^{-1}$ is true.

The following auxiliary lemma shows that the collection of the sets $\mathcal{E}_w(n)$, $n = 1, 2, \dots$ is the base of a uniformity.

Lemma 4.17. *Let n be a natural number and let*

$$(f, g),\ (g, h) \in \mathcal{E}_w(2n). \tag{4.61}$$

Then $(f, h) \in \mathcal{E}_w(n)$.

Proof: It is easy to see that for each $x \in B(n)$

$$|f(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)| \le n^{-1}.$$

Assume that $x \in X$ and $\min\{f(x), h(x)\} \le n$. To complete the proof of the lemma it is sufficient to show that $|f(x) - h(x)| \le n^{-1}$.

We may assume that $f(x) \le n$. This inequality, B(ii) and (4.61) imply that

$$|f(x) - g(x)| \le (2n)^{-1},\ g(x) \le n + 1/2,\ |h(x) - g(x)| \le (2n)^{-1} \text{ and}$$

$$|f(x) - h(x)| \le n^{-1}.$$

Lemma 4.17 is proved.

For the set $\mathcal{L}$ we consider the uniformity determined by the base $\mathcal{E}_w(n)$, $n = 1, 2, \dots$. Clearly this uniformity is metrizable (by a metric h_w). We equip the set $\mathcal{L}$ with the topology induced by this uniformity. This topology is called the weak topology. Clearly $\mathcal{L}_c$ is a closed subset of $\mathcal{L}$ with the weak topology.

Proposition 4.18. *The metric space $(\mathcal{L}, h_w)$ is complete.*

Proof: Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{L}$ be a Cauchy sequence. It is sufficient to show that it converges in $\mathcal{L}$.

For each natural number n there is a natural number $q(n)$ such that

$$(f_i, f_j) \in \mathcal{E}_w(n) \text{ for all integers } i, j \ge q(n). \tag{4.62}$$

We may assume that

$$q(n+1) > q(n) + 4,\ n = 1, 2, \dots. \tag{4.63}$$

It follows from the definition of $q(n)$ (see (4.62)), properties B(i) and B(ii) that for each natural number n and each pair of integers $i, j \ge q(n)$ the following properties hold:

(a) For each $x \in B(n)$,

$$|f_j(x) - f_i(x)| \le n^{-1}. \tag{4.64}$$

(b) For each $x \in X$ satisfying $\min\{f_i(x), f_j(x)\} \le n$ the relation (4.64) holds.

This implies that for all $x \in X$ there is a number

$$f(x) = \lim_{j\to\infty} f_j(x). \tag{4.65}$$

Property (a) and (4.65) imply that the following property holds:

(c) For each natural number n and each integer $j \ge q(n)$,

$$|f_j(x) - f(x)| \le n^{-1} \text{ for all } x \in B(n). \tag{4.66}$$

Assume that $n \ge 1$ and $j \ge q(n+1)$ are integers and that $x \in X$. We show that if

$$\inf\{f_j(x), f(x)\} \le n, \tag{4.67}$$

then

$$|f_j(x) - f(x)| \le n^{-1}. \tag{4.68}$$

Assume that (4.67) holds. There are two cases: (1) $f_j(x) \le n$; (2) $f(x) \le n$. Consider the case (1). Then in view of property (b) for each natural number $i > j$ the inequality $|f_i(x) - f_j(x)| \le n^{-1}$ is valid. Combined with (4.65) this implies (4.68).

Consider the case (2). Then

$$f(x) \le n. \tag{4.69}$$

There exists an integer

$$i > q(n+1) + j \tag{4.70}$$

such that

$$|f(x) - f_i(x)| \le [n^{-1} - (n+1)^{-1}]/4. \tag{4.71}$$

In view of (4.71) and (4.69),

$$f_i(x) \le n + 1. \tag{4.72}$$

Property (b) (see (4.64)), (4.72) and (4.70) imply that

$$|f_j(x) - f_i(x)| \le (n+1)^{-1}. \tag{4.73}$$

By (4.73) and (4.71), $|f_j(x) - f(x)| \le n^{-1}$. Hence we have shown that the following property holds:

(d) For each natural number n, each natural number $j \ge q(n+1)$ and each $x \in X$ satisfying (4.67), the inequality (4.68) is valid.

Since the sequence $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on bounded subsets of X (see property (c)) we conclude that f is lower semicontinuous and satisfies assumption A(ii). By property (d), f is bounded from below and satisfies assumption A(i). Proposition 4.18 is proved.

For the set $\mathcal{L}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_s(\epsilon) = \{(f,g) \in \mathcal{L} \times \mathcal{L} : \ |f(x) - g(x)| \le \epsilon\}, \ x \in X \tag{4.74}$$

where ϵ is a positive number. Evidently this uniform space is metrizable (by a metric h_s) and complete.

For the space $\mathcal{L} \times S(X)$ we consider the product topology induced by the metric

$$d((f,A),(g,B)) = h_s(f,g) + H_s(A,B), \ (f,A), \ (g,B) \in \mathcal{L} \times S(X)$$

which is called a strong topology and consider the product topology induced by the metric

$$d_w((f,A),(g,B)) = h_w(f,g) + H_w(A,B), \ (f,A), \ (g,B) \in \mathcal{L} \times S(X)$$

which is called a weak topology.

Put $\mathcal{A} = \mathcal{L} \times S(X)$ and for each $a = (h,A) \in \mathcal{L} \times S(X)$ define $f_a : \ X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = h(x) \text{ for all } x \in A \text{ and } f_a(x) = \infty \text{ for all } x \in X \setminus A.$$

Let a function $\phi : \ X \to R^1 \cup \{\infty\}$ satisfy

$$\phi(x) \to \infty \text{ as } ||x|| \to \infty. \tag{4.75}$$

Define

$$\mathcal{L}(\phi) = \{f \in \mathcal{L} : \ f(x) \ge \phi(x) \text{ for all } x \in X\}.$$

We consider the topological subspaces $\mathcal{L}_c \times S(X), \ \mathcal{L}(\phi) \times S(X) \subset \mathcal{L} \times S(X)$ with the relative weak and strong topologies.

We prove the following result.

Theorem 4.19. *The minimization problem for f_a is well-posed with respect to $\mathcal{L} \times S(X)$ (respectively $\mathcal{L}_c \times S(X)$, $\mathcal{L}(\phi) \times S(X)$) for a generic $a \in \mathcal{L} \times S(X)$ (respectively $\mathcal{L}_c \times S(X)$, $\mathcal{L}(\phi) \times S(X)$).*

4.8 Proof of Theorem 4.19

We use the notations introduced in Sections 4.1 and 4.7. Consider the complete metric space $(\mathcal{L}, h_w)$. Denote by τ_0^f the topology induced by the metric h_0 and by τ_w^f the topology induced by the metric h_w. It is easy to see that the topology τ_w^f is stronger than the topology τ_0^f. We can easily prove the following auxiliary results.

Lemma 4.20. *Let $f \in \mathcal{L}$. Then the following assertions hold:*

1. There exist a neighborhood V of f in the space $\mathcal{L}$ with the topology τ_w^f and a real number a_0 such that

$$g(x) \geq a_0 \text{ for each } g \in V \text{ and each } x \in X. \tag{4.76}$$

2. For each positive number c there exist a positive number r and a neighborhood U of f in the space $\mathcal{L}$ with the topology τ_w^f such that $g(x) \geq c$ for each $g \in U$ and each $x \in X \setminus B(r)$.

Proposition 4.21. *Let τ be a Hausdorff topology in $\mathcal{L}$ which is stronger than τ_0^f and which has the following property:*

For each $f \in \mathcal{L}$ and each positive number c there exist a positive number r and a neighborhood U of f in the space $\mathcal{L}$ with the topology τ such that $g(x) \geq c$ for each $g \in U$ and each $x \in X \setminus B(r)$.

Then the topology τ is stronger than the topology τ_w^f.

Let $\phi : X \to R^1$ be a function such that $\phi(x) \to \infty$ as $||x|| \to \infty$. Clearly, $\mathcal{L}(\phi)$ is a closed subset of $\mathcal{L}$ with the topology τ_0^f and the topologies τ_w^f and τ_0^f induce the same relative topology for the subspace $\mathcal{L}(\phi) \subset \mathcal{L}$.

Proposition 4.22. *Let $f \in \mathcal{L}$, $A \in S(X)$ and $\epsilon, \gamma \in (0,1)$. Then there exist $\bar{A} \in S(X)$, $\bar{f} \in \mathcal{L}$, $\bar{x} \in X$, a positive number η and a neighborhood U of $(\bar{f}, \bar{A})$ in $\mathcal{L} \times S(X)$ with the weak topology such that*

$$\bar{A} = A \cup \{\bar{x}\},\ \rho(\bar{x}, A) < \epsilon_0,\ \bar{f}(x) = f(x) + \epsilon_0 \min\{1, \rho(x, \bar{x})\},\ x \in X \tag{4.77}$$

with $\epsilon_0 \in (0, \epsilon)$ and for each $(g, D) \in U$ and each $z \in D$ satisfying

$$g(z) \leq \inf\{g(y) :\ y \in D\} + \eta \tag{4.78}$$

the following inequalities hold:

$$\rho(z, \bar{x}) \leq \gamma,\ |g(z) - f(\bar{x})| \leq \gamma. \tag{4.79}$$

Proof: Fix positive numbers

$$\epsilon_0 < 2^{-1} \min\{\epsilon, \gamma\},\ \epsilon_1 < 4^{-1}\epsilon_0,\ \eta < 16^{-1}\epsilon_1\epsilon_0. \tag{4.80}$$

Put

$$\Omega = \{x \in X :\ \rho(x, A) < \epsilon_0\}. \tag{4.81}$$

Assumption A(ii) implies that there exists an open nonempty set $E \subset X$ such that

$$E \subset \Omega,\ f(z) \leq \inf\{f(u) :\ u \in \Omega\} + 16^{-1}\eta \text{ for all } z \in E. \tag{4.82}$$

Choose $\bar{x} \in E$ and define $\bar{A}$, $\bar{f}$ by (4.77). In view of assumption A(i) there exists an integer

$$n_1 > 4 + |f(\bar{x})| + 8^{-1}\gamma. \tag{4.83}$$

Lemma 4.20 implies that there exists a neighborhood V_1 of $\bar{f}$ in the topology τ_w^f and constants $c_0 \in R^1$ and $c_1 > 1$ such that

$$\inf\{g(x) : \ x \in X\} \geq c_0 \text{ for all } g \in V_1 \tag{4.84}$$

and

$$g(x) \geq 2n_1 + 4 \text{ for each } g \in V_1 \text{ and each } x \in X \setminus B(c_1). \tag{4.85}$$

Fix an integer

$$n_2 > 2c_1 + 2|c_0| + 2n_1 + 16(\epsilon_0 - \rho(\bar{x}, A))^{-1} + 8\rho(\bar{x}, \theta) + 8\eta^{-1} \tag{4.86}$$

such that

$$\{z \in X : \ \rho(z, \bar{x}) \leq 8n_2^{-1}\} \subset E. \tag{4.87}$$

Define

$$U = \{(g, B) \in V_1 \times S(X) : \ (\bar{f}, g) \in \mathcal{E}_w(n_2), \ (\bar{A}, B) \in \mathcal{G}_w(n_2)\}. \tag{4.88}$$

We show that for all $(g, D) \in U$

$$\inf\{g(u) : \ u \in D\} = \inf\{g(u) : \ u \in D \cap B(n_2/2 - 2)\}. \tag{4.89}$$

Let $(g, D) \in U$. In view of (4.88) there exists $y \in D$ such that

$$\rho(y, \bar{x}) \leq 2n_2^{-1}. \tag{4.90}$$

By (4.90), (4.87) and (4.82),

$$y \in E \subset \Omega, \ |f(y) - f(\bar{x})| \leq 8^{-1}\eta. \tag{4.91}$$

By (4.77), (4.90) and (4.91),

$$|\bar{f}(y) - \bar{f}(\bar{x})| \leq 8^{-1}\eta + 2\epsilon_0 n_2^{-1}. \tag{4.92}$$

By (4.90), (4.86), (4.88) and (4.83) $|g(y) - \bar{f}(y)| \leq n_2^{-1}$ and

$$\inf\{g(u) : \ u \in D\} \leq g(y) \leq \bar{f}(\bar{x}) + n_2^{-1} + 8^{-1}\eta + 2\epsilon_0 n_2^{-1} < n_1. \tag{4.93}$$

Thus (4.88) and (4.85) imply that

$$\inf\{g(u) : \ u \in D\} = \inf\{g(u) : \ u \in D \text{ and } g(u) \leq n_1 + 2\} \tag{4.94}$$

$$= \inf\{g(u) : \ u \in D \cap B(c_1)\} = \inf\{g(u) : \ u \in D \cap B(n_2/2 - 2)\}$$

and (4.89) holds. We have shown (see (4.93), (4.83)) that for all $(g, D) \in U$

$$\inf\{g(u) : \ u \in D\} \leq f(\bar{x}) + 2^{-1}\eta < n_1. \tag{4.95}$$

Assume that $(g, D) \in U$, $z \in D$ and (4.78) holds. By (4.78), (4.95), (4.85), (4.88) and (4.86),

$$z \in B(c_1) \subset B(n_2/2 - 1). \tag{4.96}$$

It follows from (4.96) and (4.88) that

$$|g(z) - \bar{f}(z)| \leq n_2^{-1}, \ \rho(z, \bar{A}) \leq n_2^{-1}. \tag{4.97}$$

By (4.97), (4.77), (4.81), (4.86) and (4.82),

$$z \in \Omega, \ f(z) \geq f(\bar{x}) - 16^{-1}\eta. \tag{4.98}$$

It follows from (4.77), (4.97), (4.95), (4.78), (4.98) and (4.87) that

$$\begin{aligned} & f(z) + \epsilon_0 \min\{1, \rho(z, \bar{x})\} \\ &= \bar{f}(z) \leq g(z) + n_2^{-1} \leq f(\bar{x}) + \eta + 2^{-1}\eta + n_2^{-1} \\ &\leq f(z) + 16^{-1}\eta + \eta + 2^{-1}\eta + n_2^{-1} \leq f(z) + 2\eta. \end{aligned} \tag{4.99}$$

By (4.99) and (4.82),

$$\rho(z, \bar{x}) \leq 2\eta\epsilon_0^{-1} \leq 8^{-1}\epsilon_1 < \gamma.$$

(4.95), (4.78), (4.97) and (4.98) imply that

$$-\gamma < -n_2^{-1} - 16^{-1}\eta < \bar{f}(z) - f(\bar{x}) - n_2^{-1} \leq g(z) - f(\bar{x}) \leq \eta + 2^{-1}\eta < \gamma.$$

This completes the proof of Proposition 4.22.

By Proposition 4.22 (H1) holds for the spaces $\mathcal{L} \times S(X)$, $\mathcal{L}_c \times S(X)$ and $\mathcal{L}(\phi) \times S(X)$. This implies Theorem 4.19.

It should be mentioned that Theorem 4.19 has been obtained in [110].

4.9 A generic existence result in optimization

In this section we consider a minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in C$$

where C is a nonempty closed subset of a complete metric space X and $f : X \to R^1$ belongs to a complete metric space of continuous functions defined on X which are bounded from below on C. We assume that the set C is the closure of its interior and show that for a generic function f the minimization problem has a unique solution and this solution is an interior point of C.

We use the convention that $\infty/\infty = 1$.

Let (X, ρ) be a complete metric space and C be a nonempty closed subset of X. For each $x \in X$ and each positive number r put

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\}.$$

We assume that C is the closure of its interior.

For each $f : X \to R^1$ and each subset $K \subset X$ set

$$\inf(f;K) = \inf\{f(x) : \ x \in K\}.$$

Denote by $\mathcal{M}$ the set of all continuous functions $f : X \to R^1$ which are bounded from below on C. For each $f, g \in \mathcal{M}$ define

$$\tilde{d}(f,g) = \sup\{|f(x) - g(x)| : \ x \in X\} \tag{4.100}$$

and

$$d(f,g) = \tilde{d}(f,g)(1 + \tilde{d}(f,g))^{-1}.$$

Clearly, the metric space $(\mathcal{M}, d)$ is complete. It is easy to see that for each $f \in \mathcal{M}$, $\inf(f;C)$ is finite.

Let $f \in \mathcal{M}$. According to the definition made in Section 4.1 we say that the problem minimize $f(x)$ subject to $x \in C$ is strongly well-posed if $\inf(f;C)$ is attained at a unique point $x_f \in C$ and the following assertion holds:

For each positive number ϵ there exists a positive number δ such that for each $g \in \mathcal{M}$ satisfying $d(f,g) \le \delta$ and each $z \in C$ which satisfies $g(z) \le \inf(g;C) + \delta$ the inequalities $\rho(x_f, z) \le \epsilon$ and $|g(z) - f(x_f)| \le \epsilon$ hold.

We prove the following result.

Theorem 4.23. *There exists a set $\mathcal{F} \subset (\mathcal{M}, d)$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, d)$ such that for each $f \in \mathcal{M}$ the problem minimize $f(x)$ subject to $x \in C$ is strongly well-posed and its unique minimizer is an interior point of C.*

Note that Theorem 4.23 was obtained in [121].

4.10 A basic lemma for Theorem 4.23

For each $A \subset X$ denote by $\mathrm{int}(A)$ the interior of A, by $\mathrm{cl}(A)$ the closure of A and by $\mathrm{bd}(A)$ the set $\mathrm{cl}(A) \setminus \mathrm{int}(A)$.

For each $x \in X$ each $A \subset X$ set

$$\rho(x, A) = \inf(\rho(x,y) : \ y \in A\}.$$

We need the following well-known result [58].

Proposition 4.24. *Let A and B be nonempty closed subsets of (X, ρ) such that $A \cap B = \emptyset$. Then there exists a continuous function $h : X \to [0,1]$ such that $h(x) = 1$ for all $x \in A$ and $h(x) = 0$ for all $x \in B$.*

The following result is our basic lemma for Theorem 4.23.

Lemma 4.25. *Let $f \in \mathcal{M}$ and ϵ be a positive number. Then there exist $f_* \in \mathcal{M}$ satisfying $d(f, f_*) \leq \epsilon$ and a neighborhood $\mathcal{U}$ of f_* in $(\mathcal{M}, d)$ such that for each $g \in \mathcal{U}$*

$$\inf(g; \mathrm{int}(C)) < \inf(g; \mathrm{bd}(C)). \tag{4.101}$$

Proof: Fix

$$\delta \in (0, 8^{-1} \min\{1, \epsilon\}) \tag{4.102}$$

and choose $x_0 \in C$ such that

$$f(x_0) \leq \inf(f; C) + \delta/4. \tag{4.103}$$

Since f is continuous there exists a neighborhood V_0 of x_0 in (X, ρ) such that

$$|f(x_0) - f(z)| \leq \delta/4 \text{ for all } z \in V. \tag{4.104}$$

Since C is the closure of $\mathrm{int}(C)$ there exists $x_* \in X$ such that

$$x_* \in V \cap \mathrm{int}(C). \tag{4.105}$$

In view of (4.105) and (4.104),

$$|f(x_0) - f(x_*)| \leq \delta/4. \tag{4.106}$$

By (4.103) and (4.106),

$$f(x_*) \leq f(x_0) + \delta/4 \leq \inf(f; C) + \delta/2. \tag{4.107}$$

Define $f_1 : X \to R^1$ by

$$f_1(x) = \max\{f(x), f(x_*)\} + 8^{-1}\epsilon \min\{1, \rho(x, x_*)\}, \ x \in X. \tag{4.108}$$

It is easy to see that $f_1 \in \mathcal{M}$ and that for each $x \in X$, $f(x) \leq f_1(x)$. It follows from (4.108), (4.107) and (4.102) that for each $x \in C$

$$\begin{aligned} 0 \leq f_1(x) - f(x) &\leq 8^{-1}\epsilon + \max\{f(x), f(x_*)\} - f(x) \\ &\leq 8^{-1}\epsilon + \max\{f(x), \inf(f; C) + \delta/2\} - f(x) \\ &\leq 8^{-1}\epsilon + \delta/2 \leq 3 \cdot 16^{-1}\epsilon. \end{aligned}$$

Therefore

$$0 \leq f_1(x) - f(x) < 3 \cdot 16^{-1}\epsilon \text{ for all } x \in C. \tag{4.109}$$

Since x_* is an interior point of C there exists a positive number γ such that

$$B(x_*, 4\gamma) \subset C. \tag{4.110}$$

Thus

$$\rho(x_*, \mathrm{bd}(C)) \geq 2\gamma. \tag{4.111}$$

Put

$$A = \{x \in C : \ \rho(x, \mathrm{bd}(C)) \ge 2^{-1} \cdot 3\gamma\} \tag{4.112}$$

and

$$B = X \setminus \{x \in C : \ \rho(x, \mathrm{bd}(C)) > \gamma\}. \tag{4.113}$$

It is easy to see that A is a closed subset of X, B is a closed subset of X and

$$A \cap B = \emptyset. \tag{4.114}$$

Proposition 4.24 implies that there exists a continuous function $\phi : X \to [0, 1]$ such that

$$\phi(x) = 1, \ x \in A \text{ and } \phi(x) = 0, \ x \in B. \tag{4.115}$$

Define

$$f_*(x) = f_1(x)\phi(x) + (1 - \phi(x))(f(x) + \delta), \ x \in X. \tag{4.116}$$

It is easy to see that $f_* \in \mathcal{M}$. It follows from (4.109), (4.115) and (4.116) that for each $x \in C$,

$$0 \le f_*(x) - f(x) = \phi(x)(f_1(x) - f(x)) + (1 - \phi(x))\epsilon \le \epsilon.$$

If $x \in X \setminus C$, then (4.113) and (4.115) imply that

$$x \in B, \ \phi(x) = 0 \text{ and } f_*(x) = f(x) + \epsilon.$$

Hence

$$0 \le f_*(x) - f(x) \le \epsilon \text{ for all } x \in X \text{ and } d(f, f_*) \le \epsilon. \tag{4.117}$$

We show that for each $x \in C$, $f_*(x) \ge f_*(x_*)$. Let $x \in C$. By (4.116), (4.108), (4.107) and (4.102),

$$f_*(x) \ge \phi(x) \max\{f(x), f(x_*)\} +$$

$$(1 - \phi(x))(f(x) + \epsilon) \ge$$

$$\phi(x)f(x_*) + (1 - \phi(x))(f(x) + \epsilon) \ge$$

$$\phi(x)f(x_*) + (1 - \phi(x))(\inf(f; C) + \epsilon) \ge f(x_*)$$

and

$$f_*(x) \ge f(x_*) \tag{4.118}$$

for all $x \in C$. It follows from (4.111), (4.112), (4.115), (4.116) and (4.108) that

$$x_* \in A, \ \phi(x_*) = 1 \text{ and } f_*(x_*) = f_1(x_*) = f(x_*). \tag{4.119}$$

By (4.119) and (4.118),

$$f_*(x) \ge f(x_*) = f_*(x_*) \text{ for all } x \in C. \tag{4.120}$$

It follows from (4.113), (4.115), (4.116), (4.107) and (4.102) that for all $x \in \mathrm{bd}(C)$,

$$x \in B,\ \phi(x) = 0$$

and

$$f_*(x) = f(x) + \epsilon \geq \inf(f; C) + \epsilon > f(x_*) + \delta/2.$$

Together with (4.119) and (4.120) this inequality implies that

$$\inf(f_*; \mathrm{bd}(C)) \geq f(x_*) + \delta/2 = f_*(x_*) + \delta/2 =$$

$$\inf(f_*; C) + \delta/2.$$

Define

$$\mathcal{U} = \{g \in \mathcal{M} : \tilde{d}(g, f_*) \leq \delta/8\}.$$

It is easy to see that for each $g \in \mathcal{U}$,

$$|f_*(z) - g(z)| \leq \delta/8,\ z \in X,$$

$$|\inf(f_*; C) - \inf(g; C)| \leq \delta/8,$$

$$|\inf(f_*, \mathrm{bd}(C)) - \inf(g, \mathrm{bd}(C))| \leq \delta/8$$

and

$$\inf(g; \mathrm{bd}(C)) \geq -\delta/8 + \inf(f_*; \mathrm{bd}(C)) \geq -\delta/8 +$$

$$\inf(f_*; C) + \delta/2 \geq \delta/2 - \delta/8 + \inf(g; C) - \delta/8 \geq \inf(g; C) + \delta/4.$$

Hence

$$\inf(g; \mathrm{bd}(C)) \geq \inf(g; C) + \delta/4 \text{ for all } g \in \mathcal{U}.$$

This completes the proof of Lemma 4.25.

Lemma 4.25 implies the following result.

Proposition 4.26. *There is an everywhere dense open subset $\mathcal{F}_1$ of $(\mathcal{M}, d)$ such that for each $f \in \mathcal{F}_1$,*

$$\inf(f; C) < \inf(f; \mathrm{bd}(C)).$$

Proof: Lemma 4.25 implies that for each $f \in \mathcal{M}$ and each natural number i there exists a nonempty open set $\mathcal{U}(f, i) \subset \mathcal{M}$ such that

$$\mathcal{U}(f, i) \cap \{g \in \mathcal{M} :\ d(f, g) \leq i^{-1}\} \neq \emptyset$$

and

$$\inf(g; C) < \inf(g; \mathrm{bd}(C)) \text{ for each } g \in \mathcal{U}(f, i).$$

Set

$$\mathcal{F}_1 = \cup\{\mathcal{U}(f, i) :\ f \in \mathcal{M},\ i = 1, 2, \dots\}.$$

It is easy to see that $\mathcal{F}_1$ is an open everywhere dense subset of $\mathcal{M}$. Clearly, for each $g \in \mathcal{F}_1$

$$\inf(g; C) < \inf(g; \mathrm{bd}(C)).$$

This completes the proof of Proposition 4.26.

4.11 An auxiliary result

Proposition 4.27. *There exists a set $\mathcal{F}_0 \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that for each $f \in \mathcal{F}_0$ the problem minimize $f(x)$ subject to $x \in C$ is strongly well-posed.*

We obtain Proposition 4.27 as a realization of the variational principle established in Section 4.1 (Theorem 4.1). In view of Theorem 4.1, Proposition 4.27 follows from the next lemma.

Lemma 4.28. *For any $f \in \mathcal{M}$, any positive number ϵ and any positive number γ there exists a nonempty open set W in $\mathcal{M}$, $x \in C$, $\alpha \in R^1$ and a positive number η such that for each $h \in W$*

$$d(f, h) < \epsilon$$

and for each $h \in W$, if $z \in C$ is such that $h(z) \le \inf(h; C) + \eta$, then

$$\rho(x, z) \le \gamma \text{ and } |h(z) - \alpha| \le \gamma.$$

Proof: Let $f \in \mathcal{M}$ and let ϵ and γ be positive numbers. Fix

$$\eta \in (0, 64^{-1} \min\{1, \epsilon\} \min\{1, \gamma\}), \tag{4.121}$$

fix a point $\bar{x} \in C$ satisfying

$$f(\bar{x}) \le \inf(f; C) + \eta \tag{4.122}$$

and define

$$\bar{f}(z) = f(z) + 8^{-1}\epsilon \min\{1, \rho(z, \bar{x})\}, \; z \in X. \tag{4.123}$$

It is easy to see that $\bar{f} \in \mathcal{M}$. Denote by W an open neighborhood of $\bar{f}$ in $\mathcal{M}$ such that

$$W \subset \{h \in \mathcal{M} : \; \tilde{d}(\bar{f}, h) \le \eta/8\}. \tag{4.124}$$

In view of (4.123),

$$d(f, \bar{f}) \le \tilde{d}(\bar{f}, f) \le 8^{-1}\epsilon.$$

Together with (4.124) this implies that for each $h \in W$,

$$d(f, h) \le \tilde{d}(f, h) \le \tilde{d}(f, \bar{f}) + \tag{4.125}$$

$$\tilde{d}(\bar{f}, h) \le \epsilon/8 + \eta/8 < \epsilon.$$

Let $h \in W$. By (4.124),

$$|\bar{f}(z) - h(z)| < \eta \text{ for all } z \in X. \tag{4.126}$$

In view of (4.126),

$$|\inf(\bar{f}; C) - \inf(h; C)| \le \eta. \tag{4.127}$$

Assume that $z \in C$ and

$$h(z) \leq \inf(h; C) + \eta. \tag{4.128}$$

It follows from (4.123), (4.126), (4.127), (4.128), (4.122) and (4.121) that

$$f(z) + 8^{-1}\epsilon \min\{1, \rho(z, \bar{x})\} = \bar{f}(z) \leq h(z) + \eta \leq$$

$$\inf(h; C) + 2\eta \leq \inf(\bar{f}; C) + 3\eta \leq \bar{f}(\bar{x}) + 3\eta =$$

$$f(\bar{x}) + 3\eta \leq f(z) + 4\eta$$

and

$$\min\{1, \rho(z, \bar{x})\} \leq 32\eta\epsilon^{-1} \leq \min\{1, \gamma\}/2.$$

Thus

$$\rho(z, \bar{x}) \leq \gamma/2. \tag{4.129}$$

By (4.128), (4.127) and (4.121),

$$|h(z) - \inf(\bar{f}; C)| \leq |h(z) - \inf(h; C)| + |\inf(h; C) - \inf(\bar{f}; C)| \leq 2\eta \leq \gamma$$

and $|h(z) - \inf(\bar{f}; C)| \leq \gamma$. Together with (4.129) this inequality implies that

$$|h(z) - \inf(\bar{f}; C)| \leq \gamma, \; \rho(z, \bar{x}) \leq \gamma/2.$$

Lemma 4.28 is proved.

4.12 Proof of Theorem 4.23

It follows from Proposition 4.26 that there exists an everywhere dense open subset $\mathcal{F}_1$ of $(\mathcal{M}, d)$ such that for each $f \in \mathcal{F}_1$,

$$\inf(f; C) < \inf(f; \mathrm{bd}(C)). \tag{4.130}$$

Proposition 4.27 implies that there exists a set $\mathcal{F}_0 \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, d)$ such that for each $f \in \mathcal{F}_0$ the problem minimize $f(x)$ subject to $x \in C$ is strongly well-posed. Put

$$\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_1.$$

It is easy to see that $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, d)$.

Let $f \in \mathcal{F}$. Then inequality (4.130) holds and the minimization problem $f(x) \to \min,$, $x \in C$ is strongly well-posed. Denote by x_f its unique solution. In view of (4.130) x_f is an interior point of C. This completes the proof of Theorem 4.23.

4.13 Generic existence of solutions for problems with constraints

In this section we consider a class of minimization problems with a functional constraint on a complete metric space and a class of minimization problems with a nonfunctional constraint on a Banach space. For these two classes we show that minimization problems are generically solvable. The main results of the section are obtained as realizations of a variational principle extending the variational principle introduced in Section 4.1.

More precisely, we consider two classes of minimization problems. Let (X, ρ) be a complete metric space and let $C_l(X)$ be the set of all lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$. Denote by $C_{bl}(X)$ the set of all bounded from below functions $f \in C_l(X)$.

Let n be a natural number. We study the existence of a solution of the minimization problem

$$\text{minimize f(x) subject to } g_i(x) \le 0,\ i = 1, \dots, n,\ x \in X \tag{P3}$$

with $f \in C_{bl}(X)$ and $g_i \in C_l(X)$, $i = 1, \dots n$. This is our first class of minimization problems. We will endow the set $C_{bl}(X) \times (C_l(X))^n$ with an appropriate complete uniformity and show that for a generic $(f, (g_1, \dots, g_n)) \in C_{bl}(X) \times (C_l(X))^n$ the minimization problem (P3) has a unique solution. We also show that an analogous result holds for the subspace of all continuous functions $(f, (g_1, \dots, g_n)) \in C_{bl}(X) \times (C_l(X))^n$.

Now we describe the second class of minimization problems.

Let $(X, ||\cdot||)$ be a Banach space. Denote by $\mathcal{L}$ the set of all bounded from below lower semicontinuous functions $f : X \to R^1$. Let $S(X)$ be the set of all nonempty closed convex subsets of X. The spaces $\mathcal{L}$ and $S(X)$ will be equipped with appropriate complete uniformities. We consider the following minimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in A \tag{P4}$$

where $A \in S(X)$ and $f \in \mathcal{L}$. We show that for a generic pair $(f, A) \in \mathcal{L} \times S(X)$ the minimization problem (P4) has a unique solution. Note that in Sections 4.7 and 4.8 the generic existence of solutions of the problem (P4) was studied for pairs (f, A) with a continuous function f and a closed nonempty set A.

In the next section we discuss a general variational principle which is a generalization of the variational principle obtained in Section 4.1. Our generic existence results will be obtained as realizations of this principle.

4.14 An auxiliary variational principle

For each function $f : Y \to [-\infty, \infty]$ where Y is a nonempty set we define

$$\text{dom}(f) = \{y \in Y : -\infty < f(y) < \infty\}, \ \inf(f) = \inf\{f(y) : \ y \in Y\}.$$

We use the notation and definitions introduced in Section 4.1.

We usually consider topological spaces with two topologies where one is weaker than the other. (Note that they may coincide.) We refer to them as the weak and the strong topologies, respectively. If (X, d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). Assume that X_1 and X_2 are topological spaces and that each of them is endowed with a weak and a strong topology. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies of X_1 and X_2 and a strong topology which is the product of the strong topologies of X_1 and X_2. If $Y \subset X_1$, then we consider the topological subspace Y with the relative weak and strong topologies (unless other topologies are introduced). If (X_i, d_i), $i = 1, 2$ are metric spaces with the metrics d_1 and d_2, respectively, then the space $X_1 \times X_2$ is endowed with the metric d defined by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2), \ (x_i, y_i) \in X \times Y, \ i = 1, 2.$$

The proofs of our generic existence results consist in verifying that the hypotheses (H1) (see Section 4.1) hold for corresponding data spaces. To simplify the verification of (H1) in this section we describe the assumptions A(i)–A(iv) introduced in [105]. In [105] we showed that they imply (H1). Thus to verify (H1) we need to show that the assumptions A(i)–A(iv) are valid.

Let (X, ρ) be a metric space with the topology generated by the metric ρ and let $(\mathcal{A}_1, d_1)$, $(\mathcal{A}_2, d_2)$ be metric spaces. For the space $\mathcal{A}_i$ $(i = 1, 2)$ we consider the topology generated by the metric d_i. This topology is called the strong topology. In addition to the strong topology we consider a weak topology on $\mathcal{A}_i$, $i = 1, 2$.

Assume that with every $a \in \mathcal{A}_1$ a lower semicontinuous function $\phi_a : X \to R^1 \cup \{\infty\}$ is associated and with every $a \in \mathcal{A}_2$ a set $S_a \subset X$ is associated. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \text{ for all } x \in S_{a_2}, \ f_a(x) = \infty \text{ for all } x \in X \setminus S_{a_2}. \tag{4.131}$$

Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 : \ \inf(f_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty.

We use the following hypotheses introduced in [105]:

A(i) For each $a_1 \in \mathcal{A}_1$, $\inf(\phi_{a_1}) > -\infty$ and for each $a \in \mathcal{A}_1 \times \mathcal{A}_2$ the function f_a is lower semicontinuous.

A(ii) For each $a \in \mathcal{A}_1$ and each $D, \epsilon > 0$ there is a neighborhood $\mathcal{U}$ of a in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{\phi_a(x), \phi_b(x)\} \le D$ the relation $|\phi_a(x) - \phi_b(x)| \le \epsilon$ holds.

A(iii) For each $\gamma \in (0, 1)$ there exist positive numbers $\epsilon(\gamma)$ and $\delta(\gamma)$ such that $\epsilon(\gamma), \delta(\gamma) \to 0$ as $\gamma \to 0$ and the following property holds:

For each $\gamma \in (0, 1)$, each $a \in \mathcal{A}_1$, each nonempty set $Y \subset X$ and each $\bar{x} \in Y$ for which

$$\phi_a(\bar{x}) \le \inf\{\phi_a(z) : \ z \in Y\} + \delta(\gamma) < \infty \tag{4.132}$$

there is $\bar{a} \in \mathcal{A}_1$ such that the following conditions hold:

$$d_1(a, \bar{a}) \le \epsilon(\gamma), \ \phi_{\bar{a}}(z) \ge \phi_a(z), \ z \in X, \ \phi_{\bar{a}}(\bar{x}) \le \phi_a(\bar{x}) + \delta(\gamma); \tag{4.133}$$

for each $y \in Y$ satisfying

$$\phi_{\bar{a}}(y) \le \inf\{\phi_{\bar{a}}(z) : \ z \in Y\} + 2\delta(\gamma) \tag{4.134}$$

the inequality $\rho(y, \bar{x}) \le \gamma$ is valid.

A(iv) For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying $\inf(f_a) < \infty$ and each $\epsilon, \delta > 0$ there exist $\bar{a}_2 \in \mathcal{A}_2$, $\bar{x} \in S_{\bar{a}_2}$ and an open set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology such that

$$d_2(a_2, \bar{a}_2) < \epsilon, \ \mathcal{U} \cap \{b \in \mathcal{A}_2 : \ d_2(b, a_2) < \epsilon\} \ne \emptyset, \tag{4.135}$$

$$\phi_{a_1}(\bar{x}) \le \inf\{\phi_{a_1}(z) : \ z \in S_{\bar{a}_2}\} + \delta < \infty \tag{4.136}$$

and

$$\bar{x} \in S_b \subset S_{\bar{a}_2} \text{ for all } b \in \mathcal{U}. \tag{4.137}$$

Assume that A(iii) holds. We show that the numbers $\epsilon(\gamma)$ and $\delta(\gamma)$ can be chosen such that $0 < \delta(\gamma) \le \epsilon(\gamma) \le \gamma$.

Let $\epsilon(\gamma)$ and $\delta(\gamma)$, $\gamma \in (0, 1)$ be as guaranteed by A(iii). Assume that $\gamma \in (0, 1)$. Since $\lim_{t \to 0} \epsilon(t) = 0$ and $\lim_{t \to 0} \delta(t) = 0$ there exist $\gamma_1 \in (0, \gamma)$ and $\gamma_0 \in (0, \gamma_1)$ such that $\epsilon(\gamma_1) < \gamma$ and $\epsilon(\gamma_0), \delta(\gamma_0) < \epsilon(\gamma_1)$. Set $\bar{\epsilon}(\gamma) = \epsilon(\gamma_1)$ and $\bar{\delta}(\gamma) = \delta(\gamma_0)$. Clearly $\bar{\delta}(\gamma) < \bar{\epsilon}(\gamma) < \gamma$.

Assume that $a \in \mathcal{A}_1$, Y is a nonempty subset of X and $\bar{x} \in Y$ satisfies $\phi_a(\bar{x}) \le \inf\{\phi_a(z) : \ z \in Y\} + \bar{\delta}(\gamma) < \infty$. By A(iii) and the equality $\bar{\delta}(\gamma) = \delta(\gamma_0)$ there exists $\bar{a} \in \mathcal{A}_1$ such that the following conditions hold:

$$d_1(a, \bar{a}) \le \epsilon(\gamma_0) < \epsilon(\gamma_1) = \bar{\epsilon}(\gamma), \ \phi_{\bar{a}}(z) \ge \phi_a(z), \ z \in X,$$

$$\phi_{\bar{a}}(\bar{x}) \le \phi_a(\bar{x}) + \delta(\gamma_0) = \phi_a(\bar{x}) + \bar{\delta}(\gamma);$$

for each $y \in Y$ satisfying

$$\phi_{\bar{a}}(y) \le \inf\{\phi_{\bar{a}}(z) : \ z \in Y\} + 2\delta(\gamma_0)$$

the inequality $\rho(y, \bar{x}) \le \gamma_0 \le \gamma$ is valid. Therefore A(iii) holds with $\epsilon(\gamma) = \bar{\epsilon}(\gamma)$ and $\delta(\gamma) = \bar{\delta}(\gamma)$.

Proposition 4.29. *Assume that A(i)–A(iv) hold. Then (H1) holds for the space* $\mathcal{A}$.

Proof. Let $a = (a_1, a_2) \in \mathcal{A}$ and let $\epsilon, \gamma > 0$. We may assume that $\inf(f_a) < \infty$. Choose a positive number

$$\gamma_0 < 8^{-1} \min\{1, \epsilon, \gamma\}. \tag{4.138}$$

Let $\epsilon(\gamma_0)$, $\delta(\gamma_0) > 0$ be as guaranteed by A(iii) (namely, A(iii) is true with $\gamma = \gamma_0$, $\epsilon(\gamma) = \epsilon(\gamma_0)$, $\delta(\gamma) = \delta(\gamma_0)$). Choose

$$\delta_1 \in (0, 4^{-1}\delta(\gamma_0)). \tag{4.139}$$

By A(iv) there are $\bar{a}_2 \in \mathcal{A}_2$, $\bar{x} \in S_{\bar{a}_2}$ and an open nonempty set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology such that (4.137) holds,

$$d_2(a_2, \bar{a}_2) < \epsilon(\gamma_0),\ \mathcal{U} \cap \{b \in \mathcal{A}_2 :\ d_2(b, a_2) < \epsilon(\gamma_0)\} \neq \emptyset \tag{4.140}$$

and

$$\phi_{a_1}(\bar{x}) \leq \inf\{\phi_{a_1}(z) :\ z \in S_{\bar{a}_2}\} + \delta_1 < \infty. \tag{4.141}$$

It follows from the definition of $\epsilon(\gamma_0)$ and $\delta(\gamma_0)$, A(iii) (with $a_1 = a$ and $Y = S_{\bar{a}_2}$) and (4.341) that there is $\bar{a}_1 \in \mathcal{A}_1$ such that

$$d_1(a_1, \bar{a}_1) \leq \epsilon(\gamma_0),\ \phi_{\bar{a}_1}(z) \geq \phi_{a_1}(z),\ z \in X, \tag{4.142}$$

$$\phi_{\bar{a}_1}(\bar{x}) \leq \phi_{a_1}(\bar{x}) + \delta(\gamma_0)$$

and the following property holds:

(Pi) For each $y \in S_{\bar{a}_2}$ satisfying

$$\phi_{\bar{a}_1}(y) \leq \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{\bar{a}_2}\} + 2\delta(\gamma_0) \tag{4.143}$$

the relation $\rho(y, \bar{x}) \leq \gamma_0$ is valid.

Let $b \in \mathcal{U}$. Then by the definition of $\mathcal{U}$, (4.137) and (4.141),

$$\bar{x} \in S_b \subset S_{\bar{a}_2},\ \inf\{\phi_{a_1}(z) :\ z \in S_b\} \leq \phi_{a_1}(\bar{x}) < \infty. \tag{4.144}$$

We will show that the following property holds:

(Pii) If $y \in S_b$ satisfies

$$\phi_{\bar{a}_1}(y) \leq \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_b\} + \delta_1, \tag{4.145}$$

then

$$\rho(y, \bar{x}) \leq \gamma_0 \text{ and } |\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(\bar{x})| \leq \delta_1 + \delta(\gamma_0). \tag{4.146}$$

It follows from (4.141), (4.144) and (4.142) that

$$\begin{aligned} \phi_{a_1}(\bar{x}) - \delta_1 &\leq \inf\{\phi_{a_1}(z) :\ z \in S_{\bar{a}_2}\} \leq \inf\{\phi_{a_1}(z) :\ z \in S_b\} \\ &\leq \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_b\} \leq \phi_{\bar{a}_1}(\bar{x}) \leq \phi_{a_1}(\bar{x}) + \delta(\gamma_0) \\ &\leq \inf\{\phi_{a_1}(z) :\ z \in S_{\bar{a}_2}\} + \delta_1 + \delta(\gamma_0). \end{aligned} \tag{4.147}$$

Assume that $y \in S_b$ and (4.145) is true. It follows from (4.144), (4.145), (4.147), (4.142) and (4.139) that

$$y \in S_{\bar{a}_2},\ \phi_{\bar{a}_1}(y) \leq \inf\{\phi_{a_1}(z) :\ z \in S_{\bar{a}_2}\} + \delta(\gamma_0) + 2\delta_1$$

$$< \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{\bar{a}_2}\} + 2\delta(\gamma_0).$$

By these relations and property (Pi), $\rho(y, \bar{x}) \le \gamma_0$. (4.145), (4.147), (4.141), (4.144) and (4.142) imply that

$$|\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1 + \delta(\gamma_0). \tag{4.148}$$

Thus, (4,146) is valid. Therefore we have shown that for each $b \in \mathcal{U}$ relation (4.144) and property (Pii) hold. Choose a number

$$D > |\inf(\phi_{\bar{a}_1})| + 1 + |\phi_{\bar{a}_1}(\bar{x})|. \tag{4.149}$$

By (A2) there exists an open neighborhood $\mathcal{V}$ of $\bar{a}_1$ in $\mathcal{A}_1$ with the weak topology such that the following property holds:

(Piii) For each $b \in \mathcal{V}$ and each $x \in X$ for which $\min\{\phi_b(x), \phi_{\bar{a}_1}(x)\} \le D+2$ the relation $|\phi_{\bar{a}_1}(x) - \phi_b(x)| \le 4^{-1}\delta_1$ is true.

Property (Piii) and (4.149) imply that for each $b \in \mathcal{V}$,

$$|\phi_b(\bar{x}) - \phi_{\bar{a}_1}(\bar{x})| \le 4^{-1}\delta_1,\ \inf(\phi_b) \le \phi_b(\bar{x}) \le D. \tag{4.150}$$

Now we will show that (H1) is true with the open set $\mathcal{W} = \mathcal{V} \times \mathcal{U}$, $x = \bar{x}$, $\alpha = \phi_{\bar{a}_1}(\bar{x})$ and $\eta = 4^{-1}\delta_1$.

Assume that $b = (b_1, b_2) \in \mathcal{V} \times \mathcal{U}$. By (4.150) and (4.144)

$$\bar{x} \in S_{b_2},\ \inf(f_b) = \inf\{\phi_{b_1}(z) :\ z \in S_{b_2}\} \le \phi_{b_1}(\bar{x}) < \infty. \tag{4.151}$$

Assume now that $z \in X$ and $f_b(z) \le \inf(f_b) + 4^{-1}\delta_1$. Then

$$z \in S_{b_2},\ \phi_{b_1}(z) \le \inf\{\phi_{b_1}(y) :\ y \in S_{b_2}\} + 4^{-1}\delta_1. \tag{4.152}$$

By (4.151), (4.150) and (4.149),

$$\inf\{\phi_{b_1}(y) :\ y \in S_{b_2}\} \le \phi_{b_1}(\bar{x}) \le D,\ \inf\{\phi_{\bar{a}_1}(y) :\ y \in S_{b_2}\} \le \phi_{\bar{a}_1}(\bar{x}) \le D.$$

These inequalities imply that

$$\inf\{\phi_{b_1}(y) :\ y \in S_{b_2}\} = \inf\{\phi_{b_1}(y) :\ y \in S_{b_2} \text{ and } \phi_{b_1}(y) \le D+1\}$$

and

$$\inf\{\phi_{\bar{a}_1}(y) :\ y \in S_{b_2}\} = \inf\{\phi_{\bar{a}_1}(y) :\ y \in S_{b_2} \text{ and } \phi_{\bar{a}_1}(y) \le D+1\}.$$

It follows from these two relations and property (Piii) that

$$|\inf\{\phi_{b_1}(y) :\ y \in S_{b_2}\} - \inf\{\phi_{\bar{a}_1}(y) :\ y \in S_{b_2}\}| \le 4^{-1}\delta_1. \tag{4.153}$$

(4.153), (4.152), (4.151), (4.149) and property (Piii) imply that

$$|\phi_{\bar{a}_1}(z) - \phi_{b_1}(z)| \le 4^{-1}\delta_1, \tag{4.154}$$

$$\phi_{\bar{a}_1}(z) \le \inf\{\phi_{\bar{a}_1}(y) : \ y \in S_{b_2}\} + \delta_1. \tag{4.155}$$

It follows from (4.155), (4.152) and property (Pii) that

$$\rho(z, \bar{x}) \le \gamma_0 \text{ and } |\phi_{\bar{a}_1}(z) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1 + \delta(\gamma_0). \tag{4.156}$$

Together with (4.154), (4.139) and the definition of $\delta(\gamma_0)$ this implies that

$$|\phi_{b_1}(z) - \phi_{\bar{a}_1}(\bar{x})| \le 2\delta(\gamma_0) \le 2\gamma_0 < \gamma.$$

This completes the proof of Proposition 4.29.

Note that Proposition 4.29 was proved in [105].

Remark 4.30. In the proof of Proposition 4.29 for any $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying $\inf(f_a) < \infty$ and any $\epsilon > 0$ we constructed an open set $\mathcal{V}$ in $\mathcal{A}_1$ with the weak topology and an open set $\mathcal{U}$ in $\mathcal{A}_2$ with the weak topology which satisfy

$$\mathcal{V} \cap \{b \in \mathcal{A}_1 : \ d_1(b, a_1) < \epsilon\} \ne \emptyset \text{ and } \mathcal{U} \cap \{b \in \mathcal{A}_2 : \ d_2(b, a_2) < \epsilon\} \ne \emptyset$$

and such that $\inf(f_b) < \infty$ for each $b = (b_1, b_2) \in \mathcal{V} \times \mathcal{U}$. This implies that there exists an open set $\mathcal{F}$ in $\mathcal{A}_1 \times \mathcal{A}_2$ with the weak topology such that $\inf(f_a) < \infty$ for all $a \in \mathcal{F}$ and $\mathcal{A}$ is the closure of $\mathcal{F}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology.

4.15 The generic existence result for problem (P3)

Let (X, ρ) be a complete metric space. Consider the spaces of functions $C_l(X)$ and $C_{bl}(X)$ introduced in Section 4.13. We use the convention that $\infty - \infty = 0$. Denote by $C(X)$ the set of all continuous real-valued functions $f \in C_l(X)$ and set $C_b(X) = C(X) \cap C_{bl}(X)$. We will equip the set $C_l(X)$ with a strong and a weak topology.

For the set $C_l(X)$ we consider the uniformity determined by the following base:

$$\begin{aligned} E_s(\epsilon) = \{(g, h) \in C_l(X) \times C_l(X) : \ |g(x) - h(x)| \le \epsilon \text{ for all } x \in X \text{ and} \\ |(g(x) - h(x)) - (g(y) - h(y))| \le \epsilon\rho(x, y) \text{ for each } x, y \in \mathrm{dom}(g)\}, \end{aligned} \tag{4.157}$$

where $\epsilon > 0$. Clearly this uniform space $C_l(X)$ is metrizable (by a metric d_s) and complete. We endow the set $C_l(X)$ with the strong topology induced by this uniformity.

Now we equip the set $C_l(X)$ with a weak topology. For each $\epsilon > 0$ we set

$$\begin{aligned} E_w(\epsilon) = \{(g, h) \in C_l(X) \times C_l(X) : \\ |g(x) - h(x)| \le \epsilon + \epsilon \max\{|g(x)|, |h(x)|\} \text{ for all } x \in X\}. \end{aligned} \tag{4.158}$$

We can show in a straightforward manner that for the set $C_l(X)$ there exists a uniformity which is determined by the base $E_w(\epsilon)$, $\epsilon > 0$. It is easy to see that this uniformity is metrizable (by a metric d_w) and complete. This uniformity induces in $C_l(X)$ the weak topology. Clearly $C(X)$, $C_b(X)$ and $C_{bl}(X)$ are closed subsets of $C_l(X)$ with the strong topology.

Now we define the spaces $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{A}_1$ be either $C_{bl}(X)$ or $C_b(X)$ and let $\mathcal{A}_2 = C_1^* \times \cdots \times C_n^*$ where C_i^*, $i = 1, \dots, n$ is one of the following spaces: $C_l(X)$; $C(X)$; $C_{bl}(X)$; $C_b(X)$. For $a \in \mathcal{A}_1$ we set $\phi_a = a$ and for $g = (g_1, \dots, g_2) \in \mathcal{A}_2$ we set

$$S_g = \{x \in X : \ g_i(x) \le 0, \ i = 1, \dots, n\}. \tag{4.159}$$

For $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we define a function $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) = a_1(x), \ x \in S_{a_2}, \ f_a(x) = \infty, \ x \in X \setminus S_{a_2}. \tag{4.160}$$

Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 : \ \inf(f_a) < \infty\}$ in the space $\mathcal{A}_1 \times A_2$ with the strong topology.

Theorem 4.31. *The minimization problem for f_a is well-posed with respect to $\mathcal{A}$ for a generic $a \in \mathcal{A}$.*

Note that Theorem 4.31 has been obtained in [111].

4.16 Proof of Theorem 4.31

In the proof of Theorem 4.31 we use the following lemma which can be proved in a straightforward manner.

Lemma 4.32. *Let $a, b \in R^1$, $\epsilon \in (0, 1)$, $\Delta \ge 0$ and*

$$|a - b| < (1 + \Delta)\epsilon + \epsilon \max\{|a|, |b|\}.$$

Then

$$|a - b| < (1 + \Delta)(\epsilon + \epsilon^2(1 - \epsilon)^{-1}) + \epsilon(1 - \epsilon)^{-1} \min\{|a|, |b|\}.$$

Proof of Theorem 4.31: We need to show that (H1) holds. In view of Proposition 4.29 it is sufficient to show that assumptions A(i)–A(iv) are valid. It is easy to see that A(i) holds. We show that A(ii) is true.

Let $f \in \mathcal{A}_1 \subset C_{bl}(X)$ and let D, ϵ be positive numbers. There exists $c_0 > 0$ such that

$$f(x) \ge -c_0 \text{ for all } x \in X. \tag{4.161}$$

Fix $\epsilon_0 > 0$ such that

$$\epsilon_0(D + 2c_0 + 4) < \min\{1, \epsilon\} \tag{4.162}$$

and $\epsilon_1 > 0$ for which

$$4(\epsilon_1 + \epsilon_1(1-\epsilon_1)^{-1}) < \epsilon_0. \tag{4.163}$$

Put

$$\mathcal{U} = \{g \in C_l(X) :\ (f,g) \in E_w(\epsilon_1)\}$$

(see (4.158)).

Assume that

$$g \in \mathcal{U},\ x \in X \text{ and } \min\{f(x), g(x)\} \leq D. \tag{4.164}$$

It follows from the definition of $\mathcal{U}$ and (4.158) that

$$|f(z) - g(z)| < \epsilon_1 + \epsilon_1 \max\{|f(z)|, |g(z)|\},\ z \in X.$$

By this relation, (4.163) and Lemma 4.32, for all $z \in X$,

$$|f(z) - g(z)| < \epsilon_0 + \epsilon_0 \min\{|f(z)|, |g(z)|\}$$

and

$$g(z) \geq f(z) - \epsilon_0 - \epsilon_0|f(z)| \geq -1 - 2c_0. \tag{4.165}$$

Relations (4.165), (4.164) and (4.162) imply that

$$|f(x) - g(x)| < \epsilon_0 + \epsilon_0[\min\{f(x),\ g(x)\} + 2c_0 + 2]$$

$$< \epsilon_0 + \epsilon_0(D + 2c_0 + 2) < \epsilon.$$

Therefore A(ii) holds.

We show that the assumption A(iii) is true. Let $\gamma \in (0,1)$. Choose numbers $\epsilon(\gamma) > 0$, $\delta(\gamma) > 0$ and $\epsilon_0(\gamma) > 0$ such that

$$\epsilon(\gamma) < \gamma,\ \epsilon_0(\gamma) < \epsilon(\gamma), \tag{4.166}$$

$$d_s(g_1, g_2) \leq \epsilon(\gamma) \text{ for all } (g_1, g_2) \in E_s(\epsilon_0(\gamma)),\ \delta(\gamma) < 8^{-1}\epsilon_0(\gamma)^2.$$

Assume that $f \in \mathcal{A}_1 \subset C_{bl}(X)$, $Y \subset X$ is nonempty, $\bar{x} \in Y$ and

$$f(\bar{x}) \leq \inf\{f(z) :\ z \in Y\} + \delta(\gamma) < \infty. \tag{4.167}$$

Define $\bar{f} :\ X \to R^1 \cup \{\infty\}$ by

$$\bar{f}(x) = f(x) + \epsilon_0(\gamma) \min\{1, \rho(x, \bar{x})\},\ x \in X. \tag{4.168}$$

It is easy to see that $\bar{f} \in C_{bl}(X)$, $(f, \bar{f}) \in E_s(\epsilon_0(\gamma))$ (see (4.157)) and if $f \in C_b(X)$, then $\bar{f} \in C_b(X)$. By the definition of $\epsilon_0(\gamma)$, $d_s(f, \bar{f}) \leq \epsilon(\gamma)$. Evidently, $\bar{f}(z) \geq f(z)$, $z \in X$ and $\bar{f}(\bar{x}) = f(\bar{x})$.

Assume that $y \in Y$ and

$$\bar{f}(y) \leq \inf\{\bar{f}(z) :\ z \in Y\} + 2\delta(\gamma). \tag{4.169}$$

By (4.168), (4.169), (4.167) and (4.166),

$$f(y) + \epsilon_0(\gamma)\min\{1, \rho(y, \bar{x})\} = \bar{f}(y) \le \bar{f}(\bar{x}) + 2\delta(\gamma)$$

$$= f(\bar{x}) + 2\delta(\gamma) \le f(y) + 3\delta(\gamma),$$

$$\min\{1, \rho(y, \bar{x})\} \le 3\delta(\gamma)\epsilon_0(\gamma)^{-1} \le \epsilon_0(\gamma),\ \rho(y, \bar{x}) \le \epsilon_0(\gamma) < \gamma.$$

Hence A(iii) is true.

Now we will show that A(iv) is valid. Let $f_0 \in \mathcal{A}_1 \subset C_{bl}(X)$, $(f_1, \dots, f_n) \in \mathcal{A}_2$, $a = (f_0, (f_1, \dots, f_n))$, $\inf(f_a) < \infty$ and let $\epsilon, \delta \in (0, 1)$. There exists a positive number $\epsilon_0 < \epsilon$ such that $d_s(h_1, h_2) < \epsilon(2n)^{-1}$ for all $(h_1, h_2) \in E_s(\epsilon_0)$.

For each $r \in [0, 1]$ define $f_i^r(x) = f_i(x) - r$, $x \in X$, $i = 1, \dots, n$ and

$$v(r) = \inf\{f_0(x) :\ x \in X \text{ and } f_i^r(x) \le 0,\ i = 1, \dots, n\} \tag{4.170}$$

$$= \inf\{f_0(x) :\ x \in X \text{ and } f_i(x) \le r,\ i = 1, \dots, n\}.$$

It is easy to see that $v(r)$ is finite for all $r \in [0, 1]$ and moreover v is monotone decreasing. There exists a countable set $\sigma_0 \subset [0, 1]$ such that v is continuous at any $r \in [0, 1] \setminus \sigma_0$. Fix a positive number $r_0 < 8^{-1}\epsilon_0$ such that v is continuous at r_0. There exists a positive number $r_1 < r_0$ such that

$$|v(r_1) - v(r_0)| < 16^{-1}\delta. \tag{4.171}$$

There exists $\bar{x} \in X$ such that

$$f_i^{r_1}(\bar{x}) = f_i(\bar{x}) - r_1 \le 0,\ i = 1, \dots, n, \tag{4.172}$$

$$f_0(\bar{x}) \le \inf\{f_0(z) :\ z \in X \text{ and } f_i^{r_1}(z) \le 0,\ i = 1, \dots, n\} + 16^{-1}\delta.$$

Set

$$r_2 = (r_0 + r_1)/2,\ \bar{a}_2 = (f_1^{r_0}, \dots, f_n^{r_0}),\ \bar{b} = (f_1^{r_2}, \dots, f_n^{r_2}). \tag{4.173}$$

Fix

$$\delta_0 \in (0, \min\{4^{-1}\delta,\ 8^{-1}(r_0 - r_2)\}). \tag{4.174}$$

Lemma 4.32 and (4.158) imply that for integers $i = 1, \dots, n$ there is a neighborhood V_i of $f_i^{r_2}$ in the space $C_l(X)$ with the weak topology such that the following property holds:

(P) for each $g \in V_i$,

$$|g(x) - f_i^{r_2}(x)| < \delta_0 + \delta_0 \min\{|g(x)|, |f_i^{r_2}(x)|\} \text{ for all } x \in X. \tag{4.175}$$

Set $U = V_1 \times \cdots \times V_n$. To complete the proof of A(iv) we need to show that for all $(g_1, \dots, g_n) \in U$

$$\bar{x} \in \{x \in X :\ g_i(x) \le 0,\ i = 1, \dots, n\} \subset \{x \in X :\ f_i^{r_0}(x) \le 0,\ i = 1, \dots, n\}. \tag{4.176}$$

Assume that $(g_1, \ldots, g_n) \in U$, $x \in X$ and $g_i(x) \le 0$, $i = 1, \ldots, n$. Let $i \in \{1, \ldots, n\}$. We show that $f_i^{r_0}(x) \le 0$. Let us assume the contrary. Then

$$g_i(x) \le 0,\ f_i(x) > r_0,\ f_i^{r_2}(x) > r_0 - r_2 = 2^{-1}(r_0 - r_1) \tag{4.177}$$

and (4.177), (4.175) and (4.174) imply that

$$|g_i(x) - f_i^{r_2}(x)| < \delta_0 + \delta_0 \min\{-g_i(x),\ f_i^{r_2}(x)\},$$

$$g_i(x) > -\delta_0 + f_i^{r_2}(x)(1-\delta_0) \ge 2^{-1}(r_0 - r_1)(1-\delta_0) - \delta_0 \ge 4^{-1}(r_0 - r_1) - \delta_0 > 0.$$

The obtained contradiction proves that $f_i^{r_0}(x) \le 0$, $i = 1, \ldots, n$. Assume that $(g_1, \ldots, g_n) \in U$. We now show that $g_i(\bar{x}) \le 0$, $i = 1, \ldots, n$. Let $i \in \{1, \ldots, n\}$. It follows from (4.172) that

$$0 \ge f_i^{r_1}(\bar{x}) = f_i(\bar{x}) - r_1 = f_i^{r_2}(\bar{x}) + r_2 - r_1$$

$$= f_i^{r_2}(\bar{x}) + 2^{-1}(r_0 - r_1),\ f_i^{r_2}(\bar{x}) \le -2^{-1}(r_0 - r_1).$$

It follows from this relation, (4.175) and (4.174) that

$$|g_i(\bar{x}) - f_i^{r_2}(\bar{x})| < \delta_0 + \delta_0 |f_i^{r_2}(\bar{x})|,\ g_i(\bar{x}) < \delta_0 + f_i^{r_2}(\bar{x})(1 - \delta_0) \le$$

$$\delta_0 - 2^{-1}(r_0 - r_1)(1 - \delta_0) < \delta_0 - 4^{-1}(r_0 - r_1) < 0.$$

Therefore (4.176) is valid. This completes the proof of A(iv) and Theorem 4.31.

4.17 The generic existence result for problem (P4)

Let $(X, ||\cdot||)$ be a Banach space. Consider the set $\mathcal{L}$ of all bounded from below lower semicontinuous functions $f : X \to R^1$ and the set $S(X)$ of all nonempty closed convex subsets of X introduced in Section 4.13.

For the set $\mathcal{L}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_s(\epsilon) = \{(f, g) \in \mathcal{L} \times \mathcal{L} :\ |f(x) - g(x)| \le \epsilon,\ x \in X\}, \tag{4.178}$$

where ϵ is a positive number. It is easy to see that this uniform space is metrizable (by a metric h_s) and complete. We equip the set $\mathcal{L}$ with the strong topology induced by the metric h_s.

Let $\phi :\ X \to R^1$ and $\phi(x) \to \infty$ as $||x|| \to \infty$. Define

$$\mathcal{L}(\phi) = \{f \in \mathcal{L} :\ f(x) \ge \phi(x) \text{ for all } x \in X\}. \tag{4.179}$$

Evidently, $\mathcal{L}(\phi)$ is a closed subset of $\mathcal{L}$ with the strong topology. For the set $\mathcal{L}$ we also consider the uniformity determined by the following base:

$$\mathcal{E}_w(n) = \{(f, g) \in \mathcal{L} \times \mathcal{L} :\ |f(x) - g(x)| \le n^{-1}$$

$$\text{for all } x \in X \text{ satisfying } ||x|| \le n\}, \tag{4.180}$$

where $n = 1, 2, \ldots$. Evidently, this uniform space is metrizable (by a metric h_w). We equip the set $\mathcal{L}$ with the weak topology induced by this uniformity. For $x \in X$ and $A \subset X$ set

$$\rho(x, A) = \inf\{||x - y|| : \ y \in A\}.$$

For the set $S(X)$ we consider the uniformity determined by the following base:

$$G(n) = \{(A, B) \in S(X) \times S(X) : \ \rho(x, B) \le n^{-1} \text{ for all } x \in A \tag{4.181}$$

$$\text{and } \rho(y, A) \le n^{-1} \text{ for all } y \in B\}, \ n = 1, 2, \ldots.$$

It is well known that the space $S(X)$ with this uniformity is metrizable (by a metric H) and complete. We consider the set $S(X)$ endowed with the Hausdorff topology induced by this uniformity.

For each $a \in \mathcal{L}$ we set $\phi_a = a : \ X \to R^1$ and for each $a \in S(X)$ set $S_a = a \subset X$. For each $a = (a_1, a_2) \in \mathcal{L} \times S(X)$ define $f_a : \ X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = a_1(x), \ x \in a_2, \ f_a(x) = \infty, \ x \in X \setminus a_2.$$

It is easy to see that $\inf(f_a)$ is finite for all $a \in \mathcal{L} \times S(X)$.

Now we describe spaces $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{A}_1 = \mathcal{L}(\phi)$ and let $\mathcal{A}_2 = S(X)$. The set $\mathcal{A}_1 \times \mathcal{A}_2$ is endowed with a weak and a strong topology. The weak topology is the product of the weak topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$ and the strong topology is the product of the strong topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$. Set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

We prove the following result.

Theorem 4.33. *The minimization problem for f_a is well-posed with respect to $\mathcal{A} = \mathcal{L}(\phi) \times S(X)$ for a generic $a \in \mathcal{A}$.*

We now extend Theorem 4.33 to the space of cost functions $\mathcal{L}$. Let $\mathcal{A}_1 = \mathcal{L}$ with the strong topology induced by the metric h_s and let $\mathcal{A}_2 = S(X)$ with the Hausdorff topology induced by the metric H. The set $\mathcal{A}_1 \times \mathcal{A}_2$ is endowed with a topology which is the product of these topologies. Set $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

Theorem 4.34. *The minimization problem for f_a is well-posed with respect to $\mathcal{A} = \mathcal{L} \times S(X)$ for a generic $a \in \mathcal{A}$.*

Since the proof of Theorem 4.34 is analogous to that of Theorem 4.33 we omit it.

Note that Theorem 4.33 has been obtained in [111].

4.18 Proof of Theorem 4.33

In the proof of Theorem 4.33 we use the following auxiliary result.

Proposition 4.35. *Let $(Y, ||\cdot||)$ be a Banach space and let $B(0,1) = \{y \in Y : ||y|| \leq 1\}$. Assume that E is a closed convex subset of Y such that for all $y \in B(0,1)$*

$$\inf_{x \in E} ||y - x|| \leq 8^{-1}.$$

Then $0 \in E$.

Proof: Let us assume the contrary. Then there exists a linear continuous functional $l : Y \to R^1$ for which $||l|| = 1$ and

$$\inf\{l(y) : y \in E\} > 0.$$

There exists $y_0 \in B(0,1)$ such that $l(-y_0) \geq 1 - 8^{-1}$. There exists $y_1 \in E$ such that $||y_0 - y_1|| \leq 4^{-1}$. Then

$$l(y_1) = l(y_0) + l(y_1 - y_0) \leq l(y_0) + ||y_1 - y_0|| \leq$$

$$8^{-1} - 1 + ||y_1 - y_0|| \leq -1 + 8^{-1} + 4^{-1} \leq -2^{-1},$$

a contradiction. The obtained contradiction proves Proposition 4.35.

Proof of Theorem 4.33: We need to verify that (H1) holds (see Section 4.1). In view of Proposition 4.29 it is sufficient to verify that A(i)–A(iv) are true. It is easy to see that A(i) holds. Let $f \in \mathcal{A}_1$ and let D, ϵ be positive numbers. There exists an integer $n_0 \geq 1$ such that

$$\phi(x) \geq D + 4 \text{ for all } x \in X \text{ satisfying } ||x|| \geq n_0.$$

Fix an integer $n_1 > n_0 + 4 + \epsilon^{-1}$ and put

$$\mathcal{U} = \{g \in \mathcal{A}_1 : (f, g) \in \mathcal{E}_w(n_1)\}.$$

Assume that $g \in \mathcal{U}$, $x \in X$ and $\min\{g(x), f(x)\} \leq D$. Then $||x|| \leq n_0$ and in view of the definition of $\mathcal{E}_w(n_1)$, $|f(x) - g(x)| \leq n_1^{-1} < \epsilon$. Thus A(ii) holds. Using the same arguments as in the proof of Theorem 4.31 we can show that A(iii) holds.

We show that the assumption A(iv) is valid. Let $g \in \mathcal{A}_1 = \mathcal{L}(\phi)$, $a_2 = C \in S(X)$, $a = (g, C)$,

$$\inf(f_a) = \inf\{g(x) : x \in C\} < \infty \tag{4.182}$$

and let ϵ, δ be positive numbers. We may assume that $\epsilon, \delta < 1$. There exists an integer $n_0 \geq 1$ such that

$$n_0 > 8\max\{\epsilon^{-1}, \delta^{-1}\},\ H(A, B) \leq 8^{-1}\epsilon \text{ for each } (A, B) \in G(n_0) \tag{4.183}$$

and

$$||x|| \le n_0/2 \text{ for each } x \in X \text{ satisfying } g(x) \le |\inf(f_a)| + 4. \tag{4.184}$$

For each $r \in [0,1]$ define

$$C_r = \{x \in X : \ \rho(x,r) \le r\} \in S(X) \tag{4.185}$$

and

$$v(r) = \inf\{g(x) : \ x \in C_r\}. \tag{4.186}$$

Clearly $v(r)$ is finite for all $r \in [0,1]$ and the function v is monotone decreasing. There exists a countable set $\sigma_0 \subset [0,1]$ such that v is continuous at any $r \in [0,1] \setminus \sigma_0$. Fix a positive number $r_0 < (8n_0)^{-1}$ such that v is continuous at r_0. There exists a positive number $r_1 < r_0$ such that

$$|v(r_1) - v(r_0)| < 16^{-1}\delta. \tag{4.187}$$

Put

$$r_2 = 2^{-1}(r_1 + r_0). \tag{4.188}$$

There exists $\bar{x} \in X$ for which

$$\bar{x} \in C_{r_1}, \ g(\bar{x}) \le \inf\{g(x) : \ x \in C_{r_1}\} + 16^{-1}\delta. \tag{4.189}$$

Put

$$\bar{a}_2 = C_{r_0}, \ \bar{b} = C_{r_2}. \tag{4.190}$$

Fix an integer

$$n_1 > n_0 + 32(r_0 - r_1)^{-1}$$

and put

$$\mathcal{U} = \{B \in S(X) : \ (B, C_{r_2}) \in G(n_1)\}. \tag{4.191}$$

Clearly, $(C, C_{r_i}) \in G(n_0)$, $i = 0,1,2$. In view of (4.183), $H(C, C_{r_i}) \le 8^{-1}\epsilon$, $i = 0,1,2$. It follows from (4.189), (4.187) and (4.186) that

$$g(\bar{x}) \le v(r_1) + 16^{-1}\delta \le v(r_0) + 8^{-1}\delta.$$

Therefore in order to complete the proof of A(iv) we need to show that

$$\bar{x} \in B \subset C_{r_0} \text{ for all } B \in \mathcal{U}. \tag{4.192}$$

Let

$$B \in \mathcal{U} \tag{4.193}$$

and $y \in B$. There exists $z \in C_{r_2}$ such that $||z - y|| \le n_1^{-1}$. Since $\rho(z, C) \le r_2$ we obtain that $\rho(y, C) \le r_2 + n_1^{-1} < r_0$ (see (4.191)). Thus $y \in C_{r_0}$ and $B \subset C_{r_0}$. We show that $\bar{x} \in B$. (4.189) and (4.185) imply that

$$\{\bar{x} + z : \ z \in X \text{ and } ||z|| \le 2^{-1}(r_0 - r_1)\} \subset C_{r_2}$$

and

$$\{z \in X : \ ||z|| \le 2^{-1}(r_0 - r_1)\} \subset C_{r_2} - \bar{x}. \tag{4.194}$$

By (4.191),

$$(C_{r_2} - \bar{x}, B - \bar{x}) \in G(n_1).$$

Combined with (4.194) and Proposition 4.35 this implies that $0 \in B - \bar{x}$. Hence (4.192) holds and A(iv) is valid. This completes the proof of Theorem 4.33.

4.19 Well-posedness of a class of minimization problems

In this section we consider classes of mathematical programming problems on a complete metric space X which are identified with complete metric spaces of lower semicontinuous objective functions on the space X. We show that for any of these spaces of functions there exists an open everywhere dense subset $\mathcal{F}$ such that for each objective function f belonging to $\mathcal{F}$ there exists a neighborhood of f such that all the minimization problems with objective functions belonging to this neighborhood possess the same unique solution.

Let (X, ρ) be a complete metric space. For each function $f : X \to R^1 \cup \{\infty\}$ set

$$\inf(f) = \inf\{f(z) : \ z \in X\} \tag{4.195}$$

and

$$\mathrm{dom}(f) = \{z \in X : \ f(z) < \infty\}. \tag{4.196}$$

We use the convention that $\infty - \infty = 0$, $\infty/\infty = 1$ and that the infimum over an empty set is ∞. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\}. \tag{4.197}$$

We study classes of minimization problems on X which are identified with complete metric spaces of lower semicontinuous objective functions $f : X \to R^1 \cup \{\infty\}$. We show that for any of these spaces of functions there exists an open everywhere dense subset $\mathcal{F}$ such that each $f \in \mathcal{F}$ possesses the following property:

There exist a neighborhood $\mathcal{U}$ of f, $\bar{x} \in X$ and a number $\gamma > 0$ such that for each $g \in \mathcal{U}$ the corresponding minimization problem with the objective function g possesses a unique solution $\bar{x}$ and moreover, if $z \in X$ satisfies $g(z) \le \inf(g) + \gamma/2$, then $g(z) - g(\bar{x}) \ge \gamma\rho(z, x_f)$.

In particular, this result implies that most of the minimization problems (in the Baire category sense) possess sharp minima [17, 77].

Let $\theta \in X$ and $\phi : X \to R^1$ be a bounded from below function such that

$$\lim_{\rho(x,\theta)\to\infty} \phi(x) = \infty. \tag{4.198}$$

Denote by $\mathcal{M}$ the set of all lower semicontinuous bounded from below functions $f : X \to R^1 \cup \{\infty\}$ which are not identically ∞ and denote by $\mathcal{M}^{(\phi)}$ the set of all functions $f \in \mathcal{M}$ such that

$$f(x) \geq \phi(x) \text{ for all } x \in X. \tag{4.199}$$

Denote by $\mathcal{M}_v$ (respectively, $\mathcal{M}_c$, $\mathcal{M}_{uc}$, $\mathcal{M}_{lL}$, $\mathcal{M}_{bL}$, $\mathcal{M}_L$) the set of all finite-valued (respectively, finite-valued continuous, finite-valued uniformly continuous on all bounded sets, finite-valued locally Lipschitzian, finite-valued Lipschitzian on all bounded sets, finite-valued and Lipschitzian) functions. Put

$$\mathcal{M}_v^{(\phi)} = \mathcal{M}^{(\phi)} \cap \mathcal{M}_v,\ \mathcal{M}_c^{(\phi)} = \mathcal{M}^{\phi)} \cap \mathcal{M}_c,\ \mathcal{M}_{uc}^{(\phi)} = \mathcal{M}^{\phi)} \cap \mathcal{M}_{uc}, \tag{4.200}$$

$$\mathcal{M}_{lL}^{(\phi)} = \mathcal{M}^{\phi)} \cap \mathcal{M}_{lL},\ \mathcal{M}_{bL}^{(\phi)} = \mathcal{M}^{\phi)} \cap \mathcal{M}_{bL},\ \mathcal{M}_L^{(\phi)} = \mathcal{M}^{(\phi)} \cap \mathcal{M}_L. \tag{4.201}$$

If X is a convex closed subset of a Banach space with a norm $||\cdot||$, then we suppose that $\rho(x, y) = ||x - y||$, $x, y \in X$, denote by $\mathcal{M}_{con}$ the set of all convex functions $f \in \mathcal{M}$ and set

$$\mathcal{M}_{con}^{(\phi)} = \mathcal{M}_{con} \cap \mathcal{M}^{(\phi)},\ \mathcal{M}_{con,v}^{(\phi)} = \mathcal{M}_{con}^{(\phi)} \cap \mathcal{M}_v,\ \mathcal{M}_{con,c}^{(\phi)} = \mathcal{M}_{con}^{\phi)} \cap \mathcal{M}_c,$$

$$\mathcal{M}_{con,uc}^{(\phi)} = \mathcal{M}_{con}^{\phi)} \cap \mathcal{M}_{uc},\ \mathcal{M}_{con,lL}^{(\phi)} = \mathcal{M}_{con}^{\phi)} \cap \mathcal{M}_{lL},$$

$$\mathcal{M}_{con,bL}^{(\phi)} = \mathcal{M}_{con}^{(\phi)} \cap \mathcal{M}_{bL},\ \mathcal{M}_{con,L}^{(\phi)} = \mathcal{M}_{con}^{(\phi)} \cap \mathcal{M}_L. \tag{4.202}$$

Let $f, g \in \mathcal{M}$. If dom$(f) \neq$dom(g) set $\tilde{d}_s(f, g) = \infty$. If dom$(f)$ =dom(g) set

$$\tilde{d}_s(f, g) = \sup\{|f(x) - g(x)| :\ x \in \text{dom}(f)\}+ \tag{4.203}$$

$$\sup\{|(f - g)(x_1) - (f - g)(x_2)|\rho(x_1, x_2)^{-1} :\ x_1, x_2 \in X \text{ and } x_1 \neq x_1\}.$$

Put

$$d_s(f, g) = \tilde{d}_s(f, g)(\tilde{d}_s(f, g) + 1)^{-1}. \tag{4.204}$$

It is easy to see that $(\mathcal{M}, d_s)$ is a complete metric space and $\mathcal{M}_v$, $\mathcal{M}_c$, $\mathcal{M}_{uc}$, $\mathcal{M}_{lL}$, $\mathcal{M}_{bL}$, $\mathcal{M}_L$, $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_L^{(\phi)}$ are closed subsets of $(\mathcal{M}, d_s)$.

Let $f, g \in \mathcal{M}$. If dom$(f) \neq$ dom(g) set $\tilde{d}_w(f, g) = 1$. If dom$(f) =$ dom(g) set

$$d_{w,0}(f, g) = \sup\{|f(x) - g(x)| :\ x \in \text{dom } (f)\}, \tag{4.205}$$

for $i = 1, 2, \dots$ set

$$d_{w,i}(f, g) = \sup\{|(f - g)(x_1) - (f - g)(x_2)|\rho(x_1, x_2)^{-1} :$$

$$x_1, x_2 \in B(\theta, i) \cap \text{dom}(f) \text{ and } x_1 \neq x_2\}. \tag{4.206}$$

Finally set

$$d_w(f,g) = \sum_{i=0}^{\infty} 2^{-1-i} d_{w,i}(f,g)(1 + d_{w,i}(f,g))^{-1}. \tag{4.207}$$

Clearly, $(\mathcal{M}, d_w)$ is a metric space and for each pair $f, g \in \mathcal{M}$ the inequality $d_w(f,g) \le \tilde{d}_s(f,g)$ holds.

We prove the following two results.

Theorem 4.36. *Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}$, $\mathcal{M}_v$, $\mathcal{M}_c$, $\mathcal{M}_{uc}$, $\mathcal{M}_{lL}$, $\mathcal{M}_{bL}$, $\mathcal{M}_L$, $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_L^{(\phi)}$.*

Assume that $f \in \mathcal{A}$ and $r \in (0,1)$. Then there exist $\bar{f} \in \mathcal{A}$ and $\bar{x} \in X$ such that for each $g \in \mathcal{A}$ satisfying $d_s(\bar{f}, g) \le 2 \cdot (16)^{-2} r$ we have:

$d_s(f,g) \le r$; if $x \in X$ satisfies $g(x) \le \inf(g) + (4 \cdot 16)^{-1} r$, then

$$g(x) - g(\bar{x}) \ge (32)^{-1} r \rho(x, \bar{x}).$$

Theorem 4.37. *Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}$, $\mathcal{M}_v$, $\mathcal{M}_c$, $\mathcal{M}_{uc}$, $\mathcal{M}_{lL}$, $\mathcal{M}_{bL}$, $\mathcal{M}_L$, $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_L^{(\phi)}$.*

Then there exists an open (in the topology induced by the metric d_w) everywhere dense (in the topology induced by the metric d_s) subset $\mathcal{B} \subset \mathcal{A}$ such that the following property holds:

For each $f \in \mathcal{B}$ there exist $\gamma > 0$, $x_f \in X$ and a neighborhood $\mathcal{U}$ of f in $(\mathcal{A}, d_w)$ such that if $g \in \mathcal{U}$ and if $x \in X$ satisfies $g(x) \le \inf(g) + \gamma/2$, then

$$g(x) - g(x_f) \ge \gamma \rho(x, x_f). \tag{4.208}$$

Let $f, g \in \mathcal{M}^{(\phi)}$. If $\mathrm{dom}(f) \ne \mathrm{dom}(g)$, set $d_\phi(f,g) = 1$. If $\mathrm{dom}(f) = \mathrm{dom}(g)$, then for $i = 1, 2, \dots$ put

$$\begin{aligned} d_{\phi,i}(f,g) = \sup\{|f(x) - g(x)| :\ x \in B(\theta, i) \cap \mathrm{dom}(f)\} + \\ \sup\{|(f-g)(x_1) - (f-g)(x_2)|\rho(x_1,x_2)^{-1} : \\ x_1, x_2 \in B(\theta, i) \cap \text{ dom } (f) \text{ and} x_1 \ne x_1\} \end{aligned} \tag{4.209}$$

and set

$$d_\phi(f,g) = \sum_{i=1}^{\infty} 2^{-i} d_{\phi,i}(f,g)(d_{\phi,i}(f,g) + 1)^{-1}. \tag{4.210}$$

It is not difficult to see that $(\mathcal{M}^{(\phi)}, d_\phi)$ is a complete metric space and $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_{con}^{(\phi)}$, $\mathcal{M}_{con,v}^{(\phi)}$, $\mathcal{M}_{con,c}^{(\phi)}$, $\mathcal{M}_{con,uc}^{(\phi)}$, $\mathcal{M}_{con,lL}^{(\phi)}$, $\mathcal{M}_{con,bL}^{(\phi)}$ are closed subsets of $(\mathcal{M}^{(\phi)}, d_\phi)$.

We prove the following result.

Theorem 4.38. *Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_{con}^{(\phi)}$, $\mathcal{M}_{con,v}^{(\phi)}$, $\mathcal{M}_{con,c}^{(\phi)}$, $\mathcal{M}_{con,uc}^{(\phi)}$, $\mathcal{M}_{con,lL}^{(\phi)}$, $\mathcal{M}_{con,bL}^{(\phi)}$ and let $k > 0$. Then there exists an open everywhere dense subset $\mathcal{B}$ of the space $(\mathcal{A}, d_\phi)$ such that the following property holds:*

For each $f \in \mathcal{B}$ there exist a positive number γ, $x_f \in X$ and a neighborhood $\mathcal{U}$ of f in $(\mathcal{A}, d_\phi)$ such that if $g \in \mathcal{U}$ and if $x \in X$ satisfies $g(x) \le \inf(g) + k$, then

$$g(x) - g(x_f) \ge \gamma\rho(x, x_f).$$

Note that the results of this section have been obtained in [135].

4.20 Auxiliary results for Theorems 4.36 and 4.37

Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}$, $\mathcal{M}_v$, $\mathcal{M}_c$, $\mathcal{M}_{uc}$, $\mathcal{M}_{lL}$, $\mathcal{M}_{bL}$, $\mathcal{M}_L$, $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_L^{(\phi)}$.

Let $f \in \mathcal{A}$,

$$x_f \in \mathrm{dom}(f) \text{ and } f(x_f) = \inf(f). \tag{4.211}$$

For each $\gamma \in (0, 1)$ set

$$f_\gamma(z) = f(z) + \gamma \min\{\rho(z, x_f), 1\} \text{ for all } z \in X. \tag{4.212}$$

It is not difficult to see that the following result holds.

Lemma 4.39. *Let $\gamma \in (0, 1)$. Then $f_\gamma \in \mathcal{A}$, dom(f_γ) = dom(f), $|f_\gamma(z) - f(z)| \le \gamma$ for each $z \in$ dom(f), and for each pair $z_1, z_2 \in$ dom(f),*

$$|(f_\gamma - f)(z_1) - (f_\gamma - f)(z_2)| \le \gamma\rho(z_1, z_2).$$

Lemma 4.40. *Let $\gamma \in (0, 1)$ and $x \in X$ satisfy*

$$f_\gamma(x) < \inf(f_\gamma) + \gamma. \tag{4.213}$$

Then $\rho(x, x_f) \le \gamma^{-1}(f_\gamma(x) - \inf(f_\gamma)) < 1$.

Proof: It is easy to see that

$$\inf(f_\gamma) = \inf(f) = f(x_f). \tag{4.214}$$

(4.213) and (4.214) imply that $x \in \mathrm{dom}(f)$. By (4.212) and (4.213),

$$f(x) + \gamma \min\{\rho(x, x_f), 1\} = f_\gamma(x) < \inf(f) + \gamma,$$

$$\rho(x, x_f) < 1. \tag{4.215}$$

It follows from (4.212), (4.214) and (4.215) that

$$f_\gamma(x) - \inf(f_\gamma) = f(x) + \gamma\rho(x, x_f) - \inf(f_\gamma) \ge \gamma(x, x_f),$$

$$\rho(x, x_f) \le \gamma^{-1}(f_\gamma(x) - \inf(f_\gamma)).$$

This completes the proof of Lemma 4.40.

Lemma 4.41. *Let* $\gamma \in (0,1)$, $g \in \mathcal{A}$,

$$dom(f) = dom(g),\ |f_\gamma(z) - g(z)| \le \gamma/4 \text{ for all } z \in dom(f) \tag{4.216}$$

and let for each $z_1, z_2 \in\ dom(f)$

$$|(f_\gamma - g)(z_1) - (f_\gamma - g)(z_2)| \le 2^{-1}\gamma\rho(z_1, z_2). \tag{4.217}$$

Assume that $x \in X$ *satisfies*

$$g(x) \le \inf(g) + \gamma/4. \tag{4.218}$$

Then $g(x) - g(x_f) \ge (\gamma/2)\rho(x, x_f)$.

Proof: In view of (4.216),

$$|\inf(g) - \inf(f_\gamma)| \le \gamma/4. \tag{4.219}$$

By (4.216), (4.218) and (4.219),

$$f_\gamma(x) \le g(x) + \gamma/4 \le \inf(g) + \gamma/2 \le \inf(f_\gamma) + (3/4)\gamma.$$

Combined with Lemma 4.40 this implies that

$$\rho(x, x_f) < 1,$$

$$\rho(x, x_f) \le \gamma^{-1}(f_\gamma(x) - \inf(f_\gamma)). \tag{4.220}$$

It follows from (4.216)–(4.218) that

$$|(f_\gamma - g)(x) - (f_\gamma - g)(x_f)| \le 2^{-1}\gamma\rho(x, x_f).$$

This relation, (4.220), (4.212) and (4.211) imply that

$$g(x) - g(x_f) \ge f_\gamma(x) - f_\gamma(x_f) - \gamma\rho(x, x_f)2^{-1}$$

$$\ge \gamma\rho(x, x_f) - 2^{-1}\gamma\rho(x, x_f) \ge 2^{-1}\gamma\rho(x, x_f).$$

This completes the proof of Lemma 4.41.

Lemma 4.42. *Assume that*

$$n > \rho(x_f, \theta) + 1, \tag{4.221}$$

$\gamma \in (0,1)$, $g \in \mathcal{A}$,

$$dom(f) = dom(g),\ |f_\gamma(z) - g(z)| \le \gamma/4 \text{ for all } z \in dom(f) \tag{4.222}$$

and that for each $z_1, z_2 \in\ dom(f) \cap B(\theta, n)$

$$|(f_\gamma - g)(z_1) - (f_\gamma - g)(z_2)| \le 2^{-1}\gamma\rho(z_1, z_2). \tag{4.223}$$

Let $x \in X$ *satisfy*

$$g(x) \le \inf(g) + \gamma/4. \tag{4.224}$$

Then $g(x) - g(x_f) \ge 2^{-1}\gamma\rho(x, x_f)$.

Proof: It follows from (4.222) that $|\inf(g) - \inf(f_\gamma)| \le \gamma/4$. By this inequality, (4.224) and (4.222),

$$f_\gamma(x) \le g(x) + \gamma/4 \le \inf(g) + \gamma/2 \le \inf(f_\gamma) + (3/4)\gamma.$$

This inequality and Lemma 4.40 imply that

$$\rho(x, x_f) < 1 \text{ and } \rho(x, x_\gamma) \le \gamma^{-1}(f_\gamma(x) - \inf(f_\gamma)). \tag{4.225}$$

In view of (4.225) and (4.221),

$$\rho(x, \theta) \le \rho(x, x_f) + \rho(x_f, \theta) < n. \tag{4.226}$$

It follows from (4.221)–(4.224) and (4.226) that

$$|(f_\gamma - g)(x) - (f_\gamma - g)(x_f)| \le \gamma\rho(x, x_f)/2.$$

Together with (4.225) this implies that

$$\begin{aligned} g(x) - g(x_f) &\ge f_\gamma(x) - f_\gamma(x_f) - \gamma\rho(x, x_f)/2 \ge \gamma\rho(x, x_f) - 2^{-1}\gamma\rho(x, x_f) \\ &= 2^{-1}\gamma\rho(x, x_f). \end{aligned}$$

This completes the proof of Lemma 4.42.

4.21 Auxiliary results for Theorem 4.38

Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_L^{(\phi)}$, $\mathcal{M}_{con}^{(\phi)}$, $\mathcal{M}_{con,v}^{(\phi)}$, $\mathcal{M}_{con,c}^{(\phi)}$, $\mathcal{M}_{con,uc}^{(\phi)}$, $\mathcal{M}_{con,lL}^{(\phi)}$, $\mathcal{M}_{con,bL}^{(\phi)}$, $\mathcal{M}_{con,L}^{(\phi)}$.

Let $n, k > 0$. In view of (4.198) there is $n_1 > 4$ such that

$$\phi(z) > n + k + 4 \text{ for each } z \in X \text{ satisfying } \rho(z, \theta) > n_1 - 4. \tag{4.227}$$

Assume that

$$f \in \mathcal{A},\ \inf(f) < n,\ x_f \in \mathrm{dom}(f),\ f(x_f) = \inf(f). \tag{4.228}$$

For each $\gamma \in (0, 1)$ put

$$f_\gamma(z) = f(z) + \gamma\rho(z, x_f),\ z \in X. \tag{4.229}$$

Lemma 4.43. *Let $\gamma \in (0, 1)$. Then $f_\gamma \in \mathcal{A}$, dom(f_γ) = dom(f) and for each $z \in$ dom$(f) \cap B(\theta, n_1)$*

$$|f_\gamma(z) - f(z)| \le 2\gamma n_1.$$

Proof: It is easy to see that $f_\gamma \in \mathcal{A}$ and $\mathrm{dom}(f) = \mathrm{dom}(f_\gamma)$. Assume that $z \in \mathrm{dom}(f) \cap B(\theta, n_1)$. Combined with (4.227)–(4.229) and (4.199) this implies that

$$|f_\gamma(z) - f(z)| = \gamma\rho(z, x_f) \le \gamma\rho(z, \theta) + \gamma\rho(z, x_f) \le 2\gamma n_1.$$

This completes the proof of Lemma 4.43.

It is easy to see that the following lemma holds.

Lemma 4.44. *Let* $\gamma \in (0,1)$ *and* $x \in X$. *Then*

$$f_\gamma(x) - \inf(f_\gamma) = f_\gamma(x) - f(x_f) \geq \gamma\rho(x, x_f). \tag{4.230}$$

Lemma 4.45. *Assume that* $\gamma \in (0,1)$, $g \in \mathcal{A}$,

$$B(0, n_1) \cap dom(g) = dom(f) \cap B(\theta, n_1), \tag{4.231}$$

$$|g(z) - f_\gamma(z)| \leq \gamma/4 \text{ for all } z \in dom(g) \cap B(\theta, n_1) \tag{4.232}$$

and that for each $z_1, z_2 \in dom(f) \cap B(\theta, n_1)$

$$|(f_\gamma - g)(z_1) - (f_\gamma - g)(z_2)| \leq 2^{-1}\gamma\rho(z_1, z_2). \tag{4.233}$$

Let $x \in X$ *satisfy*

$$g(x) \leq \inf(g) + k. \tag{4.234}$$

Then $g(x) - g(x_f) \geq (\gamma/2)\rho(x, x_f)$.

Proof: (4.227) and (4.199) imply that

$$\inf\{f_\gamma(z) : z \in X \setminus B(\theta, n_1 - 4)\} \geq n + 4 + k, \tag{4.235}$$

$$\inf\{g(z) : z \in X \setminus B(\theta, n_1 - 4)\} \geq n + k + 4.$$

It follows from (4.228), (4.232) and (4.235) that

$$f(x_f) < n, \; g(x_f) \leq f_\gamma(x_f) + \gamma/4 < n + 1. \tag{4.236}$$

In view of (4.235) and (4.236),

$$\inf(f_\gamma) = \inf\{f_\gamma(z) : z \in B(\theta, n_1 - 4)\},$$

$$\inf(g) = \inf\{g(z) : z \in B(\theta, n_1 - 4)\}. \tag{4.237}$$

Combined with (4.232) this equality implies that

$$|\inf(f_\gamma) - \inf(g)| \leq \gamma/4. \tag{4.238}$$

It follows from this inequality, (4.234) and (4.236) that

$$g(x) \leq \inf(g) + k \leq g(x_f) + k < n + 1 + k.$$

By this inequality, (4.199) and (4.227),

$$\rho(x, \theta) \leq n_1 - 4. \tag{4.239}$$

It follows from this inequality, (4.232), (4.234), (4.238) and (4.231) that

$$f_\gamma(x) \leq g(x) + \gamma/4 \leq k + \inf(g) + \gamma/4 \leq \inf(f_\gamma) + \gamma/2 + k.$$

In view of Lemma 4.40,

$$\rho(x, x_f) \leq \gamma^{-1}(f_\gamma(x) - \inf(f_\gamma)). \tag{4.240}$$

Relations (4.233), (4.239), (4.238) and (4.228) imply that

$$|(f_\gamma - g)(x) - (f_\gamma - g)(x_f)| \leq 2^{-1}\gamma\rho(x, x_f).$$

By this inequality, (4.240), (4.229) and (4.228),

$$g(x) - g(x_f) \geq f_\gamma(x) - f_\gamma(x_f) - 2^{-1}\gamma\rho(x, x_f) \geq 2^{-1}\gamma\rho(x, x_f).$$

This completes the proof of Lemma 4.45.

4.22 Proofs of Theorems 4.36 and 4.37

Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}$, $\mathcal{M}_v$, $\mathcal{M}_c$, $\mathcal{M}_{uc}$, $\mathcal{M}_{lL}$, $\mathcal{M}_{bL}$, $\mathcal{M}_L$, $\mathcal{M}^{(\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_L^{\phi}$.

Proposition 4.46. *There exists an everywhere dense subset $\mathcal{A}_0$ of the metric space $(\mathcal{A}, d_s)$ such that for each $f \in \mathcal{A}_0$ the problem minimize $f(z)$ subject to $z \in X$ has a solution.*

Proof. We use the definitions and the notation introduced in Section 4.1. Set $f_a = a$ for all $a \in \mathcal{A}$. Theorem 4.1 implies that in order to prove Proposition 4.46 it is sufficient to show that (H1) is valid for the space $(\mathcal{A}, d_s)$.

Let $\epsilon, \gamma \in (0, 1)$ and let $f \in \mathcal{A}$. Fix

$$\delta \in (0, 32^{-1}\epsilon\gamma)$$

and $\bar{x} \in X$ such that

$$f(\bar{x}) \leq \inf(f) + \delta.$$

Define $\bar{f} \in \mathcal{A}$ by

$$\bar{f}(x) = f(x) + 4^{-1}\epsilon \min\{\rho(x, \bar{x}), 1\},\ x \in X$$

and put

$$\mathcal{W} = \{g \in \mathcal{A} : \tilde{d}_s(g, \bar{f}) < \delta\}.$$

Assume that $g \in \mathcal{W}$ and that $x \in X$ satisfies

$$g(z) \leq \inf(g) + \delta.$$

It is easy to see that

$$f(z) + 4^{-1}\epsilon \min\{\rho(z, \bar{x}),\ 1\} = \bar{f}(z) \leq g(z) + \delta \leq \inf(g) + 2\delta$$

$$\leq \inf(\bar{f}) + 3\delta \leq \bar{f}(\bar{x}) + 3\delta = f(\bar{x}) + 3\delta \leq f(z) + 4\delta.$$

By the relations above,

$$\rho(z, \bar{x}) \leq 16\delta\epsilon^{-1} < \gamma$$

and $|g(z) - \inf(\bar{f})| \leq 3\delta < \gamma$. Hence (H1) holds. Proposition 4.46 is proved.

Proof of Theorem 4.36: Put

$$\alpha = (16)^{-2}. \tag{4.241}$$

Assume that

$$f \in \mathcal{A},\ r \in (0, 1]. \tag{4.242}$$

Proposition 4.46 implies that there exist $f_0 \in \mathcal{A}$ and $x_0 \in X$ such that

$$d_s(f, f_0) \leq r/8,\ f_0(x_0) = \inf(f_0). \tag{4.243}$$

Put

$$\gamma = 16\alpha r \tag{4.244}$$

and define

$$f_1(z) = f_0(z) + \gamma \min\{\rho(z, x_0), 1\},\ z \in X. \tag{4.245}$$

Lemma 4.39 implies that $f_1 \in \mathcal{A}$ and

$$\tilde{d}_s(f_0, f_1) \leq 2\gamma. \tag{4.246}$$

It follows from Lemma 4.41 that the following property holds:

(C1) If $g \in \mathcal{A}$ satisfies $\mathrm{dom}(g) = \mathrm{dom}(f_1)$,

$$|f_1(z) - g(z)| \leq \gamma/4 \text{ for all } z \in \mathrm{dom}(f_1),$$

$$|(f_1 - g)(z_1) - (f_1 - g)(z_2)| \leq 2^{-1}\gamma\rho(z_1, z_2) \text{ for each } z_1, z_2 \in \mathrm{dom}(f_1)$$

and if $x \in X$ satisfies $g(x) \leq \inf(g) + \gamma/4$, then $g(x) - g(x_0) \geq 2^{-1}\gamma\rho(x, x_0)$.

Assume that $g \in \mathcal{A}$ satisfies

$$d_s(f_1, g) \leq 2\alpha r. \tag{4.247}$$

Combined with (4.246), (4.243), (4.241) and (4.244) this implies that

$$d_s(g, f) \leq d_s(g, f_1) + d_s(f_1, f_0) + d_s(f_0, f_1) \leq 2\alpha r + 2\gamma + r/8 \leq r/2. \tag{4.248}$$

It follows from (4.247), (4.204) and (4.244) that

$$\tilde{d}_s(f_1, g) = d_s(f_1, g)(1 - d_s(f_1, g))^{-1} \leq 2d_s(f_1, g) \leq 4\alpha r = \gamma/4.$$

By this relation, (4.203), (C1) and (4.244), the following property holds:

If $x \in X$ satisfies $g(x) \leq \inf(g) + 4\alpha r$, then $g(x) - g(x_0) \geq 8\alpha r\rho(x, x_0)$.

Theorem 4.36 is proved with $\bar{f} = f_1$, $\bar{x} = x_0$.

Proof of Theorem 4.37: Let $f \in \mathcal{A}$ and $r \in (0, 1]$. Proposition 4.46 implies that there exists $f_0 \in \mathcal{A}$ and $x_f \in X$ such that

$$d_s(f, f_0) \leq r/8,\ f_0(x_f) = \inf(f). \tag{4.249}$$

Fix a positive number $\gamma < r/8$ and define $f_1 \in \mathcal{A}$ by

$$f_1(z) = f_0(z) + \gamma \min\{\rho(z, x_f), 1\},\ z \in X. \tag{4.250}$$

It follows from (4.250), Lemma 4.39 and (4.204) that

$$d_s(f_1, f_0) \leq \tilde{d}_s(f_1, f_0) \leq 2\gamma < r/4. \tag{4.251}$$

By (4.251) and (4.249),

$$d_s(f_1, f) \leq d_s(f_1, f_0) + d_s(f_0, f) \leq r/4 + r/8 < r/2. \tag{4.252}$$

Lemma 4.42 implies that there exists an open neighborhood $\mathcal{U}$ of f_1 in the metric space $(\mathcal{A}, d_w)$ such that the following property holds:

If $g \in \mathcal{U}$ and if $x \in X$ satisfies $g(x) \leq \inf(g) + \gamma/4$, then $g(x) - g(x_f) \geq 2^{-1}\gamma\rho(x, x_f)$.

Theorem 4.37 is proved.

4.23 Proof of Theorem 4.38

Let $\mathcal{A}$ be one of the following spaces: $\mathcal{M}^{\phi)}$, $\mathcal{M}_v^{(\phi)}$, $\mathcal{M}_c^{(\phi)}$, $\mathcal{M}_{uc}^{(\phi)}$, $\mathcal{M}_{lL}^{(\phi)}$, $\mathcal{M}_{bL}^{(\phi)}$, $\mathcal{M}_{con}^{(\phi)}$, $\mathcal{M}_{con,v}^{(\phi)}$, $\mathcal{M}_{con,c}^{(\phi)}$, $\mathcal{M}_{con,uc}^{(\phi)}$, $\mathcal{M}_{con,lL}^{(\phi)}$, $\mathcal{M}_{con,bL}^{(\phi)}$.

Proposition 4.47. *There exists an everywhere dense subset $\mathcal{A}_0$ of the metric space $(\mathcal{A}, d_\phi)$ which is a countable intersection of open subsets of $(\mathcal{A}, d_\phi)$ such that for each $f \in \mathcal{A}_0$ the problem minimize $f(z)$ subject to $z \in X$ has a unique solution.*

Proof: In view of Theorem 4.1 in order to prove the proposition it is sufficient to show that the following property holds:

(C2) For any $f \in \mathcal{A}$, any positive number ϵ and any positive number γ there exist a nonempty open set $\mathcal{W}$ in the metric space $(\mathcal{A}, d_\phi)$, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that $\mathcal{W} \cap \{g \in \mathcal{A} :\ d_\phi(f, g) < \epsilon\} \neq \emptyset$ and that for any $g \in \mathcal{W}$ if $z \in X$ is such that $g(z) \leq \inf(g) + \eta$, then $\rho(z, x) \leq \gamma$ and $|g(z) - \alpha| \leq \gamma$.

Let $f \in \mathcal{A}$ and $\epsilon, \gamma \in (0, 1)$. There exists a natural number $n_0 > 4$ such that

$$\text{if } z \in X \text{ and } \phi(z) \leq \inf(f) + 4, \ \text{ then } \rho(z, \theta) \leq n_0 - 4, \tag{4.253}$$

$$\sum_{i=n_0}^{\infty} 2^{-i} < \epsilon/4. \tag{4.254}$$

Fix a number $\delta_1 > 0$ for which

$$\delta_1(n_0 + 1) < \epsilon/4 \tag{4.255}$$

and

$$\delta_0 \in (0, 4^{-1}\delta_1\gamma). \tag{4.256}$$

Choose $\bar{x} \in X$ which satisfies

$$f(\bar{x}) \leq \inf(f) + \delta_0. \tag{4.257}$$

(4.257) and (4.253) imply that

$$\rho(\bar{x}, \theta) \leq n_0 - 4. \tag{4.258}$$

Put

$$\bar{f}(z) = f(z) + \delta_1\rho(z, \bar{x}),\ z \in X. \tag{4.259}$$

Clearly, $\bar{f} \in \mathcal{A}$. It follows from (4.210), (4.254) and (4.255) that

$$d_\phi(f, \bar{f}) \leq \sum_{i=1}^{n_0} 2^{-i} d_{w,i}(f, \bar{f})(1 + d_{w,i}(f, \bar{f}))^{-1} + \sum_{i=n_0+1}^{\infty} 2^{-i}$$

$$\le \sum_{i=1}^{n_0} 2^{-i} d_{w,i}(f, \bar{f}) + \epsilon/4$$

$$\le \max\{d_{w,i}(f, \bar{f}) : \ i = 1, \dots, n_0\} + \epsilon/4 \le \delta_1 + \delta_1 n_0 + \epsilon/4 < \epsilon. \tag{4.260}$$

Fix $n_1 > n_0 + 4$.

Assume that $g \in \mathcal{A}$,

$$|\bar{f}(z) - g(z)| \le \delta_0, \ z \in B(\theta, n_1). \tag{4.261}$$

$$|(\bar{f} - g)(z_1) - (\bar{f} - g)(z_2)| \le \delta_0 \rho(z_1, z_2) \text{ for all } z_1, z_2 \in B(\theta, n_1). \tag{4.262}$$

Relations (4.258) and (4.261) imply that

$$|\bar{f}(\bar{x}) - g(\bar{x})| \le \delta_0. \tag{4.263}$$

In view of (4.259) and (4.257),

$$\inf(\bar{f}) = \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + 1. \tag{4.264}$$

It follows from (4.263), (4.259) and (4.257) that

$$\inf(g) \le g(\bar{x}) \le \bar{f}(\bar{x}) + \delta_0 = f(\bar{x}) + \delta_0 \le \inf(f) + 2. \tag{4.265}$$

By (4.264), (4.265) and (4.253),

$$\inf(\bar{f}) = \inf\{\bar{f}(z) : \ z \in B(\theta, n_0 + 4)\}, \tag{4.266}$$

$$\inf(g) = \inf\{g(z) : \ z \in B(\theta, n_0 + 4)\}.$$

In view of (4.266) and (4.261),

$$|\inf(\bar{f}) - \inf(g)| \le \delta_0. \tag{4.267}$$

Assume that $z \in X$ satisfies

$$g(z) \le \inf(g) + \delta_0. \tag{4.268}$$

It follows from (4.199), (4.268), (4.267), (4.259) and (4.257) that

$$\phi(z) \le g(z) \le \inf(g) + \delta_0 \le \inf(\bar{f}) + 2\delta_0 \le \bar{f}(\bar{x}) + 2\delta_0 < \inf(f) + 3.$$

Combined with (4.253) this implies that $\rho(z, \theta) \le n_0 - 4$. Together with (4.261) this implies that

$$|g(z) - \bar{f}(z)| \le \delta_0. \tag{4.269}$$

By (4.267)–(4.269),

$$\bar{f}(z) \le g(z) + \delta_0 \le \inf(g) + 2\delta_0 \le \inf(\bar{f}) + 3\delta_0.$$

The relation above, (4.257), (4.256) and (4.259) imply that

$$f(z) + \delta_1 \rho(z, \bar{x}) = \bar{f}(z) \le \inf(\bar{f}) + 3\delta_0 \le \bar{f}(\bar{x}) + 3\delta_0 = f(\bar{x}) + 3\delta_0$$
$$\le f(\bar{x}) + 3\delta_0 \le f(z) + 4\delta_0, \tag{4.270}$$
$$\rho(z, \bar{x}) \le 4\delta_0 \delta_1^{-1} < \gamma.$$

It follows from (4.269), (4.270) and (4.256) that

$$|g(z) - f(\bar{x})| \le \delta_0 + |\bar{f}(z) - f(\bar{x})| \le 4\delta_0 < \gamma.$$

Thus (C2) holds and this completes the proof of Proposition 4.47.

Proof of Theorem 4.38: Let $f \in \mathcal{A}$ and $r \in (0, 1]$. Proposition 4.47 implies that there exists $f^{(0)} \in \mathcal{A}$ and $x_f \in X$ such that

$$d_\phi(f, f^{(0)}) \le r/4, \ f^{(0)}(x_f) = \inf(f^{(0)}). \tag{4.271}$$

For each $\gamma \in (0, 1)$ put

$$f_\gamma^{(0)}(z) = f^{(0)}(z) + \gamma \rho(z, x_f), \ z \in X.$$

It is easy to see that $f_\gamma \in \mathcal{A}$ for all $\gamma \in (0, 1)$ and $d_\phi(f_\gamma^{(0)}, f^{(0)}) \to 0$ as $\gamma \to 0^+$. There exists $\gamma \in (0, 1)$ such that $d_\phi(f_\gamma^{(0)}, f^{(0)}) \le r/4$. Combined with (4.271) this implies that $d_\phi(f, f_\gamma^{(0)}) \le r/2$.

It follows from Lemma 4.45 that there exists a neighborhood $\mathcal{U}$ of $f_\gamma^{(0)}$ in $(\mathcal{A}, d_\phi)$ such that for each $g \in \mathcal{U}$ the following property holds:

if $x \in X$ satisfies $g(x) \le \inf(g) + k$, then $g(x) - g(x_f) \ge 2^{-1}\gamma\rho(x, x_f)$.

Therefore Theorem 4.38 is proved.

4.24 Generic well-posedness for a class of equilibrium problems

The study of equilibrium problems has recently been a rapidly growing area of research. See, for example, [13, 19, 65] and the references mentioned therein.

Let (X, ρ) be a complete metric space. In this section we consider the following equilibrium problem:

$$\text{To find } x \in X \text{ such that } f(x, y) \ge 0 \text{ for all } y \in X, \tag{P}$$

where f belongs to a complete metric space of functions $\mathcal{A}$ defined below. We show that for most elements of this space of functions $\mathcal{A}$ (in the sense of Baire category) the equilibrium problem (P) possesses a unique solution. In other words, the problem (P) possesses a unique solution for a generic (typical) element of $\mathcal{A}$.

Set

$$\rho_1((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2), \ x_1, x_2, y_1, y_2 \in X.$$

Clearly, $(X \times X, \rho_1)$ is a complete metric space.

Denote by $\mathcal{A}_0$ the set of all continuous functions $f : X \times X \to R^1$ such that

$$f(x,x) = 0 \text{ for all } x \in X. \tag{4.272}$$

We equip the set $\mathcal{A}_0$ with the uniformity determined by the base

$$U(\epsilon) = \{(f,g) \in \mathcal{A}_0 \times \mathcal{A}_0 : \ |f(z) - g(z)| \le \epsilon \text{ for all } z \in X \times X\}, \tag{4.273}$$

where ϵ is a positive number. Clearly, the space $\mathcal{A}_0$ with this uniformity is metrizable (by a metric d) and complete.

Denote by $\mathcal{A}$ the set of all $f \in \mathcal{A}_0$ for which the following properties hold:

(P1) For each positive number ϵ there exists $x_\epsilon \in X$ such that $f(x_\epsilon, y) \ge -\epsilon$ for all $x \in X$.

(P2) For each positive number ϵ there exists a positive number δ such that $|f(x,y)| \le \epsilon$ for all $x, y \in X$ satisfying $\rho(x,y) \le \delta$.

Evidently, $\mathcal{A}$ is a closed subset of X . We equip the space $\mathcal{A}$ with the metric d and consider the topological subspace $\mathcal{A} \subset \mathcal{A}_0$ with the relative topology.

For each $x \in X$ and each subset $D \subset X$ put

$$\rho(x, D) = \inf\{\rho(x,y) : \ y \in D\}.$$

For each $x \in X$ and each positive number r put

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\},$$

$$B^o(x,r) = \{y \in X : \ \rho(x,y) < r\}.$$

Assume that the following property holds:

(P3) There exists a number $\Delta > 0$ such that for each $y \in X$ and each pair of real numbers t_1, t_2 satisfying $0 < t_1 < t_2 < \Delta$ there exists $z \in X$ such that $\rho(z,y) \in [t_1, t_2]$.

We will prove the following result.

Theorem 4.48. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the following properties hold:*

(i) There exists a unique $x_f \in X$ such that

$$f(x_f, y) \ge 0 \textit{ for all } x, y \in X;$$

(ii) for each positive number ϵ there exist a positive number δ and a neighborhood V of f in $\mathcal{A}$ such that for each $h \in V$ and each $x \in X$ satisfying $\inf\{h(x,y) : \ y \in X\} > -\delta$ *the inequality $\rho(x_f, x) < \epsilon$ holds.*

In other words, for a generic (typical) $f \in \mathcal{A}$ the problem (P) is well-posed.

Theorem 4.48 has been obtained in [138].

4.25 An auxiliary density result

Lemma 4.49. *Let $f \in \mathcal{A}$ and $\epsilon \in (0,1)$. Then there exist $f_0 \in \mathcal{A}$ and $x_0 \in X$ such that $(f, f_0) \in U(\epsilon)$ and $f(x_0, y) \geq 0$ for all $y \in X$.*

Proof: It follows from (P1) that there exists $x_0 \in X$ such that

$$f(x_0, y) \geq -\epsilon/16 \text{ for all } y \in X. \tag{4.274}$$

Put

$$E_1 = \{(x,y) \in X \times X : \ f(x,y) \geq -\epsilon/16\}, \tag{4.275}$$
$$E_2 = \{(x,y) \in (X \times X) \setminus E_1 : \ f(x,y) \geq -\epsilon/8\},$$
$$E_3 = (X \times X) \setminus (E_1 \cup E_2).$$

For each $(y_1, y_2) \in E_1$ there exists a positive number $r_1(y_1, y_2) < 1$ such that

$$f(z_1, z_2) > -\epsilon/14 \text{ for all } z_1, z_2 \in X \text{ satisfying}$$
$$\rho(z_i, y_i) \leq r_1(y_1, y_2), \ i = 1, 2. \tag{4.276}$$

For each $(y_1, y_2) \in E_2$ there exists a positive number $r_1(y_1, y_2) < 1$ such that

$$f(z_1, z_2) > -\epsilon/6 \text{ for all } z_1, z_2 \in X \text{ satisfying} \tag{4.277}$$
$$\rho(z_i, y_i) \leq r_1(y_1, y_2), \ i = 1, 2.$$

For each $(y_1, y_2) \in E_3$ there exists a positive number $r_1(y_1, y_2) < 1$ such that

$$f(z_1, z_2) < -\epsilon/8 \text{ for all } z_1, z_2 \in X \text{ satisfying}$$
$$\rho(z_i, y_i) \leq r_1(y_1, y_2), \ i = 1, 2. \tag{4.278}$$

For each $(y_1, y_2) \in X \times X$ put

$$U(y_1, y_2) = B^o(y_1, r_1(y_1, y_2)) \times B^o(y_2, r_1(y_1, y_2)). \tag{4.279}$$

For any $(y_1, y_2) \in E_1 \cup E_2$ set

$$g_{y_1, y_2}(z) = \max\{f(z), \ 0\}, \ z \in X \times X \tag{4.280}$$

and for any $(y_1, y_2) \in E_3$ set

$$g_{y_1, y_2}(z) = f(z), \ z \in X \times X. \tag{4.281}$$

It is easy to see that $\{U(y_1, y_2) : \ y_1, y_2 \in X\}$ is an open covering of $X \times X$. Since any metric space is paracompact there exists a continuous locally finite partition of unity $\{\phi_\beta : \ \beta \in \mathcal{B}\}$ subordinated to the covering $\{U(y_1, y_2) : y_1, y_2 \in X\}$. Namely, for any $\beta \in \mathcal{B}$, $\phi_\beta : X \times X \to [0, 1]$ is a continuous function and there exist $y_1(\beta), \ y_2(\beta) \in X$ such that $\text{supp}(\phi_\beta) \subset U(y_1(\beta), y_2(\beta))$ and that

$$\sum_{\beta \in \mathcal{B}} \phi_\beta(z) = 1 \text{ for all } z \in X \times X.$$

Define

$$f_0(z) = \sum_{\beta \in \mathcal{B}} \phi_\beta(z) g_{(y_1(\beta), y_2(\beta))}(z), \ z \in X \times X. \tag{4.282}$$

It is easy to see that f_0 is well-defined, continuous and satisfies

$$f_0(z) \geq f(z) \text{ for all } z \in X \times X. \tag{4.283}$$

Let $(z_1, z_2) \in E_1$. Then

$$f(z_1, z_2) \geq -\epsilon/16. \tag{4.284}$$

Assume that $\beta \in \mathcal{B}$ and that $\phi_\beta(z_1, z_2) > 0$. Then

$$(z_1, z_2) \in \ \mathrm{supp}(\phi_\beta) \subset U(y_1(\beta), y_2(\beta)). \tag{4.285}$$

If $(y_1(\beta), y_2(\beta)) \in E_3$, then by (4.278), (4.279) and (4.285), $f(z_1, z_2) < -\epsilon/8$, a contradiction (see (4.284)). Then $(y_1(\beta), y_2(\beta)) \in E_1 \cup E_2$ and (4.280) implies that

$$g_{y_1(\beta), y_2(\beta)}(z_1, z_2) = \max\{f(z_1, z_2), \ 0\}.$$

Since this equality holds for any $\beta \in \mathcal{B}$ satisfying $\phi_\beta(z_1, z_2) > 0$ the equality (4.282) implies that

$$f_0(z_1, z_2) = \max\{f(z_1, z_2), \ 0\} \tag{4.286}$$

for all $(z_1, z_2) \in E_1$.

By (4.274), (4.275) and (4.286),

$$f_0(x_0, y) \geq 0, \ y \in X. \tag{4.287}$$

It follows from (4.272), (4.280), (4.281) and (4.282) that

$$f_0(x, x) = 0, \ x \in X. \tag{4.288}$$

Assume that

$$(z_1, z_2) \in E_2. \tag{4.289}$$

Then (4.275) and (4.289) imply that $f(z_1, z_2) \geq -\epsilon/8$. Combined with (4.280) and (4.282) this implies that

$$f_0(z_1, z_2) \leq \sum_{\beta \in \mathcal{B}} \phi_\beta(z_1, z_2)(f(z_1, z_2) + \epsilon/8) = f(z_1, z_2) + \epsilon/8.$$

Together with (4.283) this implies that

$$f(z_1, z_2) \leq f_0(z_1, z_2) \leq f(z_1, z_2) + \epsilon/8 \tag{4.290}$$

for all $(z_1, z_2) \in E_2$.

Let

$$(z_1, z_2) \in E_3 \tag{4.291}$$

and assume that

$$\beta \in \mathcal{B},\ \phi_\beta(z_1, z_2) > 0. \tag{4.292}$$

Then (4.292) implies that

$$(z_1, z_2) \in \ \mathrm{supp}(\phi_\beta) \subset U(y_1(\beta), y_2(\beta)). \tag{4.293}$$

It follows from (4.293) and the choice of $U(y_1(\beta), y_2(\beta))$ (see (4.276)–(4.279)) that

$$(y_1(\beta), y_2(\beta)) \notin E_1$$

and in view of (4.277), (4.279), (4.280) and (4.281),

$$g_{y_1(\beta), y_2(\beta)}(z_1, z_2) \le f(z_1, z_2) + \epsilon/6.$$

Since the inequality above holds for any $\beta \in \mathcal{B}$ satisfying (4.292) it follows from (4.282) that

$$f_0(z_1, z_2) \le f(z_1, z_2) + \epsilon/6.$$

Combined with (4.283), (4.284) and (4.286) this implies that for all $(z_1, z_2) \in X \times X$

$$f(z_1, z_2) \le f_0(z_1, z_2) \le f(z_1, z_2) + \epsilon/6. \tag{4.294}$$

It follows from (4.288) that $f_0 \in \mathcal{A}_0$. By (4.287) f_0 possesses (P1). Since f possesses (P2), (4.280), (4.281) and (4.282) imply that f_0 possesses (P2). Therefore $f_0 \in \mathcal{A}$ and Lemma 4.49 now follows from (4.287) and (4.294).

4.26 A perturbation lemma

Lemma 4.50. *Let $\epsilon \in (0, 1)$, $f \in \mathcal{A}$ and let $x_0 \in X$ satisfy*

$$f(x_0, y) \ge 0 \text{ for all } y \in X. \tag{4.295}$$

Then there exist $g \in \mathcal{A}$ and a positive number δ such that

$$g(x_0, y) \ge 0 \text{ for all } y \in X,\ |(g - f)(x, y)| \le \epsilon/4 \text{ for all } x, y \in X$$

and if $x \in X$ satisfies $\inf\{g(x, y) :\ y \in X\} > -\delta$, then $\rho(x_0, x) < \epsilon/8$.

Proof: It follows from (P2) that there exists a positive number

$$\delta_0 < \min\{16^{-1}\epsilon, 16^{-1}\Delta\} \tag{4.296}$$

such that

$$|f(y, z)| \le \epsilon/16 \text{ for all } y, z \in X \text{ satisfying } \rho(y, z) \le 4\delta_0. \tag{4.297}$$

Put

$$\delta = 2^{-1}\delta_0. \tag{4.298}$$

Define

$$\phi(t) = 1,\ t \in [0, \delta_0],\ \phi(t) = 0,\ t \in [2\delta_0, \infty), \tag{4.299}$$
$$\phi(t) = 2 - t\delta_0^{-1},\ t \in (\delta_0, 2\delta_0)$$

and

$$f_1(x, y) = -\phi(\rho(x, y))\rho(x, y) + (1 - \phi(\rho(x, y)))f(x, y),\ x, y \in X. \tag{4.300}$$

It is easy to see that f_1 is continuous and

$$f_1(x, x) = 0 \text{ for all } x \in X.$$

(4.299) and (4.300) imply that

$$f_1(x, y) = -\rho(x, y) \text{ for all } x, y \in X \text{ satisfying } \rho(x, y) \le \delta_0. \tag{4.301}$$

Let $x, y \in X$. We estimate $|f(x, y) - f_1(x, y)|$. If $\rho(x, y) \ge 2\delta_0$, then (4.299) and (4.300) imply that

$$|f_1(x, y) - f(x, y)| = 0. \tag{4.302}$$

Assume that

$$\rho(x, y) \le 2\delta_0. \tag{4.303}$$

It follows from (4.296) and (4.303) that

$$|f(x, y)| \le \epsilon/16. \tag{4.304}$$

In view of (4.296), (4.299), (4.300), (4.303) and (4.304),

$$|f_1(x, y) - f(x, y)| \le \rho(x, y) + |f(x, y)| \le 2\delta_0 + \epsilon/16 < \epsilon/4.$$

Combined with (4.302) this implies that

$$|f_1(x, y) - f(x, y)| < \epsilon/4 \text{ for all } x, y \in X. \tag{4.305}$$

Assume that $x \in X$. It follows from (P3) and (4.296) that there exists $y \in X$ such that

$$\rho(y, x) \in [2^{-1}\delta_0, \delta_0]. \tag{4.306}$$

By (4.306) and (4.301),

$$f_1(x, y) = -\rho(y, x) \le -2^{-1}\delta_0$$

and

$$\inf\{f_1(x, z) :\ z \in X\} \le -2^{-1}\delta_0 \tag{4.307}$$

for all $x \in X$. Put

$$g(x,y) = \phi(\rho(x,x_0))f(x,y) + (1-\phi(\rho(x,x_0)))f_1(x,y),\ x,y \in X. \quad (4.308)$$

It is easy to see that the function g is continuous and

$$g(x,x) = 0 \text{ for all } x \in X. \quad (4.309)$$

It follows from (4.295), (4.308) and (4.299) that

$$g(x_0,y) = f(x_0,y) \geq 0 \text{ for all } y \in X.$$

Since the function f possesses (P2) it follows from the relation above, (4.301) and (4.308) that g possesses the property (P2). Thus $g \in \mathcal{A}$.

In view of (4.299), (4.305) and (4.308) for all $x, y \in X$

$$|(f-g)(x,y)| \leq |f_1(x,y) - f(x,y)| \leq \epsilon/4.$$

Assume that

$$x \in X \text{ and } \inf\{g(x,y) :\ y \in X\} > -2^{-1}\delta_0 = -\delta. \quad (4.310)$$

If $\rho(x_0,x) \geq 2\delta_0$, then (4.299) and (4.308) imply that

$$g(x,y) = f_1(x,y) \text{ for all } y \in Y$$

and combined with (4.307) this implies that

$$\inf\{g(x,y) :\ y \in X\} \leq -2^{-1}\delta_0.$$

This inequality contradicts (4.310). The contradiction we have reached proves that

$$\rho(x_0,x) < 2\delta_0 < \epsilon/8.$$

Lemma 4.50 is proved.

4.27 Proof of Theorem 4.48

Denote by E the set of all $f \in \mathcal{A}$ for which there exists $x \in X$ such that $f(x,y) \geq 0$ for all $y \in X$. Lemma 4.49 implies that E is an everywhere dense subset of $\mathcal{A}$.

Let $f \in E$ and $n \geq 1$ be an integer. There exists $x_f \in X$ such that

$$f(x_f,y) \geq 0 \text{ for all } y \in X. \quad (4.311)$$

Lemma 4.50 implies that there exist $g_{f,n} \in \mathcal{A}$ and $\delta_{f,n} > 0$ such that

$$g_{f,n}(x_f,y) \geq 0 \text{ for all } y \in X,\ |(g_{f,n}-f)(x,y)| \leq (4n)^{-1} \text{ for all } x,y \in X \quad (4.312)$$

and that the following property holds:

(P4) For each $x \in X$ satisfying $\inf\{g_{f,n}(x,y) : \ y \in X\} > -\delta_{f,n}$ the inequality $\rho(x_f, x) < (4n)^{-1}$ holds.

Denote by $V(f,n)$ the open neighborhood of $g_{f,n}$ in $\mathcal{A}$ such that

$$V(f,n) \subset \{h \in \mathcal{A} : \ (h, g_{f,n}) \in U(4^{-1}\delta_{f,n})\}. \tag{4.313}$$

Assume that

$$x \in X, \ h \in V(f,n), \ \inf\{h(x,y) : \ y \in X\} > -2^{-1}\delta_{f,n}. \tag{4.314}$$

It follows from (4.273), (4.313) and (4.314) that

$$\inf\{g_{f,n}(x,y) : \ y \in X\} \geq \inf\{h(x,y) : \ y \in X\} - 4^{-1}\delta_{f,n} > -\delta_{f,n}. \tag{4.315}$$

By (4.315) and (P4),

$$\rho(x_f, x) < (4n)^{-1}. \tag{4.316}$$

Thus we have shown that the following property holds:

(P5) For each $x \in X$ and each $h \in V(f,n)$ satisfying (4.314) the inequality $\rho(x_f, x) < (4n)^{-1}$ holds.

Put

$$\mathcal{F} = \cap_{k=1}^{\infty} \cup \{V(f,n) : \ f \in E \text{ and an integer } n \geq k\}. \tag{4.317}$$

It is clear that $\mathcal{F}$ is a countable intersection of open everywhere dense subset of $\mathcal{A}$. Let

$$\xi \in \mathcal{F}, \ \epsilon > 0. \tag{4.318}$$

Fix an integer $k > 8(\epsilon^{-1} + 1)$. There exist $f \in E$ and an integer $n \geq k$ such that

$$\xi \in V(f,n). \tag{4.319}$$

By property (P4), (4.313) and (4.319), for each $x \in X$ satisfying

$$\inf\{\xi(x,y) : \ y \in X\} > -2^{-1}\delta_{f,n} \tag{4.320}$$

we have

$$\inf\{g_{f,n}(x,y) : \ y \in X\} > -2^{-1}\delta_{f,n} - 4^{-1}\delta_{f,n} > -\delta_{f,n}$$

and

$$\rho(x_f, x) < (4n)^{-1} < \epsilon/8.$$

Thus we have shown that the following property holds:

(P6) For each $x \in X$ satisfying (4.320) the inequality $\rho(x_f, x) < \epsilon/8$ holds.

It follows from (P1) that there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ such that

$$\liminf_{i \to \infty} (\inf\{\xi(x_i, y) : \ y \in X\}) \geq 0. \tag{4.321}$$

It follows from (4.321) and (P6) that for all large enough natural numbers i, j,

$$\rho(x_i, x_j) \leq \rho(x_i, x_f) + \rho(x_f, x_j) < \epsilon/4.$$

Since ϵ is any positive number we conclude that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists

$$x_\xi = \lim_{i\to\infty} x_i. \tag{4.322}$$

By (4.321) and (4.322), for all $y \in X$

$$\xi(x_\xi, y) = \lim_{i\to\infty} \xi(x_i, y) \geq 0. \tag{4.323}$$

We have also shown that any sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying (4.321) converges. This implies that if $x \in X$ satisfies $\xi(x, y) \geq 0$ for all $y \in X$, then $x = x_\xi$. It follows from (P6) and (4.323) that

$$\rho(x_\xi, x_f) \leq \epsilon/8. \tag{4.324}$$

Let $x \in X$ and $h \in V(f, n)$ satisfy (4.314). (P5) implies that $\rho(x_f, x) < (4n)^{-1}$. Combined with (4.324) this implies that

$$\rho(x, x_\xi) \leq \rho(x, x_f) + \rho(x_f, x_\xi) < (4n)^{-1} + \epsilon/8 < \epsilon.$$

This completes the proof of Theorem 4.48.

4.28 Comments

In this chapter we use the generic approach in order to show that solutions of minimization problems exist generically for different classes of problems. Any of these classes of problems is identified with a space of functions equipped with a natural complete metric and it is shown that there exists a G_δ everywhere dense subset of the space of functions such that for any element of this subset the corresponding minimization problem possesses a unique solution and that any minimizing sequence converges to this unique solution. These results are obtained as realizations of variational principles which are extensions or concretization of the variational principle established in [52].

5

Well-Posedness and Porosity

5.1 σ-porous sets in a metric space

We recall the concept of porosity [10, 26, 27, 84, 97, 98, 112].

Let (Y, d) be a complete metric space. We denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous with respect to d (or just porous if the metric is understood) if there exist $\alpha \in (0, 1]$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists $z \in Y$ for which

$$B_d(z, \alpha r) \subset B_d(y, r) \setminus E.$$

In the above definition of porosity it is known that the point y can be assumed to belong to E.

A subset of the space Y is called σ-porous with respect to d (or just σ-porous if the metric is understood) if it is a countable union of porous (with respect to d) subsets of Y.

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category. Also, every Banach space contains a set of the first category which is not σ-porous.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

Let (X, ρ) be a metric space. For each $x \in X$ and each $r > 0$ set

$$B_\rho(x, r) = \{y \in X : \rho(x, y) \leq r\}. \tag{5.1}$$

We begin with the following simple proposition. It shows that porosity is equivalent to another property which is easier to verify.

A.J. Zaslavski, *Optimization on Metric and Normed Spaces*, Springer Optimization and Its Applications 44, DOI 10.1007/978-0-387-88621-3_5,

Proposition 5.1. *A set $E \subset X$ is porous with respect to ρ if and only if the following property holds:*

(P1) There exist $\alpha \in (0,1]$ and $r_0 > 0$ such that for every $x \in X$ and every $r \in (0, r_0]$ there is $y \in X$ which satisfies

$$\rho(x,y) \le r \text{ and } B_\rho(y, \alpha r) \cap E = \emptyset.$$

Proof: Assume that the property (P1) holds with $\alpha \in (0,1]$ and $r_0 > 0$. We need to show that E is porous. Choose a positive number $\gamma < 4^{-1}$. Let $x \in X$ and $r \in (0, r_0]$. By the property (P1) there exists $y \in X$ such that

$$\rho(y,x) \le \gamma r \text{ and } B_\rho(y, \alpha\gamma r) \cap E = \emptyset. \tag{5.2}$$

It is easy to see that

$$B_\rho(y, \alpha\gamma r) \subset B_\rho(x, \alpha\gamma r + \gamma r) \subset B_\rho(x, r). \tag{5.3}$$

(5.2) and (5.3) imply that E is porous. Proposition 5.1 is proved.

Assume now that X is a nonempty set and $\rho_1, \rho_2 : X \times X \to [0, \infty)$ are metrics which satisfy $\rho_1(x,y) \le \rho_2(x,y)$ for all $x, y \in X$.

The following definition was introduced in [112].

A subset $E \subset X$ is called porous with respect to the pair (ρ_1, ρ_2) (or just porous if the pair of metrics is understood) if there exist $\alpha \in (0,1]$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $x \in X$ there exists $y \in X$ for which

$$\rho_2(y,x) \le r \text{ and } B_{\rho_1}(y, \alpha r) \cap E = \emptyset. \tag{5.4}$$

A subset of the space X is called σ-porous with respect to (ρ_1, ρ_2) (or just σ-porous if the pair of metrics is understood) if it is a countable union of porous (with respect to (ρ_1, ρ_2)) subsets of X.

We use this generalization of the porosity notion because in applications a space is usually endowed with a pair of metrics and one of them is weaker than the other. Note that porosity of a set with respect to one of these two metrics does not imply its porosity with respect to the other metric. However, the following proposition shows that if a subset $E \subset X$ is porous with respect to (ρ_1, ρ_2), then E is porous with respect to any metric which is weaker than ρ_2 and stronger than ρ_1.

Proposition 5.2. *Let k_1 and k_2 be positive numbers and $\rho : X \times X \to [0, \infty)$ be a metric such that $k_1\rho(x,y) \ge \rho_1(x,y)$ and $k_2\rho(x,y) \le \rho_2(x,y)$ for all $x, y \in X$. Assume that a set $E \subset X$ is porous with respect to (ρ_1, ρ_2). Then E is porous with respect to ρ.*

Proof: We may assume without loss of generality that $k_1 > 1$ and $k_2 < 1$. Since E is porous with respect to (ρ_1, ρ_2) there exist $\alpha \in (0,1]$ and $r_0 > 0$ such that for every $x \in X$ and every $r \in (0, r_0]$ there is $y \in X$ satisfying (5.4).

Fix a positive number $\gamma < k_2$. Let $x \in X$ and $r \in (0, r_0]$. By our choice of α and r_0 there exists $y \in X$ such that

$$\rho_2(y, x) \leq \gamma r \text{ and } B_{\rho_1}(y, \gamma\alpha r) \cap E = \emptyset.$$

It is easy to see that

$$\rho(y, x) \leq k_2^{-1}\rho_2(x, y) \leq k_2^{-1}\gamma r < r$$

and

$$B_\rho(y, k_1^{-1}\gamma\alpha r) \subset B_{\rho_1}(y, \gamma\alpha r) \subset X \setminus E.$$

These relations and Proposition 5.1 imply that E is porous with respect to ρ. Proposition 5.2 is proved.

Note that in our definition of porosity with respect to (ρ_1, ρ_2) we do not use completeness assumptions. However, in the chapter the metric space (X, ρ_2) will always be complete. This implies that a σ-porous set with respect to (ρ_1, ρ_2) is an everywhere dense subset of the metric space (X, ρ_2).

We usually consider metric spaces, say X, with two metrics, say d_w and d_s, such that $d_w(x, y) \leq d_s(x, y)$ for all $x, y \in X$. (Note that they can coincide.) We refer to them as the weak and strong metrics, respectively. The strong metric induces the strong topology and the weak metric induces the weak topology. If the set X is equipped with one metric d, then we also consider X with weak and strong metrics which coincide. If (X, d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). If X is endowed with weak and strong metrics, then usually Y is also endowed with these metrics.

In the sequel a set X equipped with two metrics d_1 and d_2 satisfying $d_1(x, y) \leq d_2(x, y)$ for all $x, y \in X$ will be denoted by (X, d_1, d_2).

If (X_i, d_i), $i = 1, 2$, are metric spaces with the metrics d_1 and d_2, respectively, then the space $X_1 \times X_2$ will be endowed with the metric $d_1 \times d_2$ defined by

$$(d_1 \times d_2)((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$
$$(x_1, x_2), (y_1, y_2) \in X_1 \times X_2. \tag{5.5}$$

Assume now that X_1 and X_2 are metric spaces and each of them is endowed with weak and strong metrics. Then for the product $X_1 \times X_2$ we also introduce a pair of metrics: a weak metric which is defined by (5.5) using the weak metrics of X_1 and X_2 and a strong metric which is defined by (5.5) using the strong metrics of X_1 and X_2.

5.2 Well-posedness of optimization problems

We use the convention that $\infty - \infty = 0$. For each function $f : X \to [-\infty, \infty]$, where X is nonempty, we set

$$\inf(f) = \inf\{f(x) :\ x \in X\}.$$

We consider a metric space (X, ρ) which is called the domain space and a topological space $\mathcal{A}$ with the topology τ which is called the data space. We always consider the set X with the topology generated by the metric ρ.

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on (X, ρ) is well-posed if $\inf(f_a)$ is finite and attained at a unique point x_a and the following assertion holds:

For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $z \in X$ satisfying $f_a(z) \le \inf(f_a) + \delta$ the inequality $\rho(z, x_a) \le \epsilon$ holds.

The following property was studied in Chapter 4.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $(\mathcal{A}, \tau)$ if $\inf(f_a)$ is finite and attained at a unique point x_a and the following assertion holds:

For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the topology τ and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(z, x_a) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

If $(\mathcal{A}, d)$ is a metric space and τ is a topology generated by the metric d, then "strongly well-posedness with respect to $(\mathcal{A}, \tau)$" will be sometimes replaced by "strongly well-posedness with respect to $(\mathcal{A}, d)$".

The following proposition is the main result of this section.

Proposition 5.3. *Assume that $a \in \mathcal{A}$, the minimization problem for f_a on (X, ρ) is well-posed and that there exists $D > \inf(f_a)$ for which the following property holds:*

(P2) For each $\epsilon > 0$ there exists a neighborhood $\mathcal{U}$ of a in $\mathcal{A}$ with the topology τ such that for each $b \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{f_a(x),\ f_b(x)\} \le D$ the relation $|f_a(x) - f_b(x)| \le \epsilon$ holds.

Then the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $(\mathcal{A}, \tau)$.

Proof: There exists a unique $x_a \in X$ such that

$$f(x_a) = \inf(f_a) \in R^1.$$

Let $\epsilon > 0$. There exists a number $\delta \in (0, \epsilon)$ such that the following property holds:

(P3) For each $z \in X$ satisfying $f_a(z) \le \inf(f_a) + \delta$ the inequality $\rho(z, x_a) \le \epsilon$ holds.

We may assume without loss of generality that

$$\inf(f_a) + 2\delta < D. \tag{5.6}$$

By property (P2) there exists a neighborhood $\mathcal{U}$ of a in $\mathcal{A}$ with the topology τ such that for each $b \in \mathcal{U}$ and each $x \in X$ satisfying

$$\min\{f_a(x),\ f_b(x)\} \le D \tag{5.7}$$

the inequality $|f_a(x) - f_b(x)| \le \delta/4$ holds.

Clearly,

$$\inf(f_a) = \inf(f_a(x) :\ x \in X \text{ and } f_a(x) \le \inf(f_a) + \delta/4\}. \tag{5.8}$$

Let $b \in \mathcal{U}$. It follows from the definition of $\mathcal{U}$ and (5.6) that

$$|f_a(x) - f_b(x)| \le \delta/4$$

for each $x \in X$ satisfying

$$f_a(x) \le \inf(f_a) + \delta/4 < D.$$

Together with (5.8) this implies that

$$\begin{aligned}\inf(f_b) &\le \inf\{f_b(x) :\ x \in X \text{ and } f_a(x) \le \inf(f_a) + \delta/4\}\\ &\le \inf\{f_a(x) + \delta/4 :\ x \in X \text{ and } f_a(x) \le \inf(f_a) + \delta/4\}\\ &= \inf(f_a) + \delta/4.\end{aligned} \tag{5.9}$$

It is easy now to see that

$$\inf(f_b) = \inf\{f_b(x) :\ x \in X \text{ and } f_b(x) \le \inf(f_a) + \delta/2\}. \tag{5.10}$$

It follows from the definition of $\mathcal{U}$ and (5.6) that if $x \in X$ satisfies

$$f_b(x) \le \inf(f_a) + \delta/2 < D,$$

then $|f_a(x) - f_b(x)| \le \delta/4$. Together with (5.9) and (5.10) this implies that

$$\begin{aligned}-\infty < \inf(f_a) &\le \inf\{f_a(x) :\ x \in X \text{ and } f_b(x) \le \inf(f_a) + \delta/2\}\\ &\le \inf\{f_b(x) + \delta/4 :\ x \in X \text{ and } f_b(x) \le \inf(f_a) + \delta/2\}\\ &= \inf(f_a) + \delta/4.\end{aligned} \tag{5.11}$$

By (5.11) and (5.9),

$$|\inf(f_a) - \inf(f_b)| \le \delta/4. \tag{5.12}$$

Assume now that $z \in X$ and

$$f_b(z) \le \inf(f_b) + \delta/4. \tag{5.13}$$

By (5.12) and (5.6),

$$f_b(z) \le \inf(f_a) + \delta/2 < D. \tag{5.14}$$

It follows from the definition of $\mathcal{U}$ and (5.14) that

$$|f_b(z) - f_a(z)| \le \delta/4. \tag{5.15}$$

(5.13) and (5.12) imply that

$$|f_b(z) - f_a(x_a)| = |f_b(z) - \inf(f_a)|$$

$$\leq |f_b(z) - \inf(f_b)| + |\inf(f_b) - \inf(f_a)| \leq \delta/4 + \delta/4 < \epsilon. \tag{5.16}$$

By (5.15) and (5.14),

$$f_a(z) \leq f_b(z) + \delta/4 \leq \inf(f_a) + \delta/2 + \delta/4.$$

Combined with property (P3) this implies that $\rho(z, x_a) \leq \epsilon$.

Thus we have constructed the neighborhood $\mathcal{U}$ of a in $\mathcal{A}$ with the topology τ such that for each $b \in \mathcal{U}$, $\inf(f_b)$ is finite (see (5.12)) and if $z \in X$ satisfies (5.13), then the inequalities $\rho(z, x_a) \leq \epsilon$ and $|f_b(z) - f_a(x_a)| \leq \epsilon$ hold (see (5.16)). Therefore the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $(\mathcal{A}, \tau)$ by definition. Proposition 5.3 is proved.

Proposition 5.3 has been obtained in [112].

5.3 A variational principle

We consider a metric space (X, ρ) which is called the domain space and a set $\mathcal{A}$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$. Assume that $d_1, d_2 : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are metrics such that $d_1(a, b) \leq d_2(a, b)$ for all $a, b \in \mathcal{A}$.

We use the following basic hypotheses about the functions.

(H1) If $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in X.

(H2) For each $\epsilon > 0$ and each integer $m \geq 1$ there exist numbers $\delta > 0$ and $r_0 > 0$ such that the following property holds:

For each $a \in \mathcal{A}$ satisfying $\inf(f_a) \leq m$ and each $r \in (0, r_0]$ there exist $\bar{a} \in \mathcal{A}$ and $\bar{x} \in X$ such that

$$d_2(a, \bar{a}) \leq r, \ \inf(f_{\bar{a}}) \leq m + 1$$

and for each $z \in X$ satisfying $f_{\bar{a}}(z) \leq \inf(f_{\bar{a}}) + \delta r$ the inequality $\rho(z, \bar{x}) \leq \epsilon$ holds.

(H3) For each integer $n \geq 1$ there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$, each $a, b \in \mathcal{A}$ satisfying $d_1(a, b) \leq \alpha r$ and each $x \in X$ satisfying $\min\{f_a(x), f_b(x)\} \leq n$ the relation $|f_a(x) - f_b(x)| \leq r$ is valid.

Clearly, if (X, ρ) is complete, then (H1) holds. Note that for classes of optimal control problems considered in [112] the domain space is not complete. Fortunately, in [112] instead of completeness assumption we can use (H1) and this hypothesis holds for spaces of integrands which satisfy the Cesari growth condition.

For each integer $n \geq 1$ denote by $\mathcal{A}_n$ the set of all $a \in \mathcal{A}$ which have the following property:

(P4) $\inf(f_a)$ is finite and there exist $\bar{x} \in X$ and $\delta > 0$ such that if $z \in X$ satisfies $f_a(z) \leq \inf(f_a) + \delta$, then $\rho(z, \bar{x}) \leq 1/n$.

Proposition 5.4. *Assume that (H1) holds and that for each integer $n \geq 1$ the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous with respect to (d_1, d_2). Then the set $\mathcal{A} \setminus (\cap_{n=1}^{\infty} \mathcal{A}_n)$ is σ-porous with respect to (d_1, d_2) and for each $a \in \cap_{n=1}^{\infty} \mathcal{A}_n$ the minimization problem for f_a on (X, ρ) is well-posed.*

Proof: It is easy to see that the set

$$\mathcal{A} \setminus (\cap_{n=1}^{\infty} \mathcal{A}_n) = \cup_{n=1}^{\infty} (\mathcal{A} \setminus \mathcal{A}_n)$$

is σ-porous with respect to (d_1, d_2).

Let $a \in \cap_{n=1}^{\infty} \mathcal{A}_n$. It is sufficient to show that the minimization problem for f_a on (X, ρ) is well-posed. Evidently, $\inf(f_a)$ is finite. By property (P4) for each integer $n \geq 1$ there exist $x_n \in X$ and $\delta_n > 0$ such that the following property holds:

(P5) If $z \in X$ satisfies $f_a(z) \leq \inf(f_a) + \delta_n$, then $\rho(z, x_n) \leq 1/n$.

Assume that $\{z_i\}_{i=1}^{\infty} \subset X$ and $\lim_{i \to \infty} f_a(z_i) = \in (f_a)$. It follows from (P5) that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence. By (H1) there exists $\bar{x} \in X$ such that $\bar{x} = \lim_{i \to \infty} z_i$. Since f_a is a lower semicontinuous function we obtain that $f_a(\bar{x}) = \inf(f_a)$. Clearly, f_a does not have another minimizer for otherwise we would be able to construct a nonconvergent sequence $\{z_i\}_{i=1}^{\infty}$. We have shown that if $\{z_i\}_{i=1}^{\infty} \subset X$ and $\lim_{i \to \infty} f_a(z_i) = \inf(f_a)$, then $\rho(z_i, \bar{x}) \to 0$ as $i \to \infty$. This implies that the minimization problem for f_a on (X, ρ) is well-posed. Proposition 5.4 is proved.

Theorem 5.5. *(Variational principle). Assume that (H1), (H2) and (H3) hold and that* $\inf(f_a)$ *is finite for each $a \in \mathcal{A}$. Then there exists a set $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{B}$ is σ-porous with respect to (d_1, d_2) and that for each $a \in \mathcal{B}$ the minimization problem for f_a on (X, ρ) is well-posed.*

Proof: We recall that $\mathcal{A}_n$ is the set of all $a \in \mathcal{A}$ which have the property (P4) $(n = 1, 2, \dots)$. By Proposition 5.4 in order to prove the theorem it is sufficient to show that for each integer $n \geq 1$ the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous with respect to (d_1, d_2).

Let $m, n \geq 1$ be integers. Set

$$\Omega_{mn} = \{a \in \mathcal{A} \setminus \mathcal{A}_n : \ \inf(f_a) \leq m\}.$$

To prove the theorem it is sufficient to show that Ω_{mn} is porous with respect to (d_1, d_2). By (H3) there exist

$$\alpha_1 \in (0,1),\ r_1 \in (0,1/2) \tag{5.17}$$

such that for each $r \in (0, r_1]$, each $b_1, b_2 \in \mathcal{A}$ satisfying $d_1(b_1, b_2) \leq \alpha_1 r$ and each $x \in X$ satisfying

$$\min\{f_{b_1}(x),\ f_{b_2}(x)\} \leq m + 4 \tag{5.18}$$

the inequality $|f_{b_1}(x) - f_{b_2}(x)| \leq r$ holds. By (H2) there exist

$$\alpha_2 \in (0,1),\ r_2 \in (0,1] \tag{5.19}$$

such that the following property holds:

(P6) For each $a \in \mathcal{A}$ satisfying $\inf(f_a) \leq m+2$ and each $r \in (0, r_2]$ there exist $\bar{a} \in \mathcal{A}$ and $\bar{x} \in X$ such that

$$d_2(a, \bar{a}) \leq r,\ \inf(f_{\bar{a}}) \leq m + 3 \tag{5.20}$$

and for each $z \in X$ satisfying $f_{\bar{a}}(z) \leq \inf(f_{\bar{a}}) + 4\alpha_2 r$ the inequality $\rho(z, \bar{x}) \leq n^{-1}$ is valid.

Define

$$\bar{r} = \alpha_1 \alpha_2 r_2,\ \bar{\alpha} = \alpha_1 \alpha_2 / 4. \tag{5.21}$$

Let $a \in \mathcal{A}$ and $r \in (0, \bar{r}]$. Consider the set

$$E_0 = \{b \in \mathcal{A}:\ d_2(b, a) \leq r/4\}. \tag{5.22}$$

There are two cases:

$$E_0 = \{b \in \mathcal{A}:\ \inf(f_b) \leq m + 2\} = \emptyset, \tag{5.23}$$

$$E_0 \cap \{b \in \mathcal{A}:\ \inf(f_b) \leq m + 2\} \neq \emptyset. \tag{5.24}$$

Assume that (5.23) holds. We will show that for each $b \in \mathcal{A}$ satisfying $d_1(a, b) \leq \bar{r}$ the inequality $\inf(f_b) > m$ is true.

Assume the contrary. Then there exists $b \in \mathcal{A}$ such that

$$d_1(a, b) \leq \bar{r} \text{ and } \inf(f_b) \leq m. \tag{5.25}$$

Choose $y \in X$ such that

$$f_b(y) \leq \inf(f_b) + 1/2 \leq m + 1/2. \tag{5.26}$$

It follows from the choice of α_1, r_1 (see (5.17) and (5.18)), (5.25) and (5.21) that

$$|f_a(y) - f_b(y)| \leq r_2 \alpha_2.$$

This inequality, (5.26) and (5.17) imply that

$$\inf(f_a) \leq f_a(y) \leq f_b(y) + r_1 \leq m + 1,$$

a contradiction (see (5.23) and (5.22)). Therefore if (5.23) holds, then

$$\{b \in \mathcal{A} : \; d_1(a,b) \le \bar{r}\} \cap \Omega_{mn} = \emptyset. \tag{5.27}$$

Assume now that (5.24) holds. There exists $a_1 \in \mathcal{A}$ such that

$$d_2(a,a_1) \le r/4, \; \inf(f_{a_1}) \le m+2. \tag{5.28}$$

By the choice of α_2 and r_2 (see (5.19) and the property (P6)) and (5.28), there exist $\bar{a} \in \mathcal{A}$ and $\bar{x} \in X$ such that

$$d_2(\bar{a},a_1) \le r/4, \; \inf(f_{\bar{a}}) \le m+3 \tag{5.29}$$

and the following property holds:

(P7) For each $x \in X$ satisfying $f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + \alpha_2 r$ the inequality $\rho(x,\bar{x}) \le n^{-1}$ holds.

(5.29) and (5.28) imply that

$$d_2(a,\bar{a}) \le r/2. \tag{5.30}$$

Assume that $b \in \mathcal{A}$ satisfies

$$d_1(b,\bar{a}) \le \bar{\alpha} r = \alpha_1 \alpha_2 r/4. \tag{5.31}$$

By (5.29),

$$\inf(f_{\bar{a}}) = \inf\{f_{\bar{a}}(x) : \; x \in X \text{ and } f_{\bar{a}}(x) \le m+7/2\}. \tag{5.32}$$

Let

$$x \in X \text{ and } f_{\bar{a}}(x) \le m+7/2. \tag{5.33}$$

It follows from (5.31), (5.33) and the choice of α_1 and r_1 (see (5.17) and (5.18)) that $|f_{\bar{a}}(x) - f_b(x)| \le \alpha_2 r/4$. Together with (5.32) this implies that

$$\inf(f_b) \le \inf\{f_b(x) : \; x \in X \text{ and } f_{\bar{a}}(x) \le m+7/2\}$$

$$\le \inf\{f_{\bar{a}}(x) + \alpha_2 r/4 : \; x \in X \text{ and } f_{\bar{a}}x) \le m+7/2\} = \inf(f_{\bar{a}}) + \alpha_2 r/4.$$

Thus

$$\inf(f_b) \le \inf(f_{\bar{a}}) + \alpha_2 r/4. \tag{5.34}$$

Assume now that

$$x \in X \text{ and } f_b(x) \le \inf(f_b) + \alpha_2 r/4. \tag{5.35}$$

Combined with (5.34) this implies that

$$f_b(x) \le \inf(f_{\bar{a}}) + \alpha_2 r/2. \tag{5.36}$$

It follows from (5.36), (5.29) and (5.21) that $f_b(x) \le m+7/2$. Then by (5.31) and the choice of α_1 and r_1 (see (5.17) and (5.18)), $|f_{\bar{a}}(x) - f_b(x)| \le \alpha_2 r/4$. Combined with (5.36) this implies that

$$f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + 3\alpha_2 r/4.$$

It follows from this relation and the property (P7) that $\rho(x, \bar{x}) \le n^{-1}$. Since this inequality holds for any $x \in X$ satisfying (5.35) we conclude that $b \in \mathcal{A}_n$. Thus we have shown that $b \in \mathcal{A}_n$ for each $b \in \mathcal{A}$ satisfying $d_1(b, \bar{a}) \le \bar{\alpha} r$. Therefore

$$\{b \in \mathcal{A} : \ d_1(b, \bar{a}) \le \bar{\alpha} r\} \cap \Omega_{mn} = \emptyset. \tag{5.37}$$

Combined with (5.27) and (5.30) this implies that in both cases (5.37) is true with $\bar{a} \in \mathcal{A}$ satisfying $d_2(a, \bar{a}) \le r/2$. (Note that if (5.23) holds, then $\bar{a} = a$.) Thus Ω_{mn} is porous. This completes the proof of Theorem 5.5.

Since the property (P2) for the space $(\mathcal{A}, d_1)$ (see Proposition 5.3) follows from (H3) we obtain that Theorem 5.5 and Proposition 5.3 imply the following result.

Theorem 5.6. *Assume that (H1), (H2) and (H3) hold and that* $\inf(f_a)$ *is finite for each* $a \in \mathcal{A}$*. Then there exists a set* $\mathcal{B} \subset \mathcal{A}$ *such that* $\mathcal{A} \setminus \mathcal{B}$ *is* σ*-porous with respect to* (d_1, d_2) *and that for each* $a \in \mathcal{B}$ *the minimization problem for* f_a *on* (X, ρ) *is strongly well-posed with respect to* $(\mathcal{A}, d_1)$*.*

Note that Theorem 5.6 was obtained in [112].

5.4 Well-posedness and porosity for classes of minimization problems

In this section we apply our variational principle established in Section 5.3 to important classes of minimization problems.

Let (X, ρ) be a complete metric space and let $\mathcal{M}_b$ be the set of all bounded from below lower semicontinuous functions $f : X \to R^1 \cup \{\infty\}$ which are not identical infinity. For each $f, g \in \mathcal{M}_b$ set

$$\tilde{d}(f, g) = \sup\{|f(x) - g(x)| : \ x \in X\},$$

$$d(f, g) = \tilde{d}(f, g)(1 + \tilde{d}(f, g))^{-1}.$$

Clearly $d : \mathcal{M}_b \times \mathcal{M}_b \to [0, \infty)$ is a metric and the metric space $(\mathcal{M}_b, d)$ is complete. For each $a \in \mathcal{M}_b$ set $f_a = a$.

Theorem 5.7. *There exists a set* $\mathcal{B} \subset \mathcal{M}_b$ *such that* $\mathcal{M}_b \setminus \mathcal{B}$ *is* σ*-porous with respect to* d *and for each* $f \in \mathcal{B}$ *the minimization problem for* f *on* (X, ρ) *is strongly well-posed with respect to* $(\mathcal{M}_b, d)$*.*

By Theorem 5.6 in order to prove Theorem 5.7 it is sufficient to show that (H1), (H2) and (H3) hold. Evidently (H1) and (H3) are valid. The following lemma implies (H2).

Lemma 5.8. *Let $\epsilon \in (0,1)$, $r \in (0,1]$ and $f \in \mathcal{M}_b$. Then there exists $\bar{f} \in \mathcal{M}_b$ and $\bar{x} \in X$ such that $f(x) \le \bar{f}(x) \le f(x)+r/2$ for all $x \in X$ and the following property holds:*

For each $y \in X$ satisfying $\bar{f}(y) \le \inf(\bar{f}) + \epsilon r/4$ the inequality $\rho(y,\bar{x}) \le \epsilon$ is valid.

Proof: There exists $\bar{x} \in X$ such that $f(\bar{x}) \le \inf(f) + \epsilon r/4$. Define $\bar{f} \in \mathcal{M}_b$ by

$$\bar{f}(x) = f(x) + 2^{-1} r \min\{\rho(x,\bar{x}), 1\},\ x \in X.$$

Let $x \in X$ and $\bar{f}(x) \le \inf(\bar{f}) + \epsilon r/4 < \infty$. Then we have

$$f(x) + 2^{-1} r \min\{\rho(x,\bar{x}), 1\} = \bar{f}(x) \le \inf(\bar{f}) + \epsilon r/4 \le$$

$$\bar{f}(\bar{x}) + \epsilon r/4 = f(\bar{x}) + \epsilon r/4 \le f(x) + \epsilon r/2.$$

Then $\rho(x,\bar{x}) \le \epsilon$. Lemma 5.8 is proved.

To conclude the section we show that the main result of [32] is a consequence of our variational principle. Note that Theorem 2.4 of [32] is an extension of the variational principle of Deville–Godefroy–Zizler [31].

Let (X,ρ) be a complete metric space. The symbol $\mathrm{diam}(Y)$ denotes the diameter of the set Y in the metric space X. For a function $h : X \to R^1$ the symbol supp h means the set $\{x \in X :\ h(x) \ne 0\}$.

Let $(\mathcal{A}, ||\cdot||_{\mathcal{A}})$ be a Banach space of bounded continuous functions on X with the following two properties:

(a) $||a||_{\mathcal{A}} \ge \sup\{|a(x)| :\ x \in X\},\ a \in \mathcal{A}$;

(b) for every natural number n, there exists a positive constant c_n such that for any point $x \in X$ there exists a function $h_n : X \to [0,1]$, such that $h_n \in \mathcal{A}$, $||h_n||_{\mathcal{A}} \le c_n$, $h_n(x) = 1$ and diam(supp h)$< 1/n$.

We equip the space $\mathcal{A}$ with the metric d defined by $d(a,b) = ||a-b||_{\mathcal{A}}$, $a, b \in \mathcal{A}$.

Theorem 5.9. *Let $f : X \to R^1 \cup \{\infty\}$ be a lower semicontinuous bounded from below function such that $\inf(f) < \infty$. Then there exists a set $\mathcal{B} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{B}$ is σ-porous with respect to the metric d and for each $a \in \mathcal{B}$ the minimization problem*

$$\textit{minimize } (f+a)(x) \textit{ subject to } x \in X$$

on (X,ρ) is strongly well-posed with respect to $(\mathcal{A}, d)$.

Proof: By Theorem 5.6 it is sufficient to show that (H1), (H2) and (H3) hold. Clearly (H1) and (H3) are valid. We will show that (H2) holds.

Let $\epsilon \in (0,1)$ and $r \in (0,1]$. Choose a natural number n_0 such that

$$n_0^{-1} < \epsilon. \tag{5.38}$$

Let c_{n_0} be as guaranteed by the property (b). Fix a positive number γ such that

$$\gamma < 1 \text{ and } 4\gamma c_{n_0} < 1. \tag{5.39}$$

Let $a \in \mathcal{A}$. Choose $\bar{x} \in X$ such that

$$(a + f)(\bar{x}) \leq \inf(f + a) + \gamma r. \tag{5.40}$$

By the property (b) there exists $h \in \mathcal{A}$ such that

$$h : X \to [0, 1],\ ||h||_{\mathcal{A}} \leq c_{n_0},\ h(\bar{x}) = 1 \text{ and } \mathrm{diam}(\mathrm{supp}\ h) < n_0^{-1}. \tag{5.41}$$

Set

$$\bar{a}(x) = a(x) - 4\gamma r h(x),\ x \in X. \tag{5.42}$$

It follows from (5.42), (5.41) and (5.39) that

$$||a - \bar{a}||_{\mathcal{A}} \leq 4\gamma r ||h||_{\mathcal{A}} \leq 4\gamma r c_{n_0} < r. \tag{5.43}$$

Assume that $x \in X$ and

$$(\bar{a} + f)(x) \leq \inf(\bar{a} + f) + \gamma r. \tag{5.44}$$

By (5.42) and (5.41)

$$\inf(f + \bar{a}) \leq (f + \bar{a})(\bar{x}) = (f + a)(\bar{x}) - 4\gamma r. \tag{5.45}$$

It follows from (5.42), (5.44), (5.45) and (5.40) that

$$(f + a)(x) - 4\gamma r h(x) = (f + \bar{a})(x) \leq \inf(f + \bar{a}) + \gamma r \leq$$

$$(f + a)(\bar{x}) - 4\gamma r + \gamma r \leq \inf(f + a) + \gamma r - 3\gamma r.$$

Therefore $h(x) \geq 1/2$. Combining with (5.41) and (5.38) this implies that $\rho(x, \bar{x}) < \epsilon$. Thus (H2) holds. This completes the proof of Theorem 5.9.

5.5 Well-posedness and porosity in convex optimization

Let $K \subset X$ be a nonempty closed convex subset of a Banach space $(X, ||\cdot||)$. In this section we study the minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in K \tag{P}$$

with $f \in \mathcal{M}$. Here $\mathcal{M}$ is a complete metric space of lower semicontinuous convex functions on K which will be defined below. It is known that for a generic lower semicontinuous convex function f the minimization problem (P) is well-posed (see Chapter 4). In this section we study the set of all functions $f \in \mathcal{M}$ for which the corresponding minimization problem (P) is well-posed.

We show that the complement of this set is not only of the first category but also a σ-porous set.

We use the notation and the definitions introduced in Sections 5.1–5.3.

We use the convention that $\infty - \infty = 0$, $\infty/\infty = 1$, $\lambda/\infty = 0$ for any $\lambda \in R^1$ and $\ln(\infty) = \infty$. For each function $f : Y \to [-\infty, \infty]$, where Y is nonempty, we set $\inf(f) = \inf\{f(x) : \ x \in Y\}$.

Let $K \subset X$ be a nonempty closed convex subset of a Banach space $(X, ||\cdot||)$. Denote by $\mathcal{M}$ the set of all lower semicontinuous convex functions $f : K \to [0, \infty]$ which are not identically ∞ and such that

$$f(x) \geq c_0(f)||x|| - c_1(f), \ x \in K \tag{5.46}$$

where $c_0(f) > 0$ and $c_1(f) > 0$ depend only on f.

For each $f, g \in \mathcal{M}$ define

$$\tilde{d}(f, g) = \sup\{|\ln(f(x) + 1) - \ln(g(x) + 1)| : \ x \in K\} \tag{5.47}$$

and

$$d(f, g) = \tilde{d}(f, g)(\tilde{d}(f, g) + 1)^{-1}. \tag{5.48}$$

Clearly, the metric space $(\mathcal{M}, d)$ is complete. Denote by $\mathcal{M}_c$ the set of all finite-valued continuous functions $f \in \mathcal{M}$. It is easy to see that $\mathcal{M}_c$ is a closed subset of $(\mathcal{M}, d)$.

It is clear that $\tilde{d}(f, g) \leq \epsilon$ where $f, g \in \mathcal{M}$ and ϵ is a positive number if and only if

$$(f(x) + 1)(g(x) + 1)^{-1} \in [e^{-\epsilon}, e^{\epsilon}]$$

for all $x \in K$. It means that $f, g \in \mathcal{M}$ are close with respect to d if and only if the function

$$x \to (f(x) + 1)(g(x) + 1)^{-1}, \ x \in K$$

is close to the function which is identically 1 with respect to the topology of the uniform convergence.

We recall the following notion introduced in Section 4.1.

Let $f \in \mathcal{M}$. We say that the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{M}, d)$ if $\inf(f)$ is finite and attained at a unique point $x_f \in K$ and the following assertion holds:

For each positive number ϵ there exist a neighborhood $\mathcal{V}$ of f in $(\mathcal{M}, d)$ and a positive number δ such that for each $g \in \mathcal{V}$, $\inf(g)$ is finite and if $z \in K$ satisfies $g(z) \leq \inf(g) + \delta$, then $||x_f - z|| \leq \epsilon$ and $|g(z) - f(x_f)| \leq \epsilon$.

We will prove the following result obtained in [118].

Theorem 5.10. *There exists a set $\mathcal{F} \subset \mathcal{M}$ such that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{M}, d)$, the set $\mathcal{M} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{M}, d)$ and the set $\mathcal{M}_c \setminus \mathcal{F}$ is σ-porous in $(\mathcal{M}_c, d)$.*

5.6 Proof of Theorem 5.10

For each $f \in \mathcal{M}$ and each positive number r put

$$B_d(f,r) = \{g \in \mathcal{M} : \ d(f,g) \le r\}.$$

For each integer $n \ge 1$ denote by $\mathcal{F}_n$ the set of all $f \in \mathcal{M}$ which have the following property:

(i) There exist $\delta(f,n) \in (0,1)$, $\alpha(f,n) \in R^1$, $x(f,n) \in K$ and a neighborhood $\mathcal{U}(f,n)$ of f in $(\mathcal{M},d)$ such that for each $g \in \mathcal{U}(f,n)$ and each $z \in K$ satisfying

$$g(z) \le \inf(g) + \delta(f,n)$$

the following inequalities hold:

$$||z - x(f,n)|| \le 1/n, \ |\alpha(f,n) - g(z)| \le 1/n. \tag{5.49}$$

Define

$$\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}_n. \tag{5.50}$$

Let $f \in \mathcal{F}$. We will show that the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{M},d)$. For any integer $n \ge 1$ let $\delta(f,n) \in (0,1)$, $\alpha(f,n) \in R^1$, $x(f,n) \in K$ and a neighborhood $\mathcal{U}(f,n)$ of f in $(\mathcal{M},d)$ be as guaranteed in property (i). Assume that $\{z_i\}_{i=1}^{\infty} \subset K$ and

$$\lim_{i\to\infty} f(z_i) = \inf(f). \tag{5.51}$$

Let $n \ge 1$ be an integer. By (5.51) and the property (i), for all sufficiently large natural numbers i,

$$f(z_i) \le \inf(f) + \delta(f,n)$$

and

$$||z_i - x(f,n)|| \le 1/n, \ |\alpha(f,n) - f(z_i)| \le 1/n. \tag{5.52}$$

Hence $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists $x_f \in K$ such that

$$x_f = \lim_{i\to\infty} z_i. \tag{5.53}$$

In view of (5.51) and the lower semicontinuity of f,

$$f(x_f) = \inf(f). \tag{5.54}$$

It is easy to see that f does not have another minimizer for otherwise we would be able to construct a nonconvergent sequence $\{z_i\}_{i=1}^{\infty}$.

By (5.51)–(5.54), for all integers $n \ge 1$,

$$||x_f - x(f,n)|| \le 1/n, \ |\alpha(f,n) - f(x_f)| \le 1/n. \tag{5.55}$$

Let $\epsilon \in (0,1)$. Fix an integer $n > 4/\epsilon$. Let

$$g \in \mathcal{U}(f,n),\ z \in K \text{ and } g(z) \leq \inf(g) + \delta(f,n). \tag{5.56}$$

It follows from (5.56), the property (i) and the definition of $\mathcal{U}(f,n)$, $\delta(f,n)$ and $x(f,n)$ that

$$||z - x(f,n)|| \leq 1/n,\ |\alpha(f,n) - g(z)| \leq 1/n.$$

Together with (5.55) these inequalities imply that

$$||x_f - z|| \leq 2/n < \epsilon,\ |g(z) - f(x_f)| \leq 2/n < \epsilon.$$

Therefore the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{M}, d)$. Now we will show that the set $\mathcal{M} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{M}, d)$ and the set $\mathcal{M}_c \setminus \mathcal{F}$ is σ-porous in $(\mathcal{M}_c, d)$.

We need the following auxiliary result.

Lemma 5.11. *Let M be a positive number, $\epsilon \in (0,1)$ and*

$$0 < \lambda < (2e(M+1))^{-1}. \tag{5.57}$$

Then for each $f, g \in \mathcal{M}$ and each $x \in K$ which satisfy

$$d(f,g) \leq \lambda\epsilon,\ \min\{f(x), g(x)\} \leq M \tag{5.58}$$

the inequality $|f(x) - g(x)| \leq \epsilon$ holds.

Proof: Assume that $f, g \in \mathcal{M}$, $x \in K$ and (5.58) holds. We may assume without loss of generality that $g(x) \geq f(x)$. By (5.48), (5.57) and (5.58),

$$\tilde{d}(f,g) = d(f,g)(1 - d(f,g))^{-1} \leq 2\lambda\epsilon.$$

Together with (5.47) this relation implies that for all $z \in K$

$$f(z) + 1 \leq g(z) + 1 \leq e^{2\lambda\epsilon}(f(z) + 1)$$

and in view of (5.58) and the mean value theorem

$$0 \leq g(z) - f(z) \leq e^{2\lambda\epsilon}(f(z)+1) - (f(z)+1) =$$

$$(e^{2\lambda\epsilon} - 1)(f(z)+1) \leq (e^{2\lambda\epsilon} - 1)(M+1) = 2\lambda\epsilon(M+1)e^t$$

with $t \in [0, 2\lambda\epsilon] \subset [0,1]$. Hence

$$0 \leq g(z) - f(z) \leq 2\lambda\epsilon(M+1)e^t \leq 2\lambda\epsilon(M+1)e.$$

Together with (5.57) these inequalities imply that $|f(z) - g(z)| \leq \epsilon$. This completes the proof of Lemma 5.11.

For each natural number m define

$$E_m = \{f \in \mathcal{M} :\ \inf(f) \le m \text{ and } f(x) \ge ||x||/m - m \text{ for all } x \in K\}. \quad (5.59)$$

It is easy to see that $\cup_{m=1}^{\infty} E_m = \mathcal{M}$ and

$$\mathcal{M} \setminus \mathcal{F} = \cup_{n=1}^{\infty}(\mathcal{M} \setminus \mathcal{F}_n) = \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} (E_m \setminus \mathcal{F}_n).$$

In order to complete the proof of the theorem it is sufficient to show that for each pair of natural numbers m, n the set $E_m \setminus \mathcal{F}_n$ is porous in $(\mathcal{M}, d)$ and the set $(E_m \cap \mathcal{M}_c) \setminus \mathcal{F}_n$ is porous in $(\mathcal{M}_c, d)$.

Let $m, n \ge 1$ be integers. Fix an integer $p \ge 1$ and real numbers $\alpha_0 > 0$, $\alpha_1 > 0$ and $\alpha > 0$ such that

$$p > 3m^2,\ \alpha_0 < (4p)^{-1},\ \alpha_1 = (1280npe)^{-1}\alpha_0 \text{ and } \alpha = \alpha_1/2. \quad (5.60)$$

Let

$$f \in E_m \setminus \mathcal{F}_n \text{ and } r \in (0, 1]. \quad (5.61)$$

Put

$$\delta_0 = \alpha_0 r (40n)^{-1}. \quad (5.62)$$

Fix $\bar{x} \in K$ for which

$$f(\bar{x}) \le \inf(f) + \delta_0 \quad (5.63)$$

and define

$$\bar{f}(x) = f(x) + 4^{-1}\alpha_0 r||x - \bar{x}||,\ x \in K. \quad (5.64)$$

It is clear that

$$\bar{f} \in \mathcal{M} \text{ and if } f \in \mathcal{M}_c, \text{ then } \bar{f} \in \mathcal{M}_c. \quad (5.65)$$

We will show that

$$B_d(\bar{f}, \alpha r) \subset B(f, r) \cap \mathcal{F}_n. \quad (5.66)$$

In view of (5.47), (5.48) and (5.64),

$$d(f, \bar{f}) \le \tilde{d}(f, \bar{f}) \le \sup\{|\ln(1 + f(x)) - \ln(1 + \bar{f}(x))| :\ x \in K\} = \quad (5.67)$$
$$\sup\{\ln((1 + f(x) + 4^{-1}\alpha_0 r||x - \bar{x}||)(1 + f(x))^{-1}) :\ x \in K\} \le$$
$$\sup\{\ln(1 + 4^{-1}\alpha_0 r||x - \bar{x}||(1 + f(x))^{-1}) :\ x \in K\} \le$$
$$\sup\{(4(1 + f(x)))^{-1}\alpha_0 r||x - \bar{x}|| :\ x \in K\}.$$

It follows from (5.59) and (5.61)–(5.63) that

$$||\bar{x}||/m - m \le f(\bar{x}) \le \inf(f) + \delta_0 \le m + 1$$

and

$$||\bar{x}|| \le m(2m + 1). \quad (5.68)$$

By (5.67) and (5.68),

$$d(f, \bar{f}) \le 4^{-1}\alpha_0 r \sup\{(||\bar{x}|| + ||x||)(1 + f(x))^{-1} :\ x \in K\} \le \quad (5.69)$$

$$4^{-1}\alpha_0 r(||\bar{x}|| + \sup\{||x||(1+f(x))^{-1} : x \in K\}) \le$$
$$4^{-1}\alpha_0 r(3m^2) + 4^{-1}\alpha_0 r \sup\{||x||(1+f(x))^{-1} : x \in K\}.$$

We will show that for all $x \in K$

$$||x||(1+f(x))^{-1} \le p. \tag{5.70}$$

Let $x \in K$. If $||x|| \le p$, then

$$||x||(1+f(x))^{-1} \le ||x|| \le p$$

and (5.70) holds. Now assume that $||x|| > p$. Then (5.59)–(5.61) imply that

$$f(x) \ge ||x||/m - m = (||x|| - m^2)/m > ||x||(2m)^{-1}$$

and

$$||x||(1+f(x))^{-1} \le ||x||f(x)^{-1} \le ||x||(||x||(2m)^{-1})^{-1} = 2m < p.$$

Thus in both cases the inequality (5.70) holds. In view of (5.69), (5.70) and (5.60),

$$d(f, \bar{f}) \le 4^{-1}3\alpha_0 r m^2 + 4^{-1}\alpha_0 r p \le 4^{-1} r\alpha_0(3m^2 + p) < 2^{-1}\alpha_0 r p \le r/8$$

and

$$d(f, \bar{f}) \le r/8. \tag{5.71}$$

It follows from (5.63) and (5.64) that

$$\inf(f) \le \inf(\bar{f}) \le \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + \delta_0. \tag{5.72}$$

We will show that the following property holds:

(ii) If $z \in K$ satisfies

$$\bar{f}(z) \le \inf(\bar{f}) + 4\delta_0, \tag{5.73}$$

then $||z - \bar{x}|| < 1/n$.

Assume that $z \in K$ satisfies (5.73). By (5.64), (5.73), (5.72), (5.63) and (5.62),

$$f(z) + 4^{-1}\alpha_0 r||z - \bar{x}|| = \bar{f}(z) \le \inf(\bar{f}) + 4\delta_0 \le$$
$$f(\bar{x}) + 4\delta_0 \le \inf(f) + 5\delta_0 \le f(z) + 5\delta_0$$

and

$$||z - \bar{x}|| \le 20\delta_0(\alpha_0 r)^{-1} < 1/n.$$

Thus the property (ii) holds.

It follows from (5.61)–(5.63) and (5.59) that

$$f(\bar{x}) \le \inf(f) + \delta_0 \le m + 1. \tag{5.74}$$

Let $g \in \mathcal{M}$ satisfy

$$d(g, \bar{f}) \le \alpha_1 r. \tag{5.75}$$

In view of (5.75), (5.60) and (5.72), $d(g, \bar{f}) < \delta_0(16me)^{-1}$. This inequality, (5.74) and Lemma 5.11 with $\epsilon = \delta_0$, $M = m + 1$, $\lambda = (16me)^{-1}$ imply that

$$|g(\bar{x}) - \bar{f}(\bar{x})| \le \delta_0.$$

Together with (5.61)–(5.64) and (5.59) this inequality implies that

$$\inf(g) \le g(\bar{x}) \le \bar{f}(\bar{x}) + \delta_0 = f(\bar{x}) + \delta_0 \le$$

$$\inf(f) + 2\delta_0 \le m + 1/2.$$

Therefore

$$\inf(g) \le m + 1/2 \text{ for any } g \in B_d(\bar{f}, \alpha_1 r). \tag{5.76}$$

Assume that

$$g_1, g_2 \in B_d(\bar{f}, \alpha_1 r),\ z \in K \text{ and } g_1(z) \le \inf(g_1) + 1/2. \tag{5.77}$$

In view of (5.77) and (5.76),

$$g_1(z) \le m + 1. \tag{5.78}$$

By (5.77), (5.60) and (5.62),

$$d(g_1, g_2) \le 2\alpha_1 r \le (8me)^{-1}\delta_0.$$

By this inequality, (5.78) and Lemma 5.11 with $\epsilon = \delta_0$, $M = m + 1$, $\lambda = (8me)^{-1}$,

$$|g_1(z) - g_2(z)| \le \delta_0 \tag{5.79}$$

and

$$\inf(g_2) \le g_1(z) + \delta_0. \tag{5.80}$$

We have shown that the following implication holds:

(iii) (5.77) implies the inequality (5.79).

Since the inequality (5.80) holds for any $z \in K$ satisfying $g_1(z) \le \inf(g_1) + 1/2$ we obtain that

$$\inf(g_2) \le \inf\{g_1(z) + \delta_0 :\ z \in K \text{ and } g_1(z) \le \inf(g_1) + 1/2\} =$$

$$\inf(g_1) + \delta_0.$$

Since g_1 and g_2 are arbitrary elements of $B_d(\bar{f}, \alpha_1 r)$ we conclude that

$$|\inf(g_1) - \inf(g_2)| \le \delta_0 \text{ for each } g_1, g_2 \in B_d(\bar{f}, \alpha_1 r). \tag{5.81}$$

Assume that

$$g \in B_d(\bar{f}, \alpha_1 r),\ z \in K \text{ and } g(z) \le \inf(g) + \delta_0. \tag{5.82}$$

It follows from (5.82), implication (iii), (5.77) and (5.79) that

$$|g(z) - \bar{f}(z)| \le \delta_0. \tag{5.83}$$

In view of (5.81) and (5.82),

$$|\inf(g) - \inf(\bar{f})| \le \delta_0. \tag{5.84}$$

By (5.82), (5.84) and (5.62),

$$|g(z) - \inf(\bar{f})| \le 2\delta_0 < 1/n. \tag{5.85}$$

It follows from (5.82)–(5.84) that

$$\bar{f}(z) \le g(z) + \delta_0 \le \inf(g) + 2\delta_0 \le \inf(\bar{f}) + 3\delta_0.$$

By this inequality and property (ii),

$$||z - \bar{x}|| < 1/n. \tag{5.86}$$

Thus we have shown that the following implication holds:

(iv) (5.82) implies the inequalities (5.85) and (5.86).

Assume that

$$h \in B_d(\bar{f}, \alpha r). \tag{5.87}$$

We will show that $h \in B_d(f, r) \cap \mathcal{F}_n$. In view of (5.87), (5.71) and (5.60),

$$d(h, f) \le d(h, \bar{f}) + d(\bar{f}, f) \le \alpha r + r/8 < r. \tag{5.88}$$

In order to prove that $h \in \mathcal{F}_n$ we need to show that the property (i) holds with $f = h$. Put

$$\mathcal{U}(h, n) = B_d(h, \alpha r),\ \delta(h, n) = \delta_0,\ \alpha(h, n) = \inf(\bar{f}),\ x(h, n) = \bar{x}.$$

Let

$$g \in B_d(h, \alpha r),\ z \in K \text{ and } g(z) \le \inf(g) + \delta_0. \tag{5.89}$$

It follows from (5.87), (5.89) and the definition of α (see (5.60)) that

$$g \in B_d(\bar{f}, \alpha_1 r). \tag{5.90}$$

By (5.90), (5.89) and implication (iv), (5.85) and (5.86) hold. Thus $h \in \mathcal{F}_n$. Together with (5.88) this fact implies that $h \in \mathcal{F}_n \cap B_d(f, r)$. Since h is an arbitrary element of $B_d(\bar{f}, \alpha r)$ we conclude that

$$B_d(\bar{f}, \alpha r) \subset \mathcal{F}_n \cap B_d(f, r).$$

We have shown that the set $E_m \setminus \mathcal{F}_n$ is porous in $(\mathcal{M}, d)$ and the set $(E_m \cap \mathcal{M}_c) \setminus \mathcal{F}_n$ is porous $(\mathcal{M}_c, d)$. Theorem 5.10 is proved.

5.7 A porosity result in convex minimization

We use the convention that $\infty - \infty = 0$ and $\infty/\infty = 1$. Let X be a reflexive Banach space with the norm $||\cdot||$ and let

$$C_\infty = \cap_{i=1}^{\infty} C_i \neq \emptyset$$

where $C_{i+1} \subset C_i$ for each $i = 1, 2, \dots$ and where each C_i is a closed convex subset of X. Let $\varphi : C_1 \to R^1$ satisfy

$$\phi(x) \to \infty \text{ as } ||x|| \to \infty.$$

Denote by $\mathcal{M}$ the set of all convex lower semicontinuous functions $f : C_1 \to R^1 \cup \{\infty\}$ which are not identically infinity on C_∞ and satisfy

$$f(x) \geq \phi(x) \text{ for all } x \in C_1.$$

For each $f \in \mathcal{M}$ and each nonempty set $C \subset C_1$ set

$$\inf(f; C) = \inf\{f(x) : \ x \in C\}.$$

It is well-known that for each $f \in \mathcal{M}$ and each $i \in \{1, 2, \dots\} \cup \{\infty\}$, the minimization problem

$$\text{minimize } f(x) \text{ subject to } \in C_i \qquad (\mathrm{P}_\mathrm{i}^\mathrm{f})$$

has a solution. Denote by $\mathcal{M}_1$ the set of all finite-valued functions $f \in \mathcal{M}$ and by $\mathcal{M}_2$ the set of all finite-valued continuous functions $f \in \mathcal{M}$. Next we endow the set $\mathcal{M}$ with a metric d. For each $f, g \in \mathcal{M}$ and each integer $m \geq 1$ put

$$d_m(f, g) = \sup\{|f(x) - g(x)| : \ x \in C_1 \ \text{ and } ||x|| \leq m\}$$

and define

$$d(f, g) = \sum_{m=1}^{\infty} 2^{-m}[d_m(f, g)(d_m(f, g) + 1)^{-1}].$$

We assume that the supremum of the empty set is zero. It is easy to see that $(\mathcal{M}, d)$ is a complete metric space and that the collection of sets

$$E(m, \delta) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \ |f(x) - g(x)| \leq \delta$$

$$\text{for each } x \in C_1 \text{ satisfying } ||x|| \leq m\},$$

where $m \geq 1$ is an integer and δ is a positive number, is a base for the uniformity generated by the metric d. It is clear that $\mathcal{M}_1$ and $\mathcal{M}_2$ are closed subsets of the metric space $(\mathcal{M}, d)$. We equip all these spaces with the same metric d.

In [45] for a function $f \in \mathcal{M}_2$ we studied the convergence of solutions to the problem $(\mathrm{P}_\mathrm{i}^\mathrm{f})$ for each $i = 1, 2, \dots$ to a solution of the problem $(\mathrm{P}_\infty^\mathrm{f})$. If X is a Hilbert space with inner product $< \cdot, \cdot >$ and

$$f(x) = < x, x > \text{ for } x \in X,$$

then this convergence property was established by Semple [91]. A similar result was also obtained for certain Banach spaces and the distance function by Israel and Reich [53]. In [45] it was shown that the convergence property holds for most functions $f \in \mathcal{M}_2$. More precisely, we considered the metric space $(\mathcal{M}_2, d)$ with

$$\phi(x) = a_1||x|| - a_2 \text{ for } x \in C_1,$$

where $a_1, a_2 > 0$, and showed that there exists a subset of $\mathcal{M}_2$ which is a countable intersection of open everywhere dense sets such that for each function belonging to this subset the convergence property holds. Note that this result is true for reflexive Banach spaces but not necessarily for non-reflexive Banach spaces. For more information consider Examples 1 and 2 in [45].

In this chapter for the spaces $\mathcal{M}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ we show that the complements of subsets of functions which have the convergence property are not only of the first Baire category but are also σ-porous sets. We prove the following result which was obtained in [46].

Theorem 5.12. *Let $\mathcal{A}$ be either $\mathcal{M}$ or $\mathcal{M}_1$ or $\mathcal{M}_2$. There exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}, d)$ and such that for each $f \in \mathcal{F}$ the following properties hold.*

(P_1) There exists a unique point $x_f \in C_\infty$ such that $f(x_f) = \inf(f; C_\infty)$.

(P_2) For each $i = 1, 2, \dots,$ let $x_i \in C_i$ be such that $f(x_i) = \inf(f; C_i)$. Then $||x_i - x_f|| \to 0$ as $i \to \infty$.

(P_3) For each positive number ϵ there exist a neighborhood U of f in $(\mathcal{A}, d)$, a positive number δ and an integer $p \geq 1$ such that for each $g \in U$, each $i \in \{p, p+1, \dots\} \cup \{\infty\}$ and each $y \in C_i$ satisfying

$$g(y) \leq \inf(g; C_i) + \delta$$

the inequality $||y - x_f|| \leq \epsilon$ holds.

5.8 Auxiliary results for Theorem 5.12

Lemma 5.13. *Let $f \in \mathcal{M}$ and δ be a positive number. Choose an integer $m = m(f) \geq 1$ such that*

$$||z|| \leq m \text{ for each } z \in C_1 \text{ satisfying } \phi(z) \leq \inf(f; C_\infty) + 1$$

and let

$$U(m, \delta) = \{g \in \mathcal{M} : \ (f, g) \in E(m, \delta)\}.$$

Then for each $i \in \{1, 2, \dots\} \cup \{\infty\}$ and each $g \in U(m, \delta)$,

$$\inf(g; C_i) \leq \inf(f; C_i) + \delta.$$

Proof: Let $i \in \{1, 2, \dots\} \cup \{\infty\}$ and $z \in C_i$ be such that $f(z) \le \inf(f; C_i) + 1$. Since $C_i \subset C_1$ and $\phi(z) \le f(z) \le \inf(f, C_i) + 1 \le \inf(f, C_\infty) + 1$ it follows that $\|z\| \le m$ and hence that $|f(z) - g(z)| \le \delta$. Therefore for each $i \in \{1, 2, \dots\} \cup \{\infty\}$

$$\inf(g; C_i) \le \inf\{g(z) : \ z \in C_i \text{ and } f(z) \le \inf(f; C_i) + 1\}$$

$$\le \inf\{f(z) + \delta : \ z \in C_i \text{ and } f(z) \le \inf(f; C_i) + 1\}$$

$$\le \inf\{f(z) : \ z \in C_i \text{ and } \ f(z) \le \inf(f; C_i) + 1\} + \delta$$

$$\le \inf(f; C_i) + \delta.$$

Lemma 5.13 is proved.

Proposition 5.14. *Let $f \in \mathcal{M}$. Assume that there exists a unique point $x_f \in C_\infty$ such that $f(x_f) = \inf(f; C_\infty)$ and that the following property holds.*

(P_4) For each positive number ϵ there exist a positive number $\delta = \delta(\epsilon)$ and an integer $p = p(\epsilon) \ge 1$ such that for each $i \in \{p, p+1, \dots\} \cup \{\infty\}$ and each $y \in C_i$ satisfying $f(y) \le \inf(f; C_i) + \delta$ the inequality $||y - x_f|| \le \epsilon$ holds.

Then properties (P_1), (P_2) and (P_3) hold.

Proof: Clearly, (P_1) holds. For each natural number i choose $x_i \in C_i$ such that $f(x_i) = \inf(f; C_i)$. Evidently, (P_4) implies (P_2). We will show that property (P_3) holds. Fix a positive number ϵ. In view of property (P_4) there exist $\delta \in (0, 1/2)$ and a natural number p such that if

$$i \in \{p, p+1, \dots\} \cup \{\infty\}, \ y \in C_i \text{ and } f(y) \le \inf(f; C_i) + 3\delta,$$

then $||y - x_f|| \le \epsilon$. Fix an integer $m = m(f) \ge 1$ and define $U = U(m, \delta)$ as in Lemma 5.13. Let $i \in \{p, p+1, \dots\} \cup \{\infty\}$ and $g \in U$. Assume that $z \in C_i$ satisfies $g(z) \le \inf(g; C_i) + \delta$. Lemma 5.13 implies that

$$\phi(z) \le g(z) \le \inf(g; C_i) + \delta \le \inf(f; C_i) + 2\delta < \inf(f; C_\infty) + 1$$

and therefore $||z|| \le m$ and $|f(z) - g(z)| \le \delta$. Since

$$f(z) \le g(z) + \delta \le \inf(g; C_i) + 2\delta \le \inf(f; C_i) + 3\delta$$

it follows that $||z - x_f|| \le \epsilon$. Thus property (P_3) holds. Proposition 5.14 is proved.

It is not difficult to show in a straightforward manner that the following lemma holds. For details see [45].

Lemma 5.15. *Let $f \in \mathcal{M}$. Then*

$$\lim_{i \to \infty} \inf(f; C_i) = \inf(f; C_\infty).$$

5.9 Proof of Theorem 5.12

It is convenient to split the proof into several smaller parts. We use the notation $\mathcal{A}$ to denote either $\mathcal{M}$, $\mathcal{M}_1$ or $\mathcal{M}_2$.

Lemma 5.16. *For each integer $n \geq 1$ denote by $\mathcal{F}_n$ the set of all $f \in \mathcal{A}$ which possess the following property.*

(Q_1) There exist $x_n \in C_\infty$, a positive number δ_n and an integer $p_n \geq 1$ such that for $i \in \{p_n, p_n+1, \dots\} \cup \{\infty\}$ and each $y \in C_i$ satisfying $f(y) \leq \inf(f; C_i) + \delta_n$ the inequality $\|y - x_n\| \leq 1/n$ holds.

If $f \in \mathcal{F} = \cap_{n=1}^\infty \mathcal{F}_n$ then properties (P_1), (P_2) and (P_3) hold.

Proof: Let $x_f \in C_\infty$ satisfy $f(x_f) = \inf(f; C_\infty)$. It follows from ($Q_1$) with $i = \infty$ and $y = x_f$ that

$$\|x_f - x_n\| \leq 1/n$$

for all integers $n \geq 1$. Thus $x_f = \lim_{n\to\infty} x_n$. Therefore x_f is the unique minimizer of f on C_∞. Let ϵ be a positive number and a natural number n satisfy $n > 2/\epsilon$. For each $i \in \{p_n, p_n+1, \dots\} \cup \{\infty\}$ and each $y \in C_i$ satisfying $f(y) \leq \inf(f; C_i) + \delta_n$ it follows from property (Q_1) that

$$\|y - x_n\| \leq 1/n.$$

Thus $\|y - x_f\| \leq \epsilon$. Therefore property ($P_4$) holds and hence properties (P_1), (P_2) and (P_3) also hold. Lemma 5.16 is proved.

Remark 5.17. To complete the proof of Theorem 5.12 we need to show that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $(\mathcal{A}, d)$. Since $\mathcal{A} \setminus \mathcal{F} = \cup_{n=1}^\infty (\mathcal{A} \setminus \mathcal{F}_n)$ it is sufficient to show that the set $\mathcal{A} \setminus \mathcal{F}_n$ is σ-porous in $(\mathcal{A}, d)$ for any integer $n \geq 1$. For each natural number m denote by E_m the subset of all $f \in \mathcal{A}$ with the following property.

(Q_2) If $x \in C_1$ and $\phi(x) \leq \inf(f; C_\infty) + 1$, then $||x|| \leq m$.

Since $\cup_{m=1}^\infty E_m = \mathcal{A}$ and $\mathcal{A} \setminus \mathcal{F}_n = \cup_{m=1}^\infty (E_m \setminus \mathcal{F}_n)$ it is sufficient to show that for each pair of integers $m, n \geq 1$ the set $E_m \setminus \mathcal{F}_n$ is porous in $(\mathcal{A}, d)$.

Lemma 5.18. *Let $m \geq 1$ be an integer, $f \in E_m$ and let $x_f \in C_\infty$ satisfy $f(x_f) = \inf(f; C_\infty)$. For each positive number γ and each $x \in C_1$ set*

$$f_\gamma(x) = f(x) + \gamma||x - x_f||.$$

Then $f_\gamma \in \mathcal{A}$ and $d(f_\gamma, f) \leq \gamma$ for all positive numbers γ.

Proof: Let $\gamma > 0$. It is easy to see that $f_\gamma \in \mathcal{A}$. Since

$$\phi(x_f) \leq f(x_f) = \inf(f; C_\infty)$$

it follows from property (Q_2) that $||x_f|| \leq m$. For each natural number k

$$\begin{aligned}d_k(f_\gamma, f) &= \sup\{|f_\gamma(x) - f(x)| : \ x \in C_1 \text{ and } ||x|| \le k\}\\ &= \sup\{\gamma||x - x_f|| : \ x \in C_1 \text{ and } ||x|| \le k\}\\ &\le \gamma \sup\{||x|| + ||x_f|| : \ x \in C_1 \text{ and } ||x|| \le k\} \le \gamma(m+k)\end{aligned}$$

and thus

$$d(f_\gamma, f) \le \sum_{k=1}^{\infty} 2^{-k}\gamma(m+k)(\gamma(m+k)+1)^{-1} \le \gamma \sum_{k=1}^{\infty} 2^{-k} = \gamma.$$

Lemma 5.18 is proved.

Lemma 5.19. *Let $m, n \ge 1$ be integers. Fix $r \in (0,1]$, $\gamma = \gamma(r) = (1 - 1/2^{m+3})r$ and $\theta = \theta(r,m,n) = r/(2^{m+4}n)$. If $f \in E_m$ and $g \in \mathcal{A}$ satisfies $d(g, f_\gamma) \le \theta$, then $g \in \mathcal{F}_n$ and $d(g,f) < r$.*

Proof: Let $f \in E_m$, $g \in \mathcal{A}$ and $d(g, f_\gamma) \le \theta$. It is easy to see that $d_m(g, f_\gamma) \le 2^{m+1}\theta$. If $x \in C_i$ and $\phi(x) \le \inf(f; C_i) + 1$, then $\phi(x) \le \inf(f; C_\infty) + 1$ and since $f \in E_m$ it follows that $\|x\| \le m$. Thus

$$|g(x) - f_\gamma(x)| \le 2^{m+1}\theta.$$

Clearly, if $y \in C_i$ and $f_\gamma(y) \le \inf(f_\gamma; C_i) + 1$, then $\phi(y) \le f_\gamma(y) \le \inf(f_\gamma; C_i) + 1 \le \inf(f_\gamma; C_\infty) + 1$ and since $\inf(f_\gamma; C_\infty) = f_\gamma(x_f) = f(x_f) = \inf(f; C_\infty)$ it follows that $\phi(y) \le \inf(f; C_\infty) + 1$. Since $f \in E_m$ we deduce that $||y|| \le m$ and

$$|g(y) - f_\gamma(y)| \le 2^{m+1}\theta.$$

Since this inequality holds for any such y we conclude that

$$\inf(g; C_i) \le \inf(f_\gamma; C_i) + 2^{m+1}\theta.$$

Assume that $z \in C_i$ and $g(z) \le \inf(g; C_i) + 2^{m+1}\theta$. Then $\phi(z) \le g(z) \le \inf(g; C_i) + 2^{m+1}\theta \le \inf(f_\gamma; C_i) + 2^{m+2}\theta < \inf(f_\gamma; C_\infty) + 1 = \inf(f; C_\infty) + 1$. Hence $||z|| \le m$ and

$$|g(z) - f_\gamma(z)| \le 2^{m+1}\theta.$$

We can now deduce that $f_\gamma(z) \le g(z) + 2^{m+1}\theta \le \inf(g; C_i) + 2^{m+2}\theta \le \inf(f_\gamma; C_i) + 3 \cdot 2^{m+1}\theta \le \inf(f, C_\infty) + 3 \cdot 2^{m+1}\theta$. If we choose p so large that $\inf(f; C_i) \ge \inf(f, C_\infty) - 2^{m+1}\theta$ when $i \in \{p, p+1, \ldots\} \cup \{\infty\}$, then

$$f_\gamma(z) \le \inf(f, C_i) + 2^{m+3}\theta \le f(z) + 2^{m+3}\theta$$

and

$$\|z - x_f\| \le 2^{m+3}\theta/\gamma \le 1/n.$$

Since $z \in C_i$ and $g(z) \le \inf(g; C_i) + 2^{m+1}\theta$ for $i \in \{p, p+1, \ldots\} \cup \{\infty\}$ implies $||z - x_f|| \le 1/n$ we have shown that $g \in \mathcal{F}_n$. In order to complete the proof of the lemma we note that $d(g,f) \le d(g, f_\gamma) + d(f_\gamma, f) \le \theta + \gamma < r$.

Completion of the Proof of Theorem 5.12: By Remark 5.17 we should show that for each pair of integers $m, n \geq 1$ the set $E_m \setminus \mathcal{F}_n$ is porous in $(\mathcal{A}, d)$. Let $f \in E_m \setminus \mathcal{F}_n$. Let $r \in (0, 1]$ and let real numbers $\gamma = \gamma(r)$ and $\theta = \theta(r, m, n)$ be as in Lemma 5.19. If we define $\alpha = \alpha(m, n)$ by the formula

$$\alpha = 1/(2^{m+4}n),$$

then $\theta = \alpha r$ and for each $r \in (0, 1]$ we can see from Lemma 5.19 that

$$\{g \in \mathcal{A} : \ \ d(g, f_\gamma) \leq \alpha r\} \subset \{g \in \mathcal{A} : \ \ d(g, f) \leq r\} \cap \mathcal{F}_n.$$

Hence each sufficiently small ball $B_d(f, r) \subset (\mathcal{A}, d)$ centered at a point $f \in E_m \setminus \mathcal{F}_n$ contains a smaller ball $B_d(f_\gamma, \alpha r)$ of fixed proportional radius centered at the point f_γ and lying entirely within $\mathcal{F}_n$. Hence $E_m \setminus \mathcal{F}_n$ is porous in $(\mathcal{A}, d)$. This completes the proof of Theorem 5.12.

5.10 A porosity result for variational problems arising in crystallography

We study the structure of minimizers of variational problems considered in [43, 55, 70, 132] which describe step-terraces on surfaces of crystals. It is well-known in surface physics that when a crystalline substance is maintained at a temperature T above its *roughening temperature* T_R, then the surface stored energy integrand, usually referred to as *surface tension*, is a smooth function β of the azimuthal angle of orientation θ. Furthermore, β obeys the following:

$$\beta(-\theta) = \beta(\pi - \theta) = \beta(\theta), \ 0 < \beta(\pi/2) \leq \beta(\theta) \leq \beta(0).$$

The classical model is given by

$$J(y) = \int_0^S \beta(\theta) ds$$

where s is arclength and y is a function defined on a fixed interval $[0, L]$ whose graph is the locus under consideration:

$$y \in W^{1,1}(0, L), \ \theta = \ \arctan y' \in [-\pi/2, \pi/2],$$

while β is a positive π-periodic function which belongs to a space of functions described below. Minimization of J subject to appropriate boundary data is a parametric variational problem. It is closely related to the variational problem defining the Wulff crystal shape as that shape for a domain of prescribed area such that the boundary integral with respect to arclength involving the integrand in J [referred to as the surface tension] attains its minimum value.

For each function $f : X \to R^1$ put $\inf(f) = \inf\{f(x) : \ x \in X\}$.

Denote by $\mathcal{M}$ the set of all functions $\beta \in C^2(R^1)$ which satisfy the following assumption:

(A)

$$\beta(t) \geq 0 \text{ for all } t \in R^1, \tag{5.91}$$

$$\beta(\pi/2) \leq \beta(t) \leq \beta(0) \text{ for all } t \in R^1, \tag{5.92}$$

$$\beta(t) = \beta(-t) \text{ for all } t \in R^1, \tag{5.93}$$

$$\beta(t+\pi) = \beta(t) \text{ for all } t \in R^1, \tag{5.94}$$

$$\beta(0) + \beta''(0) \leq 0. \tag{5.95}$$

For each $\beta_1, \beta_2 \in \mathcal{M}$ put

$$\rho(\beta_1, \beta_2) = \sup\{|\beta_1^{(i)}(t) - \beta_2^{(i)}(t)| : \ t \in R^1, \ i = 0, 1, 2\}. \tag{5.96}$$

Clearly, the metric space $(\mathcal{M}, \rho)$ is complete.

Denote by $\mathcal{M}_r$ the set of all $\beta \in \mathcal{M}$ such that

$$\beta(t) > 0 \text{ for all } t \in R^1, \tag{5.97}$$

$$\beta(0) + \beta''(0) < 0. \tag{5.98}$$

It was shown in [70] that $\mathcal{M}_r$ is an open everywhere dense subset of $(\mathcal{M}, \rho)$. Let $\beta \in \mathcal{M}$. Define

$$G_\beta(z) = \beta(\arctan(z))(1+z^2)^{1/2}, \ z \in R^1. \tag{5.99}$$

It is easy to see that G_β is a continuous function and if $\beta \in \mathcal{M}_r$, then

$$G_\beta(z) \to \infty \text{ as } z \to \pm\infty. \tag{5.100}$$

Note that if $\beta \in \mathcal{M}_r$, then $\inf(G_\beta) < \beta(0)$ (see [43]).

We can rewrite the variational functional J in the form

$$J(y) = \int_0^L G_\beta(y')dx.$$

It was shown in [43] that $y \in W^{1,1}(0, L)$ is a minimizer of J if and only if

$$|y'| \in \{z \in R^1 : \ G_\beta(z) = \inf(G_\beta)\} \text{ a.e.}$$

In [70] we showed that for a generic function β the set

$$\{z \in R^1 : \ G_\beta(z) = \inf(G_\beta)\} = \{z_\beta, -z_\beta\}$$

where z_β is a unique positive number depending only on β.

More precisely, denote by $\mathcal{F}$ the set of all $\beta \in \mathcal{M}_r$ which satisfy the following condition:

(C) There is $z_\beta \in R^1$ such that

$$G_\beta(z) > G_\beta(z_\beta) \text{ for all } z \in R^1 \setminus \{z_\beta, -z_\beta\}.$$

In [70] we showed that $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, \rho)$.

We prove the following theorem which was obtained in [132].

Theorem 5.20. *$\mathcal{M} \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}, \rho)$.*

For each $\psi \in C^2(R)$ set

$$||\psi||_{C^2} = \sup\{|\psi^{(i)}(t)| : \ t \in R^1, \ i = 0, 1, 2\}.$$

5.11 The set $\mathcal{M} \setminus \mathcal{M}_r$ is porous

Proposition 5.21. *$\mathcal{M} \setminus \mathcal{M}_r$ is a porous subset of $(\mathcal{M}, \rho)$.*

Proof: Consider the function

$$\tilde{\beta}(t) = \cos(2t) + 3/2, \ t \in R^1.$$

It is easy to see that $\tilde{\beta} \in \mathcal{M}_r$,

$$\tilde{\beta}(t) \geq 1/2 \text{ for all } t \in R^1, \tag{5.101}$$

$$\tilde{\beta}(0) + \tilde{\beta}''(0) < -1, \tag{5.102}$$

$$||\tilde{\beta}||_{C^2} = 4. \tag{5.103}$$

Put

$$\alpha = 1/32. \tag{5.104}$$

Assume that

$$\beta \in \mathcal{M}, \ r \in (0, 1]. \tag{5.105}$$

Define

$$\beta_1(t) = \beta(t) + 8^{-1} r \tilde{\beta}(t), \ t \in R^1. \tag{5.106}$$

It is easy to see that $\beta_1 \in \mathcal{M}$. It follows from (5.106) and (5.103) that

$$\rho(\beta, \beta_1) = 8^{-1} r ||\tilde{\beta}||_{C^2} = 2^{-1} r. \tag{5.107}$$

By (5.106), (5.105), (5.91) and (5.101) for all $t \in R^1$,

$$\beta_1(t) \geq 8^{-1} r \tilde{\beta}(t) \geq 16^{-1} r. \tag{5.108}$$

Relations (5.106), (5.105), (5.95) and (5.102) imply that

$$\beta_1(0) + \beta_1''(0) = \beta(0) + \beta''(0) + 8^{-1} r(\tilde{\beta}(0) + \tilde{\beta}''(0))$$

$$\leq 8^{-1} r(\tilde{\beta}(0) + \tilde{\beta}''(0)) < -8^{-1} r. \tag{5.109}$$

Assume that

$$\phi \in \mathcal{M}, \ \rho(\phi, \beta_1) \leq \alpha r = r/32. \tag{5.110}$$

In view of (5.110) and (5.107),

$$\rho(\phi, \beta) \leq \rho(\phi, \beta_1) + \rho(\beta_1, \beta) \leq r/32 + r/2 < r. \tag{5.111}$$

It follows from (5.110), (5.96) and (5.108) that for all $t \in R^1$

$$\phi(t) \geq \beta_1(t) - \rho(\phi, \beta_1) \geq \beta_1(t) - r/32 \geq r/16 - 32^{-1}r = 32^{-1}r. \qquad (5.112)$$

Relations (5.96), (5.110) and (5.109) imply that

$$\phi(0) + \phi''(0) \leq \beta_1(0) + \beta_1''(0) + 2\rho(\phi, \beta_1)$$

$$\leq -8^{-1}r + 2\rho(\phi, \beta_1) \leq -8^{-1}r + r/16 = -r/16.$$

By the relation above, (5.110) and (5.112), $\phi \in \mathcal{M}_r$. Together with (5.111) this implies that

$$\{\phi \in \mathcal{M} : \rho(\phi, \beta_1) \leq \alpha r\} \subset \{\phi \in \mathcal{M} : \rho(\phi, \beta) \leq r\} \cap \mathcal{M}_r.$$

This completes the proof of Proposition 5.21.

5.12 Auxiliary results

Let $n \geq 1$ be an integer. Put

$$\Omega_n = \{z \in R^1 : 1/n \leq |z| \leq n\}.$$

Denote by $\mathcal{F}_n$ the set of all $\beta \in \mathcal{M}$ which satisfy the following condition:
(C1) There exists $z_{\beta n} \in \Omega_n$ such that

$$G_\beta(z) > G_\beta(z_{\beta n})$$

for all

$$z \in \Omega_n \setminus \{z_{\beta n}, -z_{\beta n}\}.$$

Proposition 5.22. $\mathcal{M}_r \cap (\cap_{n=1}^{\infty} \mathcal{F}_n) \subset \mathcal{F}$.

Proof: Assume that

$$f \in \mathcal{M}_r \cap (\cap_{n=1}^{\infty} \mathcal{F}_n).$$

The inclusion $\beta \in \mathcal{M}_r$ implies that

$$\lim_{|z| \to \infty} G_\beta(z) = \infty, \ \inf(G_\beta) < \beta(0) = G_\beta(0) \qquad (5.113)$$

(see (5.100) and the remark after (5.100)).

In view of (5.113) there exist an integer $k \geq 1$ and a positive number δ such that

$$G_\beta(z) \geq G_\beta(0) + 4 \text{ for all } z \in R^1 \text{ satisfying } |z| \geq k, \qquad (5.114)$$

$$G_\beta(z) > \inf(G_\beta) + \delta \text{ for all } z \in [-1/k, 1/k].$$

Inclusion $\beta \in \mathcal{F}_k$ and condition (C1) imply that there exists $z_k \in \Omega_k$ such that

$$G_\beta(z) > G_\beta(z_k) \text{ for all } z \in \Omega_k \setminus \{z_k, -z_k\}. \tag{5.115}$$

Since the function G_β is continuous (5.113) implies that G has a point of minimum. Let $z \in R^1$ satisfy

$$G_\beta(z) = \inf(G_\beta). \tag{5.116}$$

It follows from (5.114) and the definition of Ω_n that

$$1/k \leq |z| \leq k \text{ and } z \in \Omega_k.$$

Together with (5.116) and (5.115) this implies that $z \in \{z_k, -z_k\}$. Hence $\beta \in \mathcal{F}$. This completes the proof of Proposition 5.22.

Let $n, i \geq 1$ be integers. Denote by $\mathcal{F}_{ni}$ the set of all $\beta \in \mathcal{M}$ which satisfy the following condition:

(C2) There exist a positive number δ and $z_* \in \Omega_n$ such that for each $z \in \Omega_n$ satisfying

$$G_\beta(z) \leq \inf\{G_\beta(x) : \ x \in \Omega_n\} + \delta$$

the inequality

$$\min\{|z - z_*|, |z + z_*|\} \leq 1/i$$

holds.

Proposition 5.23. *Let $n \geq 1$ be an integer. Then $\cap_{i=1}^{\infty} \mathcal{F}_{ni} \subset \mathcal{F}_n$.*

Proof: Let $\beta \in \cap_{i=1}^{\infty} \mathcal{F}_{ni}$. Condition (C2) implies that for each integer $i \geq 1$ there exist

$$z_i \in \Omega_n, \ \delta_i > 0 \tag{5.117}$$

such that the following property holds:

(C3) If $z \in \Omega_n$ satisfies

$$G_\beta(z) \leq \inf\{G_\beta(y) : \ y \in \Omega_n\} + \delta_i,$$

then

$$\min\{|z - z_i|, |z + z_i|\} \leq 1/i.$$

We may assume without loss of generality that

$$z_i \geq 0 \text{ for all integers } i \geq 1. \tag{5.118}$$

Let

$$z \in \Omega_n, \ G_\beta(z) = \inf\{G_\beta(x) : \ x \in \Omega_n\}. \tag{5.119}$$

It follows from (5.119) and property (C3) for each natural number i,

$$\min\{|z - z_i|, |z + z_i|\} \le 1/i. \tag{5.120}$$

If $z = 0$, then by (5.120) $\lim_{i\to\infty} z_i = 0$. If $z > 0$, then (5.118) and (5.120) imply that $z = \lim_{i\to\infty} z_i$. If $z < 0$, then it follows from (5.118) and (5.120) that $z = -\lim_{i\to\infty} z_i$. We conclude that there exists $\lim_{i\to\infty} z_i$ and if a number z satisfies (5.119), then $z \in \{\lim_{i\to\infty} z_i, -\lim_{i\to\infty} z_i\}$. Therefore $\beta \in \mathcal{F}_n$. This completes the proof of Proposition 5.23.

5.13 Proof of Theorem 5.20

We preface the proof of Theorem 5.20 by the following auxiliary result.

Proposition 5.24. *Let n, i be integers. Then the set $\mathcal{M} \setminus \mathcal{F}_{ni}$ is a porous subset of $(\mathcal{M}, \rho)$.*

Proof: There exists $\psi \in C^\infty(R^1)$ such that

$$0 \le \psi(t) \le 1 \text{ for all } t \in R^1,\ \psi(t) = 0 \text{ if } |t| \ge 1,$$

$$\psi(t) = 1 \text{ if } |t| \le 1/2. \tag{5.121}$$

Put

$$\psi_1(t) = (1 - t^2)\psi(t),\ t \in R^1. \tag{5.122}$$

It is easy to see that $\psi_1 \in C^\infty(R^1)$,

$$0 \le \psi_1(t) \le 1 \text{ for all } t \in R^1,\ \psi_1(t) = 0 \text{ if } |t| \ge 1,$$

$$\psi_1(t) = (1 - t^2) \text{ if } |t| \le 1/2,\ \psi_1(t) < 1 \text{ for each } t \in R^1 \setminus \{0\}. \tag{5.123}$$

Fix a number $c_0 > 0$ such that

$$c_0 > \max\{8n^2 i,\ 2(\arctan(1/n))^{-1},\ 2(\arctan(n+1) - \arctan(n))^{-1}\}. \tag{5.124}$$

Put

$$\Delta = \inf\{\cos(2t) - \cos(\pi) :\ t \in [0, \arctan(n+1)]\}. \tag{5.125}$$

It is easy to see that

$$\Delta > 0. \tag{5.126}$$

Choose a number $\alpha > 0$ for which

$$\alpha \le 8^{-1} \min\{(4c_0^2 ||\psi_1||_{C^2})^{-1},\ 64^{-1}\Delta\} n^{-1}. \tag{5.127}$$

Let

$$\beta \in \mathcal{M},\ r \in (0, 1]. \tag{5.128}$$

Consider the function

$$\tilde{\beta}(t) = \cos(2t) + 3/2,\ t \in R^1. \tag{5.129}$$

Clearly, $\tilde{\beta} \in \mathcal{M}_r$,

$$\tilde{\beta}(t) \geq 1/2 \text{ for all } t \in R^1, \tag{5.130}$$

$$\tilde{\beta}(0) + \tilde{\beta}''(0) < -1. \tag{5.131}$$

Define

$$\beta_1(t) = \beta(t) + (32)^{-1} r\tilde{\beta}(t),\ t \in R^1. \tag{5.132}$$

It is easy to see that $\beta_1 \in \mathcal{M}$,

$$\rho(\beta, \beta_1) = (32^{-1})r||\tilde{\beta}||_{C^2} = 8^{-1}r. \tag{5.133}$$

By (5.132), (5.128), (5.91) and (5.130), for each $t \in R^1$,

$$\beta_1(t) \geq (32)^{-1} r\tilde{\beta}(t) \geq (64)^{-1} r. \tag{5.134}$$

It follows from (5.132), (5.95), (5.128) and (5.131) that

$$\beta_1(0) + \beta_1''(0) = \beta(0) + \beta''(0) + (32^{-1})r[\tilde{\beta}(0) + \tilde{\beta}''(0)]$$

$$\leq (32)^{-1} r[\tilde{\beta}(0) + \tilde{\beta}''(0)] = -32^{-1} r. \tag{5.135}$$

There exists

$$\bar{z} \in \Omega_n \tag{5.136}$$

for which

$$G_{\beta_1}(\bar{z}) = \inf\{G_{\beta_1}(z) :\ z \in \Omega_n\}. \tag{5.137}$$

We may assume that

$$\bar{z} > 0. \tag{5.138}$$

Put

$$\bar{\theta} = \arctan(\bar{z}). \tag{5.139}$$

It follows from (5.139), (5.138), (5.136) and the definition of Ω_n that

$$\bar{\theta} \in [\ \arctan(1/n), \arctan(n)]. \tag{5.140}$$

Put

$$c_r = r \min\{(4c_0^2||\psi_1||_{C^2})^{-1}, 64^{-1}\Delta\} \tag{5.141}$$

and define

$$\psi_2(t) = c_r(1 - \psi_1(c_0(t - \bar{\theta}))),\ t \in R^1. \tag{5.142}$$

It is easy to see that $\psi_2 \in C^\infty(R^1)$. In view of (5.142) and (5.123),

$$0 \leq \psi_2(t) \leq c_r \text{ for all } t \in R^1, \tag{5.143}$$

$$\psi_2(t) = c_r \text{ if } |t - \bar{\theta}| \geq c_0^{-1}, \tag{5.144}$$

$$\psi_2(\bar{\theta}) = 0, \tag{5.145}$$

$$\psi_2(t) > 0 \text{ for each } t \in R^1 \setminus \{\bar{\theta}\}. \tag{5.146}$$

By (5.140) and (5.124),

$$\{t \in [0, \pi/2] : \ |t - \bar{\theta}| \geq c_0^{-1}\}$$
$$= [0, \pi/2] \cap ((-\infty, \bar{\theta} - c_0^{-1}] \cup [\bar{\theta} + c_0^{-1}, \infty))$$
$$\supset [0, \pi/2] \cap ((-\infty, \ \arctan(1/n) - c_0^{-1}] \cup [\arctan(n) + c_0^{-1}, \infty))$$
$$\supset [0, \pi/2] \cap ((-\infty, 2^{-1} \ \arctan(1/n)] \cup [\arctan(n+1), \infty))$$
$$= [0, 2^{-1}\arctan(1/n)] \cup [\arctan(n+1), \pi/2]. \tag{5.147}$$

Relations (5.147) and (5.144) imply that

$$\psi_2(t) = c_r \text{ for each } t \in [0, 2^{-1}\arctan(1/n)] \cup [\ \arctan(n+1), \pi/2]. \tag{5.148}$$

In view of (5.148) there exists a function $\psi_3 : R^1 \to R^1$ such that

$$\psi_3(t) = \psi_2(t), \ t \in [0, \pi/2],$$
$$\psi_3(-t) = \psi_3(t + \pi) = \psi_3(t) \text{ for all } t \in R^1. \tag{5.149}$$

It is easy to see that $\psi_3 \in C^\infty(R^1)$. It follows from (5.149), (5.143), (5.145) and (5.146) that

$$0 \leq \psi_3(t) \leq c_r \text{ for all } t \in R^1, \tag{5.150}$$
$$\psi_3(\bar{\theta}) = 0,$$
$$\psi_3(t) > 0 \text{ for all } t \in [0, \pi/2] \setminus \{\bar{\theta}\}, \tag{5.151}$$
$$\psi_3(t) > 0 \text{ for all } t \in [-\pi/2, 0] \setminus \{-\bar{\theta}\}.$$

Put

$$\phi(t) = \beta_1(t) + \psi_3(t), \ t \in R^1. \tag{5.152}$$

It is easy to see that $\phi \in C^2(R^1)$. By the inclusion $\beta_1 \in \mathcal{M}$, (5.152), (5.150) and (5.149),

$$\phi(t) \geq 0 \text{ for all } t \in R^1, \ \phi(t) = \phi(-t) = \phi(t + \pi) \text{ for all } t \in R^1. \tag{5.153}$$

Relations (5.149) and (5.148) imply that

$$\psi_3(0) = \psi_2(0) = c_r, \ \psi_3''(0) = \psi_2''(0) = 0, \tag{5.154}$$
$$\psi_3(\pi/2) = \psi_2(\pi/2) = c_r.$$

By (5.152), (5.154), (5.135), (5.141) and (5.125),

$$\phi(0) + \phi''(0) = \beta_1(0) + \beta_1''(0) + \psi_3(0) + \psi_3''(0)$$
$$= \beta_1(0) + \beta_1''(0) + c_r \leq -32^{-1}r + c_r \leq 64^{-1}r. \tag{5.155}$$

Now we show that for all $t \in R^1$

$$\phi(\pi/2) \le \phi(t) \le \phi(0).$$

By (5.153) it is sufficient to show that this inequality holds for all $t \in [0, \pi/2]$.

Assume that $t \in [0, \pi/2]$. It follows from (5.152), (5.150), (5.92) and (5.154) that

$$\phi(t) = \beta_1(t) + \psi_3(t) \le \beta_1(t) + c_r \le \beta_1(0) + c_r = \beta_1(0) + \psi_3(0) = \phi(0). \quad (5.156)$$

Let us show that $\phi(t) \ge \phi(\pi/2)$. There are two cases:

$$|t - \bar{\theta}| \ge c_0^{-1} \quad (5.157)$$

and

$$|t - \bar{\theta}| < c_0^{-1}. \quad (5.158)$$

Assume that (5.157) holds. It follows from (5.157), (5.149) and (5.144) that

$$\psi_3(t) = \psi_2(t) = c_r.$$

Together with (5.152), (5.92) and (5.154) this equality implies that

$$\phi(t) = \beta_1(t) + \psi_3(t) = \beta_1(t) + c_r \ge \beta_1(\pi/2) + c_r = \beta_1(\pi/2)$$
$$+ \psi_3(\pi/2) = \phi(\pi/2)$$

and

$$\phi(t) \ge \phi(\pi/2).$$

Assume that (5.158) holds. By (5.158), (5.140) and (5.124),

$$2^{-1}\arctan(1/n) \le \arctan(1/n) - c_0^{-1} \le \bar{\theta} - c_0^{-1} \le t \le \bar{\theta} + c_0^{-1}$$
$$\le \arctan(n) + c_0^{-1} \le \arctan(n+1). \quad (5.159)$$

Relations (5.132), (5.92), (5.159), (5.129), (5.126) and (5.125) imply that

$$\beta_1(t) - \beta_1(\pi/2) = \beta(t) - \beta(\pi/2) + 32^{-1}r(\tilde{\beta}(t) - \tilde{\beta}(\pi/2))$$
$$\ge 32^{-1}r(\tilde{\beta}(t) - \tilde{\beta}(\pi/2)) = 32^{-1}r(\cos(2t) - \cos(\pi)) \ge 32^{-1}r\Delta. \quad (5.160)$$

By (5.152), (5.150), (5.160), (5.141) and (5.154),

$$\phi(t) = \beta_1(t) + \psi_3(t) \ge \beta_1(t) \ge \beta_1(\pi/2) + 32^{-1}r\Delta$$
$$\ge \beta_1(\pi/2) + c_r = \beta_1(\pi/2) + \psi_3(\pi/2) = \phi(\pi/2).$$

Thus in both cases $\phi(t) \ge \phi(\pi/2)$. Combined with (5.156) this implies that

$$\phi(\pi/2) \le \phi(t) \le \phi(0). \quad (5.161)$$

We proved (5.161) for all $t \in [0, \pi/2]$. It follows from (5.153) that inequality (5.161) holds for all $t \in R^1$. Together with (5.155) and (5.153) inequality

(5.161) implies that $\phi \in \mathcal{M}$. By (5.133), (5.96), (5.152), (5.149), (5.142), (5.123) and (5.124),

$$\rho(\beta, \phi) \le \rho(\beta, \beta_1) + \rho(\beta_1, \phi) \le 8^{-1}r + \rho(\beta_1, \phi)$$

$$\le 8^{-1}r + ||\psi_3||_{C^2} \le 8^{-1}r + ||\psi_2||_{C^2}$$

$$\le 8^{-1}r + c_r \max\{1, c_0||\psi_1||_{C^2}, c_0^2||\psi_1||_{C^2}\} \le 8^{-1}r + c_r c_0^2|\psi_1||_{C^2}.$$

Together with (5.141) this inequality implies that

$$\rho(\beta, \phi) \le r/8 + c_r c_0^2||\psi_1||_{C^2} \le 8^{-1}r$$

$$+r(4c_0^2||\psi_1||_{C^2})^{-1}c_0^2||\psi_1||_{C^2} \le r/8 + r/4 \le r/2. \tag{5.162}$$

In view of (5.152), (5.150) and (5.99),

$$\phi(t) \ge \beta_1(t) \text{ for all } t \in R^1$$

and

$$G_\phi(t) \ge G_{\beta_1}(t) \text{ for all } t \in R^1. \tag{5.163}$$

Now assume that

$$h \in \mathcal{M},\ \rho(h, \phi) \le \alpha r, \tag{5.164}$$

$$z \in \Omega_n,\ G_h(z) \le \inf\{G_h(y) :\ y \in \Omega_n\} + \alpha r n. \tag{5.165}$$

It follows from (5.162), (5.164) and (5.127) that

$$\rho(h, \beta) \le \rho(h, \phi) + \rho(\phi, \beta) \le \alpha r + r/2 \le r/8 + r/2 \le 3r/4. \tag{5.166}$$

By (5.99), (5.96), the definition of Ω_n and (5.164), for each $y \in \Omega_n$,

$$|G_\phi(y) - G_h(y)| = |\phi(\arctan(y))(1 + y^2)^{1/2} - h(\arctan(y))(1 + y^2)^{1/2}|$$

$$\le (1 + y^2)^{1/2}\rho(h, \phi) \le (1 + n^2)^{1/2}\rho(h, \phi) \le 2n\alpha r. \tag{5.167}$$

In particular

$$|G_\phi(z) - G_h(z)| \le 2n\alpha r. \tag{5.168}$$

Relation (5.167) implies that

$$|\inf\{G_\phi(y) :\ y \in \Omega_n\} - \inf\{G_h(y) :\ y \in \Omega_n\}| \le 2n\alpha r. \tag{5.169}$$

It follows from (5.168), (5.165) and (5.169) that

$$G_\phi(z) \le G_h(z) + 2n\alpha r \le \inf\{G_h(y) :\ y \in \Omega_n\} + 3\alpha r n$$

$$\le \inf\{G_\phi(y) :\ y \in \Omega_n\} + 5\alpha r n. \tag{5.170}$$

It follows from (5.170), (5.99), (5.139), (5.152) and (5.150) that

$$G_\phi(z) \le 5\alpha r n + G_\phi(\bar{z}) = 5\alpha r n + \phi(\arctan(\bar{z}))(1 + \bar{z}^2)^{1/2}$$

$$= 5\alpha rn + \phi(\bar{\theta})(1+\bar{z}^2)^{1/2} = 5\alpha rn + (1+\bar{z}^2)^{1/2}(\beta_1(\bar{\theta}) + \psi_3(\bar{\theta}))$$

$$= 5\alpha rn + (1+\bar{z}^2)^{1/2}\beta_1(\bar{\theta})$$

$$= 5\alpha rn + (1+\bar{z}^2)^{1/2}\beta_1(\arctan(\bar{z})) = 5\alpha rn + G_{\beta_1}(\bar{z}). \tag{5.171}$$

By (5.99), (5.152), (5.149), (5.137) and (5.165),

$$G_\phi(z) = \phi(\arctan(z))(1+z^2)^{1/2}$$

$$= \beta_1(\arctan(z))(1+z^2)^{1/2} + \psi_3(\arctan(z))(1+z^2)^{1/2}$$

$$= G_{\beta_1}(z) + \psi_2(|\arctan(z)|)(1+z^2)^{1/2}$$

$$\geq G_{\beta_1}(\bar{z}) + \psi_2(|\arctan(z)|)(1+z^2)^{1/2}.$$

Together with (5.171) the equality above implies that

$$G_{\beta_1}(\bar{z}) + \psi_2(|\arctan(z)|)(1+z^2)^{1/2} \leq G_\phi(z)$$

$$\leq 5\alpha rn + G_{\beta_1}(\bar{z})$$

and

$$\psi_2(|\arctan(z)|)(1+z^2)^{1/2} \leq 5\alpha rn.$$

This inequality implies that

$$\psi_2(|\arctan(z)|) \leq 5\alpha rn. \tag{5.172}$$

If

$$||\arctan(z)| - \bar{\theta}| \geq c_0^{-1},$$

then (5.144), (5.141) and (5.127) imply that

$$\psi_2(|\arctan(z)|) = c_r \geq 8\alpha rn.$$

The relation above contradicts (5.172). Thus

$$|\arctan(|z|) - \bar{\theta}| < c_0^{-1}. \tag{5.173}$$

By the mean value theorem,

$$||z| - \bar{z}| = |\arctan(|z|) - \bar{\theta}||(\tan)'(x)| =$$

$$|\arctan(|z|) - \bar{\theta}|(\cos(x))^{-2}, \tag{5.174}$$

where

$$x \in [\min\{\bar{\theta}, \arctan(|z|)\}, \max\{\bar{\theta}, \arctan(|z|)\}]. \tag{5.175}$$

It follows from (5.175), (5.165), (5.139), (5.136), (5.138) and the definition of Ω_n that

$$x \in [\arctan(1/n), \arctan(n)]. \tag{5.176}$$

Since

$$\cos(x)^{-2} = (\tan(x))^2 + 1 \in [1 + 1/n^2, n^2 + 1]$$

it follows from (5.174), (5.173) and (5.124) that

$$||z| - \bar{z}| \leq |\arctan(|z|) - \bar{\theta}|(n^2 + 1) \leq c_0^{-1}(n^2 + 1) \leq 1/i. \tag{5.177}$$

We have shown that $||z| - \bar{z}| \leq 1/i$. Therefore $h \in \mathcal{F}_{ni}$. Combined with (5.166) this inequality implies that

$$\{h \in \mathcal{M} : \ \rho(h, \phi) \leq \alpha r\} \subset \{h \in \mathcal{M} : \ \rho(\beta, h) \leq r\} \cap \mathcal{F}_{ni}$$

and $\mathcal{M} \setminus \mathcal{F}_{ni}$ is a porous subset of $(\mathcal{M}, \rho)$. This completes the proof of Proposition 5.24.

Completion of the Proof of Theorem 5.20: Let $n \geq 1$ be an integer. It follows from Proposition 5.24 that for each integer $i \geq 1$, $\mathcal{M} \setminus \mathcal{F}_{ni}$ is a porous subset of $(\mathcal{M}, \rho)$. By Proposition 5.23,

$$\mathcal{M} \setminus \mathcal{F}_n \subset \mathcal{M} \setminus (\cap_{i=1}^{\infty} \mathcal{F}_{ni}) = \cup_{i=1}^{\infty} (\mathcal{M} \setminus \mathcal{F}_{ni}).$$

Thus $\mathcal{M} \setminus \mathcal{F}_n$ is a σ-porous subset of $(\mathcal{M}, \rho)$ for any natural number n.

Propositions 5.21 and 5.22 imply that

$$\mathcal{M} \setminus \mathcal{F} \subset \mathcal{M} \setminus (\mathcal{M}_r \cap (\cap_{n=1}^{\infty} \mathcal{F}_n)) = (\mathcal{M} \setminus \mathcal{M}_r) \cup_{n=1}^{\infty} (\mathcal{M} \setminus \mathcal{F}_n)$$

and $\mathcal{M} \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}, \rho)$. This completes the proof of Theorem 5.20.

5.14 Porosity results for a class of equilibrium problems

Let (X, ρ) be a complete metric space. We consider the following equilibrium problem:

$$\text{To find } x \in X \text{ such that } f(x, y) \geq 0 \text{ for all } y \in X, \tag{P}$$

where f belongs to a complete metric space of functions $\mathcal{A}$ defined below. Using the notion of porosity we show that for most elements of the space of functions $\mathcal{A}$ the equilibrium problem (P) possesses a solution.

We consider the set $\mathcal{F} \subset \mathcal{A}$ which consists of all functions $f \in \mathcal{A}$ such that the problem (P) possesses a solution and show that $\mathcal{A} \setminus \mathcal{F}$ is a porous set in $\mathcal{A}$. Actually we prove three porosity results for different spaces of functions. These results were obtained in [139].

We use the following notation and definitions.

Put

$$\rho_1((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2), \ x_1, x_2, y_1, y_2 \in X.$$

It is easy to see that $(X \times X, \rho_1)$ is a complete metric space. For each $x \in X$ and each positive number r put

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\}.$$

We use the convention $\infty/\infty = 1$.

Denote by $\mathcal{M}$ the set of all functions $f : X \times X \to R^1$. For each $f, g \in \mathcal{M}$ set

$$\tilde{d}(f,g) = \sup\{|f(x,y) - g(x,y)| : \ x, y \in X\}$$
$$+ \sup\{|(f-g)(x_1,x_2) - (f-g)(y_1,y_2)|(\rho_1((x_1,x_2),(y_1,y_2)))^{-1} :$$
$$(x_1,x_2),\ (y_1,y_2) \in X \times X \text{ and } (x_1,x_2) \ne (y_1,y_2)\}, \tag{5.178}$$

$$d(f,g) = \tilde{d}(f,g)(1 + \tilde{d}(f,g))^{-1}. \tag{5.179}$$

Evidently, $(\mathcal{M}, d)$ is a complete metric space. If $f, g \in \mathcal{M}$ and $d(f,g) < 1$, then

$$\tilde{d}(f,g) = d(f,g)(1 - d(f,g))^{-1}. \tag{5.180}$$

5.15 The first porosity result

In this section we use the notation and definitions from Section 5.14.

Let the metric space (X, ρ) be compact. Set

$$\mathrm{diam}(X) = \sup\{\rho(x,y) : \ x, y \in X\}.$$

Denote by $\mathcal{M}_0$ the set of all functions $f \in \mathcal{M}$ such that the following properties hold:

(A1) $f(x,x) = 0$ for all $x \in X$;

(A2) For each $y \in X$ the function $f : (\cdot, y)$ is upper semicontinuous;

(A3) For any positive number ϵ there exists $x_\epsilon \in X$ such that $f(x_\epsilon, y) \ge -\epsilon$ for all $y \in X$.

It is easy to see that $\mathcal{M}_0$ is a closed subset of the metric space $(\mathcal{M}, d)$.

Remark 5.25. Since the space (X, ρ) is compact it is not difficult to see that $\mathcal{M}_0$ is the set of all $f \in \mathcal{M}$ which satisfy (A1) and (A2) and for which there exists $x_f \in X$ such that $f(x_f, y) \ge 0$ for all $y \in X$.

Assume that $\mathcal{A}$ is a nonempty closed subset of the metric space $(\mathcal{M}_0, d)$ such that the following property holds:

for each $f \in \mathcal{A}$ and each positive number r the function

$$(x,y) \to f(x,y) + r\rho(x,y),\ x, y \in X$$

belongs to $\mathcal{A}$.

Denote by $\mathcal{F}$ the set of all $f \in \mathcal{A}$ such that the following property holds:

there exist $\bar{x} \in X$, a positive number γ and an open neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$

$$g(\bar{x}, y) \geq \gamma(\bar{x}, y) \text{ for all } y \in X.$$

We consider the complete metric space $\mathcal{A}$ equipped with the metric d.

Theorem 5.26. *$\mathcal{A} \setminus \mathcal{F}$ is a porous set in the complete metric space $\mathcal{A}$.*

Proof: Put

$$\alpha = 32^{-1}(\text{diam}(X) + 1)^{-1}. \tag{5.181}$$

Let $f \in \mathcal{A}$ and $r \in (0, 1)$ and define

$$f_r(x, y) = f(x, y) + 8^{-1}\rho(x, y)r(\text{diam}(X) + 1)^{-1}, \ x, y \in X. \tag{5.182}$$

It is easy to see that $f_r \in \mathcal{A}$. Denote by Ω the set of all $x \in X$ such that

$$f(x, y) \geq 0 \text{ for all } y \in X.$$

Remark 5.25 implies that

$$\Omega \neq \emptyset. \tag{5.183}$$

It follows from (5.178), (5.179) and (5.182) that

$$d(f, f_r) \leq \tilde{d}(f, f_r) \leq 8^{-1}r + 8^{-1}r. \tag{5.184}$$

Assume that

$$y \in \mathcal{A}, \ d(f_r, g) \leq \alpha r. \tag{5.185}$$

In view of (5.181), (5.184) and (5.185),

$$d(g, f) \leq d(g, f_r) + d(f_r, f) \leq \alpha r + r/4 < r/2. \tag{5.186}$$

Relations (5.186), (5.181) and (5.185) imply that

$$\tilde{d}(f_r, g) = d(f_r, g)(1 - d(f_r, g))^{-1} \leq 2\alpha r. \tag{5.187}$$

For each $x \in \Omega$ and each $y \in X$ it follows from the equation

$$g(y, y) = f_r(y, y) = 0, \tag{5.188}$$

(5.178) and (5.187) that

$$|(g - f_r)(x, y)| = |(g - f_r)(x, y) - (g - f_r)(y, y)| \leq \tilde{d}(g, f_r)\rho(x, y) \leq 2\alpha r\rho(x, y). \tag{5.189}$$

Relations (5.189), (5.182), (5.181) and the definition of Ω imply that for each $x \in \Omega$ and each $y \in X$

$$\begin{aligned} g(x, y) &\geq f_r(x, y) - 2\alpha r\rho(x, y) \\ &= f(x, y) + 8^{-1}r(\text{diam}(X) + 1)^{-1}\rho(x, y) - 2\alpha r\rho(x, y) \\ &\geq f(x, y) + (\text{diam}(X) + 1)^{-1}16^{-1}r\rho(x, y) \geq (\text{diam}(X) + 1)^{-1}16^{-1}\rho(x, y). \end{aligned}$$

Theorem 5.26 is proved.

5.16 The second porosity result

We use the notation and definitions introduced in Section 5.14.

Assume that each bounded closed subset of the metric space (X, ρ) is compact. Fix $\theta \in X$.

Denote by $\mathcal{M}_1$ the set of all functions $f \in \mathcal{M}$ such that the following properties hold:

(B1) $f(x, x) = 0$ for all $x \in X$;

(B2) For all $y \in X$ the function $f : (\cdot, y) : X \to R^1$ is upper semicontinuous;

(B3) There exist positive numbers M_f, δ_f and $y_f \in X$ such that $f(x, y_f) < -\delta_f$ for all $x \in X$ such that $\rho(x, \theta) > M_f$;

(B4) For each positive number ϵ there exists $x_\epsilon \in X$ such that

$$f(x_\epsilon, y) \geq -\epsilon \text{ for all } y \in X. \tag{5.190}$$

It is easy to see that $\mathcal{M}_1$ is a closed subset of the metric space $\mathcal{M}$ with the metric d. We equip the space $\mathcal{M}_1$ with the metric d. Define

$$\psi(x, y) = \min\{\rho(x, y), 1\}, \; x, y \in X. \tag{5.191}$$

Assume that $\mathcal{A}$ is a closed subset of the metric space $\mathcal{M}_1$ with the metric d such that

$$f + r\psi \in \mathcal{A} \text{ for each } f \in \mathcal{A} \text{ and each number } r > 0. \tag{5.192}$$

We equip the space $\mathcal{A}$ with the metric d.

Proposition 5.27. *Let $f \in \mathcal{M}_1$. Then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$.*

Proof: It follows from (B4) that for each integer $k \geq 1$ there exists $x_k \in X$ such that

$$f(x_k, y) \geq -k^{-1} \text{ for all } y \in X. \tag{5.193}$$

By (B3) there exist positive numbers δ_0, M_0 and $y_0 \in X$ such that

$$f(x, y_0) < -\delta_0 \text{ for all } x \in X \text{ such that } \rho(x, \theta) > -M_0. \tag{5.194}$$

Relations (5.193) and (5.194) imply that $\rho(x_k, \theta) \leq M_0$ for all sufficiently large natural numbers k. This implies that there exists a convergent subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ with

$$\bar{x} := \lim_{k \to \infty} x_{k_i}. \tag{5.195}$$

It follows from (5.193), (5.195) and (B2) that

$$f(\bar{x}, y) \geq 0 \text{ for all } y \in X.$$

Proposition 5.27 is proved.

Denote by $\mathcal{F}$ the set of all $f \in \mathcal{A}$ such that the following property holds:

there exist $\bar{x} \in X$, a positive number γ and an open neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$

$$g(\bar{x}, y) \geq \gamma\psi(\bar{x}, y) \text{ for all } y \in X. \tag{5.196}$$

Theorem 5.28. *$\mathcal{A} \setminus \mathcal{F}$ is a porous set in the space $\mathcal{A}$ with the metric d.*

Proof: Set

$$\alpha = 32^{-1}.$$

Let $f \in \mathcal{A}$ and $r \in (0, 1]$ and define

$$f_r(x, y) = f(x, y) + 8^{-1}r\psi(x, y),\ x, y \in X. \tag{5.197}$$

It is easy to see that $f_r \in \mathcal{A}$.

Denote by Ω the set of all $x \in X$ such that

$$f(x, y) \geq 0 \text{ for all } y \in X.$$

Proposition 5.27 implies that

$$\Omega \neq \emptyset. \tag{5.198}$$

It follows from (5.178), (5.179), (5.191) and (5.197) that

$$d(f, f_r) \leq \tilde{d}(f, f_r) \leq 8^{-1}r + 8^{-1}r. \tag{5.199}$$

Assume that

$$g \in \mathcal{A} \text{ and } d(f_r, g) \leq \alpha r. \tag{5.200}$$

In view of (5.199) and (5.200),

$$d(g, f) \leq d(g, f_r) + d(f_r, f) \leq \alpha r + r/4 < r/2. \tag{5.201}$$

It follows from (5.180) and (5.200) that

$$\tilde{d}(f_r, g) = d(f_r, g)(1 - d(f_r, g))^{-1} \leq 2\alpha r. \tag{5.202}$$

Let

$$x \in \Omega \text{ and } y \in X. \tag{5.203}$$

We estimate $g(x, y)$. There are two cases: $\rho(x, y) \geq 1$; $\rho(x, y) < 1$.

Assume that

$$\rho(x, y) \geq 1. \tag{5.204}$$

It follows from (5.197), (5.202), (5.178), (5.203), the definition of Ω, (5.204) and (5.191) that

$$\begin{aligned} g(x, y) &= f_r(x, y) + (g - f_r)(x, y) \geq f(x, y) + 8^{-1}r\psi(x, y) - 2\alpha r \\ &\geq 8^{-1}r - 2\alpha r \geq 16^{-1}r = 16^{-1}r\psi(x, y). \end{aligned} \tag{5.205}$$

Consider the case

$$\rho(x, y) < 1. \tag{5.206}$$

It follows from (5.197), (5.203), the definition of Ω, (5.206) and (5.201) that

$$\begin{aligned} g(x, y) &= f_r(x, y) + (g - f_r)(x, y) \\ &= f(x, y) + 8^{-1} r\psi(x, y) + (g - f_r)(x, y) \\ &\geq 8^{-1} r\psi(x, y) + (g - f_r)(x, y). \end{aligned} \tag{5.207}$$

(B1), (5.178) and (5.202) imply that

$$\begin{aligned} |(g - f_r)(x, y)| &= |(g - f_r)(x, y) - (g - f_r)(y, y)| \\ &\leq \tilde{d}(g, f_r)\rho(x, y) \leq 2\alpha r\rho(x, y). \end{aligned} \tag{5.208}$$

Together with (5.207), (5.206) and (5.191) this implies that

$$g(x, y) \geq 8^{-1} r\rho(x, y) - 16^{-1} r\rho(x, y) = 16^{-1} r\rho(x, y) = 16^{-1} r\psi(x, y).$$

Thus in both cases

$$g(x, y) \geq 16^{-1} r\psi(x, y)$$

for all $y \in X$. This completes the proof of Theorem 5.28.

5.17 The third porosity result

We use the notation and definitions from Section 5.14.

Let (X, ρ) be a complete metric space. Define

$$\psi(x, y) = \min\{\rho(x, y), 1\}, \ x, y \in X. \tag{5.209}$$

Assume that $\mathcal{A}$ is a closed subset of the space $\mathcal{M}$ with the metric d such that for each $f \in \mathcal{A}$ the following properties hold:

(C1) $f(x, x) = 0$ for all $x \in X$;

(C2) For each number $r > 0$, $f + r\psi \in \mathcal{A}$;

(C3) For each positive number ϵ there exists $x_\epsilon \in X$ such that for all $y \in X$,

$$f(x_\epsilon, y) + \epsilon\psi(x_\epsilon, y) \geq 0. \tag{5.210}$$

We equip the space $\mathcal{A}$ with the metric d.

Proposition 5.29. *Let $\tilde{\mathcal{M}}$ be the set of all functions $f \in \mathcal{M}$ for which (C1) and (C3) hold. Then $\tilde{\mathcal{M}}$ is a closed subset of the space $\mathcal{M}$ with the metric d and (C2) holds for $\mathcal{A} = \tilde{\mathcal{M}}$.*

Proof: It is easy to see that (C2) holds for $\mathcal{A} = \tilde{\mathcal{M}}$. Let us show that $\tilde{\mathcal{M}}$ is a closed subset of $(\mathcal{M}, d)$.

Assume that $\{f_k\}_{k=1}^{\infty} \subset \tilde{\mathcal{M}}$, $f \in \mathcal{M}$ and

$$\lim_{k \to \infty} d(f_k, f) = 0. \tag{5.211}$$

Then

$$f(x, x) \geq 0 \text{ for all } x \in X. \tag{5.212}$$

We show that (C3) holds. Let $\epsilon \in (0, 1)$. There exists an integer $k \geq 1$ such that

$$d(f_k, f) \leq 8^{-1}\epsilon. \tag{5.213}$$

Relations (5.213) and (5.180) imply that

$$\tilde{d}(f_k, f) = d(f_k, f)(1 - d(f_k, f))^{-1} \leq 2d(f, f_k) \leq \epsilon/4. \tag{5.214}$$

It follows from (C3) that there exists $x_0 \in X$ such that for all $y \in X$

$$f_k(x_0, y) + 16^{-1}\epsilon\psi(x_0, y) \geq 0. \tag{5.215}$$

Let $y \in X$. If $\rho(x_0, y) \geq 1$, then (5.214), (5.178), (5.211) and (5.215) imply that

$$\begin{aligned} f(x_0, y) + \epsilon\psi(x_0, y) &= f(x_0, y) + \epsilon \geq f_k(x_0, y) - 4^{-1}\epsilon \\ &\geq f_k(x_0, y) + (\epsilon/2) = f_k(x_0, y) + (\epsilon/2)\psi(x_0, y) \geq 0. \end{aligned}$$

Assume that

$$\rho(x_0, y) < 1. \tag{5.216}$$

Relation (5.211) implies that

$$\psi(x_0, y) = \rho(x_0, y)$$

and in view of (C1) which holds with f and f_k, (5.178) and (5.214),

$$\begin{aligned} f(x_0, y) &= f_k(x_0, y) + (f - f_k)(x_0, y) \\ &= f_k(x_0, y) + (f - f_k)(x_0, y) - (f - f_k)(y, y) \\ &\geq f_k(x_0, y) - (\epsilon/4)\rho(x_0, y). \end{aligned}$$

Together with (5.211), (5.216) and (5.215) this implies that

$$\begin{aligned} f(x_0, y) + \epsilon\psi(x_0, y) &= f(x_0, y) + \epsilon\rho(x_0, y) \\ &\geq f_k(x_0, y) - (\epsilon/4)\rho(x_0, y) + \epsilon\rho(x_0, y) \\ &\geq f_k(x_0, y) + (\epsilon/2)\rho(x_0, y) = f_k(x_0, y) + (\epsilon/2)\psi(x_0, y) \geq 0 \end{aligned}$$

and

$$f(x_0, y) + \epsilon\psi(x_0, y) \geq 0 \text{ for all } y \in X.$$

Therefore

$$f \in \tilde{\mathcal{M}}.$$

This completes the proof of Proposition 5.29.

The next proposition easily follows from (C3).

Proposition 5.30. *The set of all $f \in \mathcal{A}$ for which there is $x_f \in X$ satisfying $f(x_f, y) \geq 0$ for all $y \in X$ is an everywhere dense subset of the metric space $\mathcal{A}$.*

Denote by $\mathcal{F}$ the set of all $f \in \mathcal{A}$ such that the following property holds:

there exist $\bar{x} \in X$, a positive number γ and a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ such that for each $g \in \mathcal{U}$

$$g(\bar{x}, y) \geq \gamma\psi(\bar{x}, y) \text{ for all } y \in X.$$

Theorem 5.31. *$\mathcal{A} \setminus \mathcal{F}$ is a porous set in the metric space $\mathcal{A}$ with the metric d.*

Proof: Let

$$\alpha = 32^{-1}$$

and

$$f \in \mathcal{A},\ r \in (0, 1]. \tag{5.217}$$

Proposition 5.30 implies that there exist

$$f_0 \in \mathcal{A},\ x_0 \in X \tag{5.218}$$

such that

$$\tilde{d}(f, f_0) \leq r/16, \tag{5.219}$$

$$f_0(x_0, y) \geq 0 \text{ for all } y \in X.$$

Put

$$f_1(x, y) = f_0(x, y) + 8^{-1} r\psi(x, y),\ x, y \in X. \tag{5.220}$$

It follows from (5.220) and (C2) that $f_1 \in \mathcal{A}$. By (5.219) and (5.220),

$$f_1(x_0, y) \geq 8^{-1} r\psi(x_0, y) \text{ for all } y \in X. \tag{5.221}$$

In view of (5.178), (5.179), (5.209) and (5.220),

$$d(f_1, f_0) \leq \tilde{d}(f_1, f_0) \leq 8^{-1} r + 8^{-1} r. \tag{5.222}$$

Assume that

$$g \in \mathcal{A} \text{ and } d(f_1, g) \leq \alpha r. \tag{5.223}$$

By (5.217), (5.219), (5.222) and (5.223),

$$d(g, f) \leq d(g, f_1) + d(f_1, f_0) + d(f_0, f) \leq \alpha r + 4^{-1} r + r/16 < r. \tag{5.224}$$

It follows from (5.180), (5.217) and (5.223) that

$$\tilde{d}(f_1, g) = d(f_1, g)(1 - d(f_1, g))^{-1} \leq 2\alpha r. \tag{5.225}$$

Let $y \in X$. We estimate $g(x_0, y)$. There are two cases: $\rho(x, y) \geq 1$; $\rho(x, y) < 1$.

Assume that $\rho(x,y) \geq 1$. Then (5.209), (5.225), (5.217) and (5.178) imply that

$$g(x_0,y) = f_1(x_0,y) + (g-f_1)(x_0,y) \geq 8^{-1}r\psi(x_0,y) - 16^{-1}r = 16^{-1}r\psi(x_0,y).$$

Assume that $\rho(x_0,y) < 1$. Then it follows from (5.221), (C1), (5.178), (5.225) and (5.217) that

$$\begin{aligned} g(x_0,y) &= f_1(x_0,y) + (g-f_1)(x_0,y) \\ &\geq 8^{-1}r\psi(x_0,y) + (g-f_1)(x_0,y) - (g-f_1)(y,y) \\ &\geq 8^{-1}r\rho(x_0,y) - 2\alpha r\rho(x_0,y) \\ &\geq 16^{-1}r\rho(x_0,y) = 16^{-1}r\psi(x_0,y). \end{aligned}$$

Therefore

$$g(x_0,y) \geq 16^{-1}r\psi(x_0,y)$$

for all $y \in X$. Theorem 5.31 is proved.

5.18 Comments

In this chapter we continue to consider various classes of minimization problems showing that most problems in these classes are well-posed. In order to meet this goal we use a porosity notion. As in Chapter 4 we identify a class of minimization problems with a certain complete metric space of functions, study the set of all functions for which the corresponding minimization problem is well-posed and show that the complement of this set is not only of the first category but also a σ-porous set.

6

Parametric Optimization

6.1 Generic variational principle

In this chapter we obtain our generic results as realizations of a variational principle which is introduced in this section.

We consider a complete metric space (X, ρ). Let $(\mathcal{A}, d_{\mathcal{A}})$ be a complete metric space and $\mathcal{B}$ be a nonempty set. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric $d_{\mathcal{A}}$. This topology will be called the strong topology. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology. (Note that these topologies can coincide.) We assume that with every $(a, b) \in \mathcal{A} \times \mathcal{B}$ a lower semicontinuous function f_{ab} on X is associated with values in $\bar{R} = [-\infty, \infty]$. For each function $g : X \to \bar{R}$ we set

$$\inf(g) = \inf\{g(x) : \ x \in X\}.$$

For each $x \in X$ and each $F \subset X$ we set

$$\rho(x, F) = \inf\{\rho(x, y) : \ y \in F\}.$$

We use the convention that $\infty - \infty = 0$.

Let $(a, b) \in (\mathcal{A} \times \mathcal{B})$. We say that the minimization problem for f_{ab} on (X, ρ) is well-posed in the generalized sense [40] if $\inf(f_{ab})$ is finite, the set $\{x \in X : f_{ab}(x) = \inf(f_{ab})\}$ is nonempty and compact and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying $\lim_{i \to \infty} f_{ab}(x_i) = \inf(f_{ab})$ has a convergent subsequence.

In our study we use the following basic hypotheses about the functions.

(H) For each $a \in \mathcal{A}$ and each positive number ϵ there exist $\bar{a} \in \mathcal{A}$, an open neighborhood V of $\bar{a}$ in $\mathcal{A}$ with the weak topology, a finite set $\{x_1, \dots, x_q\} \subset X$, where $q \geq 1$ is an integer, and a positive number δ such that

(i) $d_{\mathcal{A}}(a, \bar{a}) < \epsilon$;

(ii) for each $\xi \in V$ and each $b \in \mathcal{B}$, $\inf(f_{\xi b})$ is finite and if $x \in X$ satisfies $f_{\xi b}(x) \leq \inf(f_{\xi b}) + \delta$, then $\min\{\rho(x, x_i) : \ i = 1, \dots, q\} \leq \epsilon$.

The following theorem was established in [113].

A.J. Zaslavski, *Optimization on Metric and Normed Spaces*, Springer Optimization and Its Applications 44, DOI 10.1007/978-0-387-88621-3_6,

Theorem 6.1. *Assume that (H) holds. Then there exists a set* $\mathcal{F} \subset \mathcal{A}$ *which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of* $\mathcal{A}$ *such that for each* $a \in \mathcal{F}$ *and each* $b \in \mathcal{B}$ *the minimization problem for* f_{ab} *on* (X, ρ) *is well-posed in the generalized sense.*

Proof: It follows from (H) that for each $a \in \mathcal{A}$ and each integer $n \geq 1$ there exist $a(n) \in \mathcal{A}$, an open neighborhood $V(a, n)$ of $a(n)$ in $\mathcal{A}$ with the weak topology, a positive number $\delta(a, n)$ and a nonempty finite set $Q(a, n) \subset X$ such that

$$d_{\mathcal{A}}(a, a(n)) < 1/n \tag{6.1}$$

and the following property holds:

(C1) For each $\xi \in V(a, n)$ and each $b \in \mathcal{B}$, $\inf(f_{\xi b})$ is finite and if $x \in X$ satisfies

$$f_{\xi b}(x) \leq \inf(f_{\xi b}) + \delta(a, n), \tag{6.2}$$

then

$$\rho(x, Q(a, n)) \leq 1/n. \tag{6.3}$$

Define

$$\mathcal{F} = \cap_{n=1}^{\infty} [\cup \{V(a, i) : \ a \in \mathcal{A} \text{ and } i \geq n\}]. \tag{6.4}$$

It is easy to see that $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$.

Let $\xi \in \mathcal{F}$ and $b \in \mathcal{B}$. It is easy to see that $\inf(f_{\xi b})$ is finite. It follows from (6.4) that for each natural number n there exist $a_n \in \mathcal{A}$ and an integer $i_n \geq n$ such that

$$\xi \in V(a_n, i_n). \tag{6.5}$$

Assume that a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ satisfies

$$\lim_{i \to \infty} f_{\xi b}(z_i) = \inf(f_{\xi b}). \tag{6.6}$$

In view of (6.6), (6.5) and the definition of $V(a_n, i_n)$ (see the property (C1)), for each natural number n the inequality

$$\rho(z_j, Q(a_n, i_n)) \leq 1/i_n \leq 1/n \tag{6.7}$$

holds for all sufficiently large natural numbers j. Since the sets $Q(a_n, i_n)$, $n = 1, 2, \dots$ are finite we conclude that for each natural number p there exists a subsequence $\{z_{i_k}^{(p)}\}_{k=1}^{\infty}$ of the sequence $\{z_i\}_{i=1}^{\infty}$ such that the following properties hold:

For each natural number p the sequence $\{z_{i_k}^{(p+1)}\}_{k=1}^{\infty}$ is a subsequence of $\{z_{i_k}^{(p)}\}_{k=1}^{\infty}$.

For each natural number p and each pair of natural numbers $j, s \geq 1$,

$$\rho(z_{i_j}^{(p)}, z_{i_s}^{(p)}) \leq 2p^{-1}. \tag{6.8}$$

These properties imply that there exists a subsequence $\{z^*_{i_k}\}_{k=1}^{\infty}$ of the sequence $\{z_i\}_{i=1}^{\infty}$ which is a Cauchy sequence.

Let

$$x_* = \lim_{k\to\infty} z^*_{i_k}. \tag{6.9}$$

It follows from (6.6) and the lower semicontinuity of $f_{\xi b}$ that

$$f_{\xi b}(x_*) = \inf(f_{\xi b}). \tag{6.10}$$

Therefore we have shown that for each sequence $\{z_i\}_{i=1}^{\infty} \subset X$ satisfying (6.6) there exists a subsequence which converges to a point x_* satisfying (6.10). This completes the proof of Theorem 6.1.

Remark 6.2. Note that Theorem 6.1 is also valid if $\mathcal{B} = \cup_{i=1}^{\infty} C_i$ and (H) holds with $\mathcal{B} = C_i$ for all integers $i \geq 1$.

6.2 Concretization of the hypothesis (H)

We use the notations and definitions introduced in Section 6.1. The proofs of our generic existence results consist in verification in each case of the hypothesis (H). To simplify the verification of (H) in this section we introduce new assumptions and show that they imply (H). Thus to verify (H) we need to show that these new assumptions are valid. In fact this approach allows us to simplify the problem.

In the sequel we assume that

$$\inf(f_{ab}) \text{ is finite for all } (a,b) \in \mathcal{A} \times \mathcal{B}. \tag{6.11}$$

We use the following assumptions.

(A1) For each $a \in \mathcal{A}$, $\sup_{b\in\mathcal{B}} \inf(f_{ab}) < \infty$.

(A2) For each $a \in \mathcal{A}$ and each positive number ϵ there exist $\bar{a} \in \mathcal{A}$, a nonempty finite set $Q \subset X$ and a positive number δ such that $d_{\mathcal{A}}(a, \bar{a}) < \epsilon$ and the following property holds:

For each $b \in \mathcal{B}$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \leq \inf(f_{\bar{a}b}) + \delta$, the inequality $\rho(x, Q) \leq \epsilon$ holds.

(A3) For each $a \in \mathcal{A}$, each natural number n and each positive number ϵ there exists a neighborhood V of a in $\mathcal{A}$ with the weak topology such that for each $b \in \mathcal{B}$, each $\xi \in V$ and each $x \in X$ satisfying $\min\{f_{ab}(x), f_{\xi b}(x)\} \leq n$ the inequality $|f_{ab}(x) - f_{\xi b}(x)| \leq \epsilon$ holds.

Proposition 6.3. *Assume that (A1),(A2) and (A3) hold. Then (H) holds.*

Proof: Let $a \in \mathcal{A}$ and ϵ be a positive number. (A2) implies that there exist $\bar{a} \in \mathcal{A}$, a nonempty finite set $Q \subset X$ and a number $\delta \in (0,1)$ such that

$$d_{\mathcal{A}}(a, \bar{a}) < \epsilon/2 \tag{6.12}$$

and that the following property holds:

(C2) For each $b \in \mathcal{B}$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta$ the inequality $\rho(x, Q) \le \epsilon/2$ holds.

In view of (A1) there exists an integer $n \ge 1$ such that

$$n > |\sup_{b\in\mathcal{B}} \inf(f_{\bar{a}b})| + 4 + 4\epsilon. \tag{6.13}$$

It follows from (A3) that there exists a neighborhood V of $\bar{a}$ in $\mathcal{A}$ with the weak topology such that the following property holds:

(C3) For each $b \in \mathcal{B}$, each $\xi \in V$ and each $x \in X$ satisfying

$$\min\{f_{\bar{a}b}(x), f_{\xi b}(x)\} \le n$$

the inequality

$$|f_{\bar{a}b}(x) - f_{\xi b}(x)| \le \delta/8 \tag{6.14}$$

holds.

Let $\xi \in V$ and $b \in \mathcal{B}$. We will show that

$$|\inf(f_{\xi b}) - \inf(f_{\bar{a}b})| \le \delta/8. \tag{6.15}$$

Let $x \in X$ and

$$f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + 1. \tag{6.16}$$

It follows from (6.16), (6.13) and the property (C3) that (6.14) holds. Since (6.16) implies (6.14) we obtain that

$$\inf(f_{\xi b}) \le \inf\{f_{\xi b}(x) :\ x \in X \text{ and } f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + 1\} \le$$
$$\inf\{f_{\bar{a}b}(x) + \delta/8 :\ x \in X \text{ and } f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + 1\} = \delta/8 +$$
$$\inf\{f_{\bar{a}b}(x) :\ x \in X \text{ and } f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + 1\} = \delta/8 + \inf(f_{\bar{a}b}).$$

Hence

$$\inf(f_{\xi b}) \le \inf(f_{\bar{a}b}) + \delta/8. \tag{6.17}$$

Let $x \in X$ and

$$f_{\xi b}(x) \le \inf(f_{\xi b}) + 1. \tag{6.18}$$

In view of (6.18), (6.17), (6.13) and the property (C3), the inequality (6.14) is true. Since (6.18) implies (6.14) we obtain that

$$\inf(f_{\bar{a}b}) \le \inf\{f_{\bar{a}b}(x) :\ x \in X \text{ and } f_{\xi b}(x) \le \inf(f_{\xi b}) + 1\} \le$$
$$\inf\{f_{\xi b}(x) + \delta/8 :\ x \in X \text{ and } f_{\xi b}(x) \le \inf(f_{\xi b}) + 1\} =$$
$$\delta/8 + \inf\{f_{\xi b}(x) :\ x \in X \text{ and } f_{\xi b}(x) \le \inf(f_{\xi b}) + 1\} =$$
$$\delta/8 + \inf(f_{\xi b}).$$

Hence $\inf(f_{\bar{a}b}) \le \delta/8 + \inf(f_{\xi b})$. Together with (6.17) this inequality implies (6.15).

Let $x \in X$ and

$$f_{\xi b}(x) \le \inf(f_{\xi b}) + \delta/4. \tag{6.19}$$

By (6.19) and (6.15),

$$f_{\xi b}(x) \le \inf(f_{\bar{a}b}) + 3\delta/8. \tag{6.20}$$

It follows from (6.20), (6.13) and the property (C3) that (6.14) is true. In view of (6.14) and (6.20), $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta/2$. By this inequality and the property (C2), $\rho(x, Q) \le \epsilon/2$. Proposition 6.3 is proved.

Assume that $\mathcal{B}$ is a Hausdorff topological space. We use the following assumption.

(A4) Let $(a, b) \in \mathcal{A} \times \mathcal{B}$, ϵ be a positive number and let n be a natural number. Then there exists a neighborhood U of b in $\mathcal{B}$ such that for each $\xi \in U$ and each $x \in X$ satisfying $\min\{f_{ab}(x), f_{a\xi}(x)\} \le n$ the inequality $|f_{ab}(x) - f_{a\xi}(x)| \le \epsilon$ holds.

Proposition 6.4. *Assume that (A4) holds. Then the function $b \to \inf(f_{ab})$, $b \in \mathcal{B}$ is continuous for each $a \in \mathcal{A}$.*

Proof: Let $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\epsilon \in (0, 1)$. Fix an integer

$$n > |\inf(f_{ab})| + 4. \tag{6.21}$$

It follows from (A4) that there exists a neighborhood U of b in $\mathcal{B}$ such that for each $\xi \in U$ and each $x \in X$ satisfying

$$\min\{f_{ab}(x), f_{a\xi}(x)\} \le n + 1 \tag{6.22}$$

the following inequality holds:

$$|f_{ab}(x) - f_{a\xi}(x)| \le \epsilon/2. \tag{6.23}$$

Assume that $\xi \in U$. Inequality (6.21) implies that

$$\inf(f_{ab}) = \inf\{f_{ab}(x) :\ x \in X \text{ and } f_{ab}(x) \le n - 3\}. \tag{6.24}$$

Let $x \in X$ satisfy

$$f_{ab}(x) \le n - 3. \tag{6.25}$$

It follows from (6.25), the definition of U (see (6.22) and (6.23)) that

$$f_{a\xi}(x) \le f_{ab}(x) + \epsilon/2. \tag{6.26}$$

Since (6.25) implies (6.26) equality (6.24) implies that

$$\inf(f_{a\xi}) \le \inf\{f_{a\xi}(x) :\ x \in X \text{ and } f_{ab}(x) \le n - 3\} \le$$

$$\inf\{f_{ab}(x) + \epsilon/2 :\ x \in X \text{ and } f_{ab}(x) \le n - 3\} = \epsilon/2 + \inf(f_{ab}).$$

Hence

$$\inf(f_{a\xi}) \le \inf(f_{ab}) + \epsilon/2. \tag{6.27}$$

Let $x \in X$ satisfy

$$f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2. \tag{6.28}$$

It follows from (6.28), (6.27) and (6.21) that $f_{a\xi}(x) \le n-3$. By this inequality and the definition of U, (6.23) is true. Since (6.28) implies (6.23) we obtain

$$\inf(f_{ab}) \le \inf\{f_{ab}(x) : \; x \in X \text{ and } f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2\} \le$$

$$\inf\{f_{a\xi}(x) + \epsilon/2 : \; x \in X \text{ and } f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2\} =$$

$$\epsilon/2 + \inf\{f_{a\xi}(x) : \; x \in X \text{ and } f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2\} =$$

$$\epsilon/2 + \inf(f_{a\xi}).$$

Hence $\inf(f_{ab}) \le \epsilon/2 + \inf(f_{a\xi})$. Combined with (6.27) this implies that $|\inf(f_{ab}) - \inf(f_{a\xi})| \le \epsilon/2$ for all $\xi \in U$. This completes the proof of Proposition 6.4.

Corollary 6.5. *Assume that $\mathcal{B}$ is compact. Then (A4) implies (A1).*

We also use the following assumption.

(A5) Let $a \in \mathcal{A}$ and ϵ be a positive number. Then there exists a positive number δ such that for each nonempty finite set $F \subset \mathcal{B}$ there exist $\bar{a} \in \mathcal{A}$ and a nonempty compact subset A of the metric space (X, ρ) such that

$$d_{\mathcal{A}}(a, \bar{a}) < \epsilon, \tag{6.29}$$

$$\sup_{b \in \mathcal{B}} \inf(f_{\bar{a}b}) \le \sup_{b \in \mathcal{B}} \inf(f_{ab}) + 1, \tag{6.30}$$

$$f_{\bar{a}b}(x) \ge f_{ab}(x) \text{ for all } b \in \mathcal{B} \text{ and all } x \in X,$$

$$|f_{\bar{a}b_1}(x) - f_{\bar{a}b_2}(x)| \le |f_{ab_1}(x) - f_{ab_2}(x)| \tag{6.31}$$

for each $b_1, b_2 \in X$ and each $x \in X$, and the following property holds:

For each $b \in F$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta$ the inequality $\rho(x, A) \le \epsilon$ holds.

Proposition 6.6. *Assume that $\mathcal{B}$ is compact and (A4) and (A5) hold. Then (A2) holds.*

Proof: Corollary 6.5 implies that (A1) holds and

$$\sup_{b \in \mathcal{B}} \inf(f_{ab}) < \infty \text{ for all } a \in \mathcal{A}. \tag{6.32}$$

Let $a \in \mathcal{A}$ and ϵ_0 be a positive number. Let $\delta \in (0, 1)$ be as guaranteed by (A5) with $\epsilon = \epsilon_0/2$. Fix an integer

$$m > |\sup_{b \in \mathcal{B}} \inf(f_{ab})| + 4. \tag{6.33}$$

It follows from (A4) that for each $b \in \mathcal{B}$ there exists an open neighborhood $\mathcal{U}_b$ of b in $\mathcal{B}$ such that for each $\xi \in \mathcal{U}_b$ and each $x \in X$ satisfying

$$\min\{f_{a\xi}(x), f_{ab}(x)\} \le m + 4 \tag{6.34}$$

the inequality

$$|f_{a\xi}(x) - f_{ab}(x)| \le \delta/16 \tag{6.35}$$

holds. Since $\mathcal{B}$ is compact there exists a nonempty finite set $F \subset \mathcal{B}$ such that

$$\mathcal{B} = \cup\{\mathcal{U}_b : \ b \in F\}. \tag{6.36}$$

In view of the definition of δ and (A5) with $\epsilon = \epsilon_0/2$, there exist $\bar{a} \in \mathcal{A}$ and a nonempty compact set A of the metric space (X, ρ) such that

$$d_{\mathcal{A}}(a, \bar{a}) < \epsilon_0/2, \tag{6.37}$$

(6.30) and (6.31) hold and the following property holds:

(C4) For each $b \in F$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta$ the inequality $\rho(x, A) \le \epsilon_0/2$ is valid.

Since A is compact there exists a nonempty finite set $Q \subset A$ such that

$$\rho(x, Q) \le \epsilon_0/4 \text{ for all } x \in A. \tag{6.38}$$

Let $\xi \in \mathcal{B}$, $x \in X$ and

$$f_{\bar{a}\xi}(x) \le \inf(f_{\bar{a}\xi}) + \delta/16. \tag{6.39}$$

In order to complete the proof of the proposition it is sufficient to show that $\rho(x, Q) \le \epsilon_0$. In view of (6.38) we need only to show that $\rho(x, A) \le \epsilon_0/2$.

It follows from (6.36) that there exists $b \in F$ such that

$$\xi \in \mathcal{U}_b. \tag{6.40}$$

By (6.33) and (6.30),

$$\inf(f_{ab}), \ \inf(f_{a\xi}) < m - 4, \ \inf(f_{\bar{a}b}), \ \inf(f_{\bar{a}\xi}) < m. \tag{6.41}$$

Hence

$$\inf(f_{hg}) = \inf\{f_{hg}(z) : \ z \in X \text{ and } f_{hg}(z) \le m + 1\}, \tag{6.42}$$
$$(h, g) \in \{(a, b), \ (a, \xi), \ (\bar{a}, b), \ (\bar{a}, \xi)\}.$$

We will show that

$$|\inf(f_{\bar{a}b}) - \inf(f_{\bar{a}\xi})| \le \delta/16. \tag{6.43}$$

Let $z \in X$ and

$$\min\{f_{\bar{a}b}(z), f_{\bar{a}\xi}(z)\} \le m + 2. \tag{6.44}$$

Relations (6.44) and (6.30) imply that $\min\{f_{ab}(z), f_{a\xi}(z)\} \le m+2$. It follows from this inequality, (6.40) and the definition of $\mathcal{U}_b$ (see (6.34) and (6.35)) that $|f_{ab}(z) - f_{a\xi}(z)| \le \delta/16$. Together with (6.31) this inequality implies that

$$|f_{\bar{a}b}(z) - f_{\bar{a}\xi}(z)| \le |f_{ab}(z) - f_{a\xi}(z)| \le \delta/16.$$

Thus for each $z \in X$ satisfying (6.44), $|f_{\bar{a}b}(z) - f_{\bar{a}\xi}(z)| \le \delta/16$. Combining this inequality with (6.42) we see that (6.43) holds. By (6.30), (6.39) and (6.41), $f_{a\xi}(x) \le f_{\bar{a}\xi}(x) \le m+1$. It follows from this inequality, (6.40), the definition of $\mathcal{U}_b$ (see (6.34) and (6.35)) and (6.31) that

$$|f_{\bar{a}\xi}(x) - f_{\bar{a}b}(x)| \le |f_{a\xi}(x) - f_{ab}(x)| \le \delta/16.$$

Together with (6.39) and (6.43) this inequality implies that

$$f_{\bar{a}b}(x) \le f_{\bar{a}\xi}(x) + \delta/16 \le \delta/8 + \inf(f_{\bar{a}\xi}) \le \inf(f_{\bar{a}b}) + \delta/4.$$

These inequalities and the property (C4) imply that $\rho(x, A) \le \epsilon_0/2$. Proposition 6.6 is proved.

Theorem 6.1, Propositions 6.3 and 6.6 and Corollary 6.5 imply the following result which was obtained in [113].

Theorem 6.7. *Assume that $\mathcal{B}$ is compact and (A3), (A4) and (A5) hold. Then there exists a set $\mathcal{F}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{F}$ and each $b \in \mathcal{B}$ the minimization problem for f_{ab} on (X, ρ) is well-posed in the generalized sense.*

Remark 6.8. Note that Theorem 6.7 is also valid if $\mathcal{B} = \cup_{i=1}^{\infty} C_i$ and (A3), (A4) and (A5) hold with $\mathcal{B} = C_i$ for all natural numbers i.

6.3 Two generic existence results

Denote by $C_l(X)$ the set of all lower semicontinuous bounded from below functions $f : X \to R^1 \cup \{\infty\}$ which are not identically ∞. In this section we assume that $\mathcal{A}, \mathcal{B} \subset C_l(X)$,

$$f_{ab}(x) = a(x) + b(x),\ x \in X,\ a \in \mathcal{A},\ b \in \mathcal{B}$$

and that one of the following properties holds:

Any function $h \in \mathcal{A}$ is finite-valued; any function $h \in \mathcal{B}$ is finite-valued.

Fix $\theta \in X$. For the set $C_l(X)$ we consider the uniformity determined by the following base:

$$U_0(n) = \{(f, g) \in C_l(X) \times C_l(X) :\ |f(x) - g(x)| \le n^{-1}$$
$$\text{for all } x \in X \text{ such that } \rho(x, \theta) \le n \text{ or } \min\{f(x), g(x)\} \le n\},$$

where $n = 1, 2, \dots$. Clearly the space $C_l(X)$ with this uniformity is Hausdorff and has a countable base. Therefore this uniform space is metrizable (by a metric $d_0(\cdot,\cdot)$) [58].

The set $C_l(X)$ is also equipped with the uniformity determined by the following base:

$$U_1(n) = \{(f,g) \in C_l(X) \times C_l(X) : \ f(x) \le g(x) + n^{-1}$$

$$\text{and } g(x) \le f(x) + n^{-1} \text{ for each } x \in X \text{ and}$$

$$f(x) + g(y) \le f(y) + g(x) + n^{-1}\rho(x,y) \text{ for each } x, y \in X\},$$

where $n = 1, 2, \ldots$. It is easy to see that the space $C_l(X)$ with this uniformity is metrizable (by a metric d_1) and complete. For the space $\mathcal{A}$ we consider the strong and weak topologies induced by the metrics d_1 and d_0, respectively. The space $\mathcal{B}$ is equipped with the topology induced by the metric d_0.

We show that our main result in [106] is obtained as a realization of our variational principle (see Theorem 6.7). In this result the space $\mathcal{A}$ is either $C_l(X)$ or $\{f \in C_l(X) : \ f \text{ is finite-valued}\}$ or

$$\{f \in C_l(X) : \ f \text{ is finite-valued and continuous}\}$$

or

$$\{f \in C_l(X) : \ f \text{ is finite-valued and continuous and}$$

$$\sup\{|f(y) - f(x)|/\rho(x,y) : \ x, y \in X \text{ and } x \ne y\} < \infty\}.$$

Note that in all these cases the metric space $(\mathcal{A}, d_1)$ is complete.

Theorem 6.9. *Assume that $\mathcal{B}$ is a countable union of compact subsets of the metric space $(C_l(X), d_0)$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $g \in \mathcal{F}$ and each $h \in \mathcal{B}$ the problem minimize $g(x) + h(x)$ subject to $x \in X$ on (X, ρ) is well-posed in the generalized sense.*

Proof: We may assume without loss of generality that $\mathcal{B}$ is a compact subset of $(C_l(X), d_0)$. In view of Theorem 6.7 we need to show that (A3), (A4) and (A5) hold. Evidently, (A4) holds. We will show that (A3) is true.

Let $a \in \mathcal{A}$, $\epsilon \in (0,1)$ and let n be a natural number. By (A4) and Proposition 6.4 the function $b \to \inf(\xi + b)$, $b \in \mathcal{B}$ is continuous for all $\xi \in \mathcal{A}$. Since $\mathcal{B}$ is compact there exists a number

$$c_0 > |\inf(b)| + 1, \ b \in \mathcal{B}.$$

Fix an integer

$$m > n + c_0 + \epsilon^{-1}.$$

Let $b \in \mathcal{B}$, $\xi \in \mathcal{A}$, $(a, \xi) \in U_0(m)$, $x \in X$ and

$$\min\{(a+b)(x), (\xi + b)(x)\} \le n. \tag{6.45}$$

In view of (6.45) and the choice of c_0 and m,

$$\min\{a(x), \xi(x)\} \le \min\{(a+b)(x), (\xi+b)(x)\} - b(x) \le n + c_0 < m.$$

Since $(a, \xi) \in U_0(m)$, $|a(x) - \xi(x)| \le m^{-1} < \epsilon$. Hence (A3) holds.

We will show that (A5) holds. Let $a \in \mathcal{A}$ and let $\epsilon \in (0, 1)$. Fix a natural number n_0 such that the following condition holds:

$$\text{if } \xi \in \mathcal{A} \text{ and } (a, \xi) \in U_1(n_0), \text{ then } d_1(a, \xi) < \epsilon. \tag{6.46}$$

Choose numbers

$$\epsilon_0 \in (0, 8^{-1}\min\{\epsilon, n_0^{-1}\}), \ \delta \in (0, 8^{-1}\epsilon_0). \tag{6.47}$$

Let $\{b_1, \dots, b_q\} \subset \mathcal{B}$ where $q \ge 1$ is an integer. For each $i \in \{1, \dots, q\}$ choose $x_i \in X$ such that

$$(a + b_i)(x_i) \le \inf(a + b_i) + 4^{-1}\delta\epsilon_0. \tag{6.48}$$

Define $A = \{x_1, \dots, x_q\}$,

$$\bar{a}(y) = a(y) + 4^{-1}\epsilon_0 \min\{1, \rho(y, A)\}, \ y \in X. \tag{6.49}$$

It is easy to see that $\bar{a} \in \mathcal{A}$. It follows from (6.49), (6.47) and (6.46) that $(\bar{a}, a) \in U_1(n_0)$ and $d_1(\bar{a}, a) < \epsilon$. Evidently, (6.30) and (6.31) hold. Let $i \in \{1, \dots, q\}$, $x \in X$ and

$$(\bar{a} + b_i)(x) \le \inf(\bar{a} + b_i) + \delta\epsilon_0 < \infty. \tag{6.50}$$

By (6.50), (6.49) and (6.48),

$$(\bar{a} + b_i)(x) \le \inf(\bar{a} + b_i) + \delta\epsilon_0 \le (\bar{a} + b_i)(x_i) + \delta\epsilon_0 =$$

$$(a + b_i)(x_i) + \delta\epsilon_0 \le (a + b_i)(x) + 2\delta\epsilon_0.$$

Combined with (6.49) and (6.47) this implies that

$$b_i(x) + a(x) + 4^{-1}\epsilon_0 \min\{1, \rho(x, A)\} = (\bar{a} + b_i)(x) \le$$

$$(a + b_i)(x) + 2\delta\epsilon_0, \ \min\{1, \rho(x, A)\} \le 8\delta < \epsilon, \ \rho(x, A) < \epsilon.$$

Theorem 6.9 is proved.

Now we obtain an analog of Theorem 6.9 for a space $\mathcal{A}$ which is a set of convex functions defined on a closed convex subset of a Banach space.

Let X be a nonempty convex closed subset of a Banach space $(E, ||\cdot||)$, $\theta = 0 \in X$, $\rho(x, y) = ||x - y||$, $x, y \in X$, $c_0 \in R^1$ and let $c_1 > 0$.

Denote by $C_{con}(X)$ the set of all convex functions $f \in C_l(X)$ which satisfy

$$f(x) \ge c_1||x|| - c_0 \text{ for all } x \in X. \tag{6.51}$$

We consider the metric space $(\mathcal{A}, d_0)$ where $\mathcal{A}$ is either $C_{con}(X)$ or $\{f \in C_{con}(X) : \ f \text{ is finite-valued}\}$ or

$$\{f \in C_{con}(X) : \ f \text{ is finite-valued and continuous}\}.$$

The space $\mathcal{A}$ is equipped with the topology induced by the metric d_0.

Theorem 6.10. *Assume that $\mathcal{B}$ is a countable union of compact subsets of the metric space $(C_l(X),d_0)$. Then there exists a subset $\mathcal{F}$ of the complete metric space $(\mathcal{A}, d_0)$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{A}, d_0)$ such that for each $g \in \mathcal{F}$ and each $h \in \mathcal{B}$ the problem minimize $g(x) + h(x)$ subject to $x \in X$ on (X, ρ) is well-posed in the generalized sense.*

Proof: It is easy to see that the metric space $(\mathcal{A}, d_0)$ is complete and the topology induced by the metric d_0 in $\mathcal{A}$ is the topology of uniform convergence on bounded subsets of X. This topology is also induced by the uniformity with the following base:

$$U_{co}(n) = \{(f, g) \in \mathcal{A} \times \mathcal{A} : \ |f(x) - g(x)| \le n^{-1} \tag{6.52}$$

$$\text{for all } x \in X \text{ satisfying } ||x|| \le n\}$$

where $n = 1, 2, \ldots$. We may assume without loss of generality that $\mathcal{B}$ is a compact subset of $(C_l(X), d_0)$. In view of Theorem 6.7 we need to show that (A3), (A4) and (A5) hold. Evidently, (A4) holds. Analogously to the proof of Theorem 6.9 we can show that (A3) is valid.

We will show that (A5) holds. Let $a \in \mathcal{A}$ and $\epsilon \in (0, 1)$. By (A4) and Proposition 6.4, the functions $b \to \inf(a+b)$, $b \to \inf(b)$, $b \in \mathcal{B}$ are continuous. Since $\mathcal{B}$ is compact there exists a number $d_0 > 1$ such that

$$|\inf(a + b)|, \ |\inf(b)| \le d_0 \text{ for all } b \in \mathcal{B}. \tag{6.53}$$

Fix an integer $n_1 \ge 1$ such that

$$c_1 n_1 - c_0 > 2d_0 + 2. \tag{6.54}$$

Assume that

$$b \in \mathcal{B}, \ x \in X \text{ and } (a + b)(x) \le \inf(a + b) + 1. \tag{6.55}$$

Relations (6.53) and (6.55) imply that

$$a(x) \le \inf(a + b) + 1 - b(x) \le \inf(a + b) + 1 - \inf(b) \le 2d_0 + 1.$$

Combined with (6.54) and (6.51) these inequalities imply that $||x|| \le n_1$. Thus we have shown that

$$\text{if } (a + b)(x) \le \inf(a + b) + 1 \text{ with } b \in \mathcal{B}, \ x \in X, \text{ then } ||x|| \le n_1. \tag{6.56}$$

There exists a natural number $n_0 \ge 1 + n_1$ such that

$$\text{if } \xi \in \mathcal{A}, \ (a, \xi) \in U_{co}(n_0), \text{ then } d_0(a, \xi) < \epsilon. \tag{6.57}$$

Choose

$$\epsilon_0 \in (0, 8^{-1} \min\{\epsilon, n_0^{-2}\}), \ \delta \in (0, 8^{-1}\epsilon_0). \tag{6.58}$$

Let $\{b_1, \dots, b_q\} \subset \mathcal{B}$ where $q \geq 1$ is an integer. For each $i \in \{1, \dots, q\}$ choose $x_i \in X$ such that

$$(a + b_i)(x_i) \leq \inf(a + b_i) + 4^{-1}\delta\epsilon_0. \tag{6.59}$$

Let A be the convex hull of the set $\{x_1, \dots, x_q\} \cup \{0\}$ and define

$$\bar{a}(y) = a(y) + 4^{-1}\epsilon_0\rho(y, A),\ y \in X. \tag{6.60}$$

It is easy to see that $\bar{a} \in \mathcal{A}$. It follows from (6.60), (6.58) and the inclusion $0 \in A$ that

$$\sup\{|\bar{a}(y) - a(y)| :\ y \in X,\ ||y|| \leq n_0\} \leq 4^{-1}\epsilon_0 n_0 \leq 32^{-1}n_0^{-1}.$$

Hence $(\bar{a}, a) \in U_{co}(n_0)$ (see (6.52)) and in view of (6.57),

$$d_0(a, \bar{a}) < \epsilon.$$

It is not difficult to see that (6.61) is true. We will show that (6.60) holds.

Assume that (6.55) holds. It follows from (6.56) that $||x|| \leq n_1$ and by (6.60) and (6.58),

$$\bar{a}(x) + b(x) \leq a(x) + b(x) + 4^{-1}\epsilon_0||x|| \leq a(x) + b(x) + 4^{-1}\epsilon_0 n_1 \leq$$

$$a(x) + b(x) + 2^{-5}n_0^{-1}.$$

Thus we have shown that (6.55) implies that $\bar{a}(x) + b(x) \leq a(x) + b(x) + 1/2$. Thus (6.60) holds.

Let $i \in \{1, \dots, q\}$, $x \in X$ and

$$(\bar{a} + b_i)(x) \leq \inf(\bar{a} + b_i) + 4^{-1}\delta\epsilon_0.$$

Combined with (6.60) and (6.59) this inequality implies that

$$(\bar{a} + b_i)(x) \leq \inf(\bar{a} + b_i) + 4^{-1}\delta\epsilon_0 \leq (\bar{a} + b_i)(x_i) + 4^{-1}\delta\epsilon_0 =$$

$$(a + b_i)(x_i) + 4^{-1}\delta\epsilon_0 \leq \inf(a + b_i) + 2^{-1}\delta\epsilon_0 \leq (a + b_i)(x) + 2^{-1}\delta\epsilon_0,$$

$$b_i(x) + a(x) + 4^{-1}\epsilon_0\rho(x, A) = (\bar{a} + b_i)(x) \leq a(x) + b_i(x) + 2^{-1}\delta\epsilon_0$$

and

$$\rho(x, A) \leq 2\delta < \epsilon.$$

Therefore (A5) is valid. This completes the proof of Theorem 6.10.

It should be mentioned that the results of this section were obtained in [113].

6.4 A generic existence result in parametric optimization

Let $\mathcal{B}$ be a Hausdorff topological space. We assume that $\mathcal{B}$ is a countable union of its compact subsets.

Denote by $\mathcal{M}$ the set of all functions $a : \mathcal{B} \times X \to R^1 \cup \{\infty\}$ such that for each $b \in \mathcal{B}$ the following properties hold:

The function $x \to a(b, x)$, $x \in X$ belongs to $C_l(X)$;

for each natural number n and each positive number ϵ there exists a neighborhood U of b in $\mathcal{B}$ such that for each $\xi \in U$ and each $x \in X$ satisfying $\min\{a(\xi, x), a(b, x)\} \le n$ the inequality $|a(b, x) - a(\xi, x)| \le \epsilon$ holds.

For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$U_s(n) = \{(a_1, a_2) \in \mathcal{M} \times \mathcal{M} : \ |a_1(b, x) - a_2(b, x)| \le n^{-1} \quad (6.61)$$

$$\text{for all } b \in \mathcal{B} \text{ and all } x \in X\}$$

where $n \ge 1$ is an integer. The space $\mathcal{M}$ with this uniformity is metrizable (by a metric d_s) and complete.

Fix $\theta \in X$. For the set $\mathcal{M}$ we also consider the uniformity determined by the following base:

$$U_w(n) = \{(a_1, a_2) \in \mathcal{M} \times \mathcal{M} : \ |a_1(b, x) - a_2(b, x)| \le n^{-1} \quad (6.62)$$

$$\text{for each } b \in \mathcal{B} \text{ and each } x \in X \text{ such that } \rho(x, \theta) \le n$$

$$\text{or } \min\{a_1(b, x), a_2(b, x)\} \le n\},$$

where $n = 1, 2, \dots$. It is easy to see that the space $\mathcal{M}$ with this uniformity is metrizable (by a metric d_w). For the space $\mathcal{M}$ we consider the strong and weak topologies induced by the metrics d_s and d_w, respectively.

Denote by $\mathcal{M}_f$ the set of all finite-valued functions $a \in \mathcal{M}$, by $\mathcal{M}_s$ the set of all functions $a \in \mathcal{M}_f$ such that the function $x \to a(b, x)$, $x \in X$ is continuous for all $b \in \mathcal{B}$, and by $\mathcal{M}_c$ the set of all continuous functions $a \in \mathcal{M}_f$. Clearly $\mathcal{M}_f$, $\mathcal{M}_s$ and $\mathcal{M}_c$ are closed subsets of the metric space $(\mathcal{M}, d_s)$. We consider the subspaces $\mathcal{M}_f$, $\mathcal{M}_s$, $\mathcal{M}_c \subset \mathcal{M}$ equipped with the relative weak and strong topologies.

The following result was obtained in [113].

Theorem 6.11. *Let $\mathcal{A}$ be either $\mathcal{M}$ or $\mathcal{M}_f$ or $\mathcal{M}_s$ or $\mathcal{M}_c$. Then there exists a subset $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{F}$ and each $b \in \mathcal{B}$ the problem minimize $a(b, x)$ subject to $x \in X$ on (X, ρ) is well-posed in the generalized sense.*

Proof: We may assume without loss of generality that $\mathcal{B}$ is compact. For each $(a, b) \in \mathcal{A} \times \mathcal{B}$ set

$$f_{ab}(x) = a(b, x), \ x \in X.$$

Theorem 6.7 implies that we need to show that (A3), (A4) and (A5) hold. Clearly, (A3) and (A4) are valid. We will show that (A5) holds.

Let $a \in \mathcal{A}$ and $\epsilon \in (0,1)$. There exists a natural number n_0 such that

$$\text{if } \xi \in \mathcal{A},\ (a,\xi) \in U_s(n_0),\ \text{ then } d_s(a,\xi) < \epsilon. \tag{6.63}$$

Fix

$$\epsilon_0 \in (0, 8^{-1} \min\{\epsilon, n_0^{-1}\}),\ \delta \in (0, 8^{-1}\epsilon_0). \tag{6.64}$$

Let $\{b_1, \dots, b_q\} \subset \mathcal{B}$ where $q \geq 1$ is an integer. For each $i \in \{1, \dots, q\}$ choose $x_i \in X$ for which

$$a(b_i, x_i) \leq \inf\{a(b_i, x) :\ x \in X\} + 4^{-1}\delta\epsilon_0. \tag{6.65}$$

Define $A = \{x_1, \dots, x_q\}$ and

$$\bar{a}(b,y) = a(b,y) + 4^{-1}\epsilon_0 \min\{1, \rho(y, A)\},\ b \in \mathcal{B},\ y \in X. \tag{6.66}$$

It is easy to see that $\bar{a} \in \mathcal{A}$. In view of (6.66) and (6.64), $(a, \bar{a}) \in U_s(n_0)$. It follows from (6.63) that $d_s(a, \bar{a}) < \epsilon$. Clearly (6.30) and (6.31) hold.

Let $i \in \{1, \dots, q\}$, $x \in X$ and

$$\bar{a}(b_i, x) \leq \inf\{\bar{a}(b_i, z) :\ z \in X\} + 4^{-1}\delta\epsilon_0. \tag{6.67}$$

Relations (6.66), (6.67) and (6.65) imply that

$$a(b_i, x) + 4^{-1}\epsilon_0 \min\{1, \rho(x, A)\} = \bar{a}(b_i, x) \leq \bar{a}(b_i, x_i) + 4^{-1}\delta\epsilon_0 =$$

$$a(b_i, x_i) + 4^{-1}\delta\epsilon_0 \leq a(b_i, x) + 2^{-1}\delta\epsilon_0$$

and

$$\min\{1, \rho(x, A)\} \leq 2\delta < \epsilon.$$

Hence $\rho(x, A) < \epsilon$. Theorem 6.11 is proved.

6.5 Parametric optimization and porosity

Let (X, ρ) be a complete metric space. We continue to study the parametric family of the minimization problems

$$\text{minimize } f(b,x) \text{ subject to } x \in X \tag{P1}$$

with a parameter $b \in \mathcal{B}$ and $f(\cdot,\cdot) \in \mathcal{M}$. Here $\mathcal{B}$ is a Hausdorff topological space which is a countable union of its compact subsets and $\mathcal{M}$ is a complete metric space of functions on $\mathcal{B} \times X$. We also study a special case of the parametric family of the minimization problems (P1). Namely, we consider the minimization problem

$$\text{minimize } g(x) + h(x) \text{ subject to } x \in X \tag{P2}$$

with $g \in \mathcal{L}$ and $h \in M$ where M is a Hausdorff compact space of functions on X and $\mathcal{L}$ is a complete metric space of functions on X.

We study the set of all functions $f(\cdot,\cdot) \in \mathcal{M}$ for which the corresponding parametric family of the minimization problems (P1) has solutions for all parameters $b \in \mathcal{B}$. We show that the complement of this set is not only of the first category but also a σ-porous set in the space $\mathcal{M}$.

We also consider the set of all functions $g \in \mathcal{L}$ for which the minimization problems (P2) have solutions for all functions $h \in M$. We prove that the complement of this set is also a σ-porous set in $\mathcal{L}$.

These two results and their extensions are obtained by using a certain abstract class of parametric minimization problems.

In this chapter we use the notion of porosity discussed in Chapter 5. A set Y equipped with two metrics d_1 and d_2 satisfying $d_1(x,y) \le d_2(x,y)$ for all $x, y \in Y$ will be denoted by (Y, d_1, d_2).

6.6 A variational principle and porosity

We consider a complete metric space (X, ρ). Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty sets. We assume that the set $\mathcal{A}$ is equipped with two complete metrics d_w, d_s : $\mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that

$$d_w(a_1, a_2) \le d_s(a_1, a_2) \text{ for all } a_1, a_2 \in \mathcal{A}. \tag{6.68}$$

We assume that with every $(a, b) \in \mathcal{A} \times \mathcal{B}$ a lower semicontinuous bounded from below function f_{ab} on X is associated with values in $R^1 \cup \{\infty\}$ which is not identically ∞. For each function $g : X \to [-\infty, \infty]$ we set

$$\inf(g) = \inf\{g(x) :\ x \in X\}. \tag{6.69}$$

For each $x \in X$ and each $E \subset X$ set

$$\rho(x, A) = \inf\{\rho(x, y) :\ y \in A\}. \tag{6.70}$$

We use the convention that $\infty - \infty = 0$ and $\infty/\infty = 1$.

Let $(a, b) \in \mathcal{A} \times \mathcal{B}$. Recall that the minimization problem for f_{ab} on (X, ρ) is well-posed in the generalized sense [40] if $\inf(f_{ab})$ is finite, the set

$$\{x \in X :\ f_{ab}(x) = \inf(f_{ab})\}$$

is nonempty and compact and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying

$$\lim_{i \to \infty} f_{ab}(x_i) = \inf(f_{ab})$$

has a convergent subsequence.

In our study we use the following basic hypotheses about the functions.

(H1) For each $a \in \mathcal{A}$, $\sup\{\inf(f_{ab}) : b \in \mathcal{B}\} < \infty$.

(H2) For each positive number ϵ and each natural number m there exist positive numbers δ and r_0 such that the following property holds:

For each $a \in \mathcal{A}$ satisfying $\sup\{\inf(f_{ab}) : b \in \mathcal{B}\} \le m$ and each $r \in (0, r_0]$ there exist $\bar{a} \in \mathcal{A}$ and a nonempty finite set $A \subset X$ such that

$$d_s(a, \bar{a}) \le r, \ \sup_{b \in \mathcal{B}} \inf(f_{\bar{a}b}) \le m + 1$$

and for each $b \in \mathcal{B}$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta r$ the inequality $\rho(x, A) \le \epsilon$ holds.

(H3) For each natural number m there exist $\alpha \in (0, 1)$ and a positive number r_0 such that for each $r \in (0, r_0]$, each $b \in \mathcal{B}$, each $a_1, a_2 \in \mathcal{A}$ satisfying $d_w(a_1, a_2) \le \alpha r$ and each $x \in X$ satisfying $\min\{f_{a_1 b}(x), f_{a_2 b}(x)\} \le m$ the inequality $|f_{a_1 b}(x) - f_{a_2 b}(x)| \le r$ holds.

The following result was obtained in [127].

Theorem 6.12. *(variational principle) Assume that (H1)–(H3) hold. Then there exists a set* $\mathcal{F} \subset \mathcal{A}$ *such that the complement* $\mathcal{A} \setminus \mathcal{F}$ *is* σ*-porous in* $\mathcal{A}$ *with respect to the pair of metrics* (d_w, d_s) *and that for each* $a \in \mathcal{F}$ *and each* $b \in \mathcal{B}$ *the minimization problem for* f_{ab} *on* (X, ρ) *is well-posed in the generalized sense.*

Proof: For each natural number n denote by $\mathcal{A}_n$ the set of all $a \in \mathcal{A}$ which have the following property:

(C1) There exist a positive number δ and a nonempty finite set $A \subset X$ such that for each $b \in \mathcal{B}$ and each $x \in X$ which satisfies $f_{ab}(x) \le \inf(f_{ab}) + \delta$ the inequality

$$\rho(x, A) \le 1/n \tag{6.71}$$

holds.

Define $\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{A}_n$.

Let n be a natural number. We will show that the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). In view of (H1) it is sufficient to show that for each natural number m the set

$$\Omega_{nm} := \{a \in \mathcal{A} \setminus \mathcal{A}_n : \sup\{\inf(f_{ab}) : b \in \mathcal{B}\} \le m\} \tag{6.72}$$

is porous in $\mathcal{A}$ with respect to (d_w, d_s).

Let m be a natural number. It follows from (H3) that there exist

$$\alpha_1 \in (0, 1), \ r_1 \in (0, 1/2) \tag{6.73}$$

such that for each $r \in (0, r_1]$, each $b \in \mathcal{B}$, each $a_1, a_2 \in \mathcal{A}$ satisfying $d_w(a_1, a_2) \le \alpha_1 r$ and each $x \in X$ satisfying

$$\min\{f_{a_1 b}(x), f_{a_2 b}(x)\} \le m + 4 \tag{6.74}$$

the inequality $|f_{a_1 b}(x) - f_{a_2 b}(x)\} \le r$ is valid.

It follows from (H2) that there exist $r_2, \alpha_2 \in (0,1)$ such that the following property holds:

(C2) For each $a \in \mathcal{A}$ satisfying $\sup\{\inf(f_{ab}) : \ b \in \mathcal{B}\} \le m+2$ and each $r \in (0, r_2]$ there exist $\bar{a} \in \mathcal{A}$ and a nonempty finite set $A \subset X$ such that

$$d_s(a, \bar{a}) \le r, \ \sup_{b \in \mathcal{B}} \inf(f_{\bar{a}b}) \le m+3$$

and that for each $b \in \mathcal{B}$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + 4r\alpha_2$ the inequality $\rho(x, A) \le 1/n$ holds.

Fix

$$\bar{\alpha} \in (0, \alpha_1\alpha_2/16), \ \bar{r} \in (0, r_1 r_2 \bar{\alpha}). \tag{6.75}$$

Let $a \in \mathcal{A}$ and $r \in (0, \bar{r}]$. Define

$$E_0 = \{\xi \in \mathcal{A} : \ d_s(\xi, a) \le r/4\}. \tag{6.76}$$

There are two cases:

$$E_0 \cap \{\xi \in \mathcal{A} : \ \sup_{b \in \mathcal{B}} \inf(f_{\xi b}) \le m+2\} = \emptyset; \tag{6.77}$$

$$E_0 \cap \{\xi \in \mathcal{A} : \ \sup_{b \in \mathcal{B}} \inf(f_{\xi b}) \le m+2\} \neq \emptyset. \tag{6.78}$$

Assume that (6.77) holds. We show that for each $\xi \in \mathcal{A}$ satisfying $d_w(a, \xi) \le \bar{r}$ the inequality $\sup_{b \in \mathcal{B}} \inf(f_{\xi b}) > m$ holds.

Let us assume the contrary. Then there exist $\xi \in \mathcal{A}$ such that

$$d_w(a, \xi) \le \bar{r} \text{ and } \sup_{b \in \mathcal{B}} \inf(f_{\xi b}) \le m. \tag{6.79}$$

Let $b \in \mathcal{B}$. It follows from (6.79) that there exists $y \in X$ such that

$$f_{\xi b}(y) \le \inf(f_{\xi b}) + 1/2 \le m + 1/2. \tag{6.80}$$

By the definition of α_1, r_1 (see (6.73), (6.74)), (6.79), (6.80) and (6.75), $|f_{ab}(y) - f_{\xi b}(y)| \le \bar{r}\alpha_1^{-1} \le 1/4$. In view of this inequality and (6.80),

$$\inf(f_{ab}) \le f_{ab}(y) \le f_{\xi b}(y) + 1/4 \le m+1.$$

Thus $\inf(f_{ab}) \le m+1$ for all $b \in \mathcal{B}$, a contradiction (see (6.76), (6.77)). Therefore

$$\{\xi \in \mathcal{A} : \ d_w(a, \xi) \le \bar{r}\} \cap \{\xi \in \mathcal{A} : \ \sup_{b \in \mathcal{B}} \inf(f_{\xi b}) \le m\} = \emptyset.$$

It follows from (6.72) that

$$\{\xi \in \mathcal{A} : \ d_w(a, \xi) \le \bar{r}\} \cap \Omega_{nm} = \emptyset. \tag{6.81}$$

Thus we have shown that (6.77) implies (6.81).

Assume now that (6.78) is valid. Then there exists $a_1 \in \mathcal{A}$ such that

$$d_s(a, a_1) \le r/4 \text{ and } \sup_{b \in \mathcal{B}} \inf(f_{a_1 b}) \le m + 2. \tag{6.82}$$

It follows from the definition of α_2, r_2, the property (C2), (6.82) and (6.75) that there exist $\bar{a} \in \mathcal{A}$ and a nonempty finite set $A \subset X$ such that

$$d_s(a_1, \bar{a}) \le r/4, \ \sup_{b \in \mathcal{B}} \inf(f_{\bar{a}b}) \le m + 3 \tag{6.83}$$

and the following property holds:

(C3) For each $b \in \mathcal{B}$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + r\alpha_2$ the inequality $\rho(x, A) \le 1/n$ holds.

By (6.82) and (6.83),

$$d_s(a, \bar{a}) \le r/2. \tag{6.84}$$

Assume that $\xi \in \mathcal{A}$ satisfies

$$d_w(\bar{a}, \xi) \le \bar{\alpha} r. \tag{6.85}$$

We show that for all $b \in \mathcal{B}$ the inequality $\inf(f_{\xi b}) \le \inf(f_{\bar{a}b}) + \alpha_2 r/16$ holds.

Let $b \in \mathcal{B}$. Relation (6.83) implies that

$$\inf(f_{\bar{a}b}) = \inf\{f_{\bar{a}b}(x) : \ x \in X \text{ and } f_{\bar{a}b}(x) \le m + 7/2\}. \tag{6.86}$$

Let $x \in X$ satisfy

$$f_{\bar{a}b}(x) \le m + 7/2. \tag{6.87}$$

By (6.85), (6.87), (6.75) and the definition of α_1, r_1 (see (6.73), (6.74)),

$$|f_{\bar{a}b}(x) - f_{\xi b}(x)| \le \bar{\alpha} r \alpha_1^{-1} \le \alpha_2 r/16.$$

Since this inequality holds for any $x \in X$ satisfying (6.87), it follows from (6.86) that

$$\inf(f_{\xi b}) \le \inf\{f_{\xi b}(x) : \ x \in X \text{ and } f_{\bar{a}b}(x) \le m + 7/2\} \le$$

$$\inf\{f_{\bar{a}b}(x) + \alpha_2 r/16 : \ x \in X \text{ and } f_{\bar{a}b}(x) \le m + 7/2\} =$$

$$\alpha_2 r/16 + \inf(f_{\bar{a}b}).$$

Thus for all $b \in \mathcal{B}$,

$$\inf(f_{\xi b}) \le \inf(f_{\bar{a}b}) + \alpha_2 r/16. \tag{6.88}$$

Assume now that $b \in \mathcal{B}$, $x \in X$ and

$$f_{\xi b}(x) \le \inf(f_{\xi b}) + \alpha_2 r/16. \tag{6.89}$$

In view of (6.88) and (6.89),

$$f_{\xi b}(x) \le \inf(f_{\bar{a}b}) + \alpha_2 r/8. \tag{6.90}$$

By (6.90) and (6.83), $f_{\xi b}(x) \le m + 7/2$. By this inequality, (6.75), (6.85) and the definition of α_1, r_1 (see (6.73), (6.74)),

$$|f_{\bar{a}b}(x) - f_{\xi b}(x)| \le \bar{\alpha} r \alpha_1^{-1} \le r\alpha_2/16.$$

Together with (6.90) this inequality implies that

$$f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \alpha_2 r/4.$$

This relation and the property (C3) imply that $\rho(x, A) \le 1/n$. Since this inequality is valid for any $b \in \mathcal{B}$ and any $x \in X$ satisfying (6.89), we conclude that $\xi \in \mathcal{A}_n$. Thus we have shown that

$$\{\xi \in \mathcal{A} :\ d_w(\bar{a}, \xi) \le \bar{\alpha} r\} \subset \mathcal{A}_n \subset \mathcal{A} \setminus \Omega_{nm}.$$

Together with (6.81) and (6.84) this relation implies that in both cases

$$\{\xi \in \mathcal{A} :\ d_w(\bar{a}, \xi) \le \bar{\alpha} r\} \cap \Omega_{nm} = \emptyset$$

with $\bar{a} \in \mathcal{A}$ satisfying (6.84). (Note that if (6.77) holds, then $\bar{a} = a$.) Therefore the set Ω_{nm} is porous in $\mathcal{A}$ with respect to (d_w, d_s). This implies that $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) for all integers $n \ge 1$. Since $\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{A}_n$ we conclude that the set $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s).

Now assume that $a \in \mathcal{F}$ and $b \in \mathcal{B}$. We show that the minimization problem for f_{ab} on (X, ρ) is well-posed in the generalized sense.

Assume that $\{z_i\}_{i=1}^{\infty} \subset X$ satisfies

$$\lim_{i \to \infty} f_{ab}(z_i) = \inf(f_{ab}). \tag{6.91}$$

We will show that the sequence $\{z_i\}_{i=1}^{\infty}$ has a convergent subsequence.

It follows from the definition of the sets $\mathcal{A}_n$, $n = 1, 2, \dots$ (see the property (C1)) that for each natural number n there exist a number $\delta_n > 0$ and a nonempty finite set $A_n \subset X$ such that the following property holds:

(C4) If $x \in X$ satisfies $f_{ab}(x) \le \inf(f_{ab}) + \delta_n$, then $\rho(x, A_n) \le 1/n$.

Property (C4) and (6.91) imply that for any natural number n the inequality $\rho(z_j, A_n) \le 1/n$ is valid for all sufficiently large natural numbers j. This implies that for each natural number p there exists a subsequence $\{z_{i_k}^{(p)}\}_{k=1}^{\infty}$ of the sequence $\{z_i\}_{i=1}^{\infty}$ with the following properties:

For each natural number p the sequence $\{z_{i_k}^{(p+1)}\}_{k=1}^{\infty}$ is a subsequence of $\{z_{i_k}^{(p)}\}_{k=1}^{\infty}$;

For each natural number p and each pair of natural numbers j, s,

$$\rho(z_{i_j}^{(p)}, z_{i_s}^{(p)}) \le 2/p.$$

These properties imply that there exists a subsequence $\{z_{i_k}^*\}_{k=1}^\infty$ of the sequence $\{z_i\}_{i=1}^\infty$ which is a Cauchy sequence. There exists $x_* = \lim_{k\to\infty} z_{i_k}^*$. It follows from (6.91) and the lower semicontinuity of f_{ab} that

$$f_{ab}(x_*) = \inf(f_{ab}). \tag{6.92}$$

Therefore we have shown that for each sequence $\{z_i\}_{i=1}^\infty \subset X$ satisfying (6.91) there exists a subsequence $\{z_{i_k}^*\}_{k=1}^\infty$ which converges to $x_* \in X$ satisfying (6.92). Theorem 6.12 is proved.

6.7 Concretization of the variational principle

We use the notations and definitions introduced in Section 6.6. We assume that $\mathcal{B}$ is a Hausdorff topological space. In our study we use the following assumptions.

(H4) For each $(a, b) \in \mathcal{A} \times \mathcal{B}$, each natural number n and each positive number ϵ there exists a neighborhood $\mathcal{U}$ of b in $\mathcal{B}$ such that for each $\xi \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{f_{a\xi}(x), f_{ab}(x)\} \le n$ the inequality $|f_{ab}(x) - f_{a\xi}(x)| \le \epsilon$ holds.

(H5) For each positive number ϵ and each natural number m there exist positive numbers δ and r_0 such that the following property holds:

For each $a \in \mathcal{A}$ satisfying $\sup\{\inf(f_{ab}) :\ b \in \mathcal{B}\} \le m$, each $r \in (0, r_0]$ and each nonempty finite set $Q \subset \mathcal{B}$ there exist $\bar{a} \in \mathcal{A}$ and a nonempty compact set $A \subset X$ such that

$$d_s(a, \bar{a}) \le r,\ \sup_{b\in\mathcal{B}} \inf(f_{\bar{a}b}) \le m + 1,$$

$$f_{\bar{a}b}(x) \ge f_{ab}(x) \text{ for each } (b, x) \in \mathcal{B} \times X$$

and

$$|f_{\bar{a}b_1}(x) - f_{\bar{a}b_2}(x)| \le |f_{ab_1}(x) - f_{ab_2}(x)| \text{ for all } b_1, b_2 \in \mathcal{B} \text{ and all } x \in X,$$

and that for each $b \in Q$ and each $x \in X$ satisfying $f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta r$ the inequality $\rho(x, A) \le \epsilon$ holds.

Lemma 6.13. *Assume that (H4) holds and $a \in \mathcal{A}$. Then the function $b \to \inf(f_{ab})$, $b \in \mathcal{B}$ is continuous.*

Proof: Let $b \in \mathcal{B}$ and $\epsilon \in (0, 1)$. Fix an integer

$$n > |\inf(f_{ab})| + 4.$$

It follows from (H4) that there exists a neighborhood $\mathcal{U}$ of b in $\mathcal{B}$ such that for each $\xi \in \mathcal{U}$ and each $x \in X$ satisfying

$$\min\{f_{ab}(x), f_{a\xi}(x)\} \le n + 1$$

the inequality

$$|f_{ab}(x) - f_{a\xi}(x)| \le \epsilon/2 \tag{6.93}$$

holds.

Assume that $\xi \in \mathcal{U}$. We show that $\inf(f_{a\xi}) \le \inf(f_{ab}) + \epsilon/2$. It follows from the choice of n that

$$\inf(f_{ab}) = \inf\{f_{ab}(x) : \ x \in X \text{ and } f_{ab}(x) \le n-3\}. \tag{6.94}$$

Let $x \in X$ and

$$f_{ab}(x) \le n-3. \tag{6.95}$$

It follows from (6.95) and the definition of $\mathcal{U}$ (see (6.93)) that

$$|f_{ab}(x) - f_{a\xi}(x)| \le \epsilon/2 \text{ and } f_{a\xi}(x) \le f_{ab}(x) + \epsilon/2. \tag{6.96}$$

Since (6.95) implies (6.96) equality (6.94) implies that

$$\inf(f_{a\xi}) \le \inf\{f_{a\xi}(x) : \ x \in X \text{ and } f_{ab}(x) \le n-3\} \le$$

$$\inf\{f_{ab}(x) + \epsilon/2 : \ x \in X \text{ and } f_{ab}(x) \le n-3\} =$$

$$\epsilon/2 + \inf\{f_{ab}(x) : \ x \in X \text{ and } f_{ab}(x) \le n-3\} = \epsilon/2 + \inf(f_{ab}).$$

Hence

$$\inf(f_{a\xi}) \le \inf(f_{ab}) + \epsilon/2. \tag{6.97}$$

Now we show that $\inf(f_{ab}) \le \inf(f_{a\xi}) + \epsilon/2$. Let $x \in X$ and

$$f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2. \tag{6.98}$$

In view of (6.98), (6.97) and the choice of n,

$$f_{a\xi}(x) \le \inf(f_{ab}) + \epsilon \le n-3.$$

By these inequalities and the definition of $\mathcal{U}$ (see (6.93)), (6.93) holds and

$$f_{ab}(x) \le f_{a\xi}(x) + \epsilon/2. \tag{6.99}$$

Since (6.98) implies (6.99) we obtain that

$$\inf(f_{ab}) \le \inf\{f_{ab}(x) : \ x \in X \text{ and } f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2\} \le$$

$$\inf\{f_{a\xi}(x) + \epsilon/2 : \ x \in X \text{ and } f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2\} =$$

$$\epsilon/2 + \inf\{f_{a\xi}(x) : \ x \in X \text{ and } f_{a\xi}(x) \le \inf(f_{a\xi}) + \epsilon/2\} = \epsilon/2 + \inf(f_{a\xi}).$$

Therefore $\inf(f_{ab}) \le \inf(f_{a\xi}) + \epsilon/2$. Together with (6.97) this inequality implies that $|\inf(f_{ab}) - \inf(f_{a\xi})| \le \epsilon/2$ for all $\xi \in \mathcal{U}$. Lemma 6.13 is proved.

Corollary 6.14. *Assume that $\mathcal{B}$ is compact and (H4) is true. Then (H1) holds.*

Lemma 6.15. *Assume that $\mathcal{B}$ is compact and (H4) and (H5) are true. Then (H2) holds.*

Proof: Let ϵ_0 be a positive number and let m be a natural number. Let $\delta \in (0,1)$ and $r_0 \in (0,1)$ be as guaranteed by (H5) with $\epsilon = \epsilon_0/2$.

Let $a \in \mathcal{A}$ and

$$r \in (0, r_0], \ \sup_{b\in\mathcal{B}} \inf(f_{ab}) \le m. \tag{6.100}$$

It follows from (H4) that for each $b \in \mathcal{B}$ there exists an open neighborhood $\mathcal{U}_b$ of b in $\mathcal{B}$ such that for each $\xi \in \mathcal{U}_b$ and each $x \in X$ satisfying

$$\min\{f_{a\xi}(x), f_{ab}(x)\} \le m + 4 \tag{6.101}$$

the inequality

$$|f_{ab}(x) - f_{a\xi}(x)| \le \delta r/16 \tag{6.102}$$

holds. Since $\mathcal{B}$ is compact there exists a finite nonempty set $Q \subset \mathcal{B}$ such that

$$\cup\{\mathcal{U}_b : \ b \in Q\} = \mathcal{B}. \tag{6.103}$$

It follows from (H5) and the definition of δ and r_0 that there exist $\bar{a} \in \mathcal{A}$ and a nonempty compact set $A_0 \subset X$ such that

$$d_s(a, \bar{a}) \le r, \ \sup_{b\in\mathcal{B}} \inf(f_{\bar{a}b}) \le m + 1, \tag{6.104}$$

$$f_{\bar{a}b}(x) \ge f_{ab}(x) \text{ for all } b \in \mathcal{B} \text{ and all } x \in X, \tag{6.105}$$

$$|f_{\bar{a}h_1}(x) - f_{\bar{a}h_2}(x)| \le |f_{ah_1}(x) - f_{ah_2}(x)| \text{ for all } h_1, h_2 \in \mathcal{B} \text{ and all } x \in X$$

and that for each $b \in Q$ and each $x \in X$ satisfying

$$f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta r \tag{6.106}$$

the following inequality holds:

$$\rho(x, A_0) \le \epsilon_0/2. \tag{6.107}$$

There exists a nonempty finite set $A \subset A_0$ such that

$$\rho(x, A) \le \epsilon_0/4 \text{ for all } x \in A_0. \tag{6.108}$$

Let $b \in \mathcal{B}$. In view of (6.103) there exists $h \in Q$ such that

$$b \in \mathcal{U}_h.$$

We show that $|\inf(f_{\bar{a}b}) - \inf(f_{\bar{a}h})| \le \delta r/16$. Relations (6.100) and (6.104) imply that

$$\inf(f_{\bar{a}b}), \inf(f_{\bar{a}h}) \le m + 1, \ \inf(f_{ab}), \inf(f_{ah}) \le m. \tag{6.109}$$

Then

$$\inf(f_{\bar{a}\xi}) = \inf\{f_{\bar{a}\xi}(z) : \ z \in X \text{ and } f_{\bar{a}\xi}(z) \le m+2\}, \ \xi \in \{b, h\}, \tag{6.110}$$

$$\inf(f_{a\xi}) = \inf\{f_{a\xi}(z) : \ z \in X \text{ and } f_{a\xi}(z) \le m+1\}, \ \xi \in \{b, h\}.$$

Let $z \in X$ and

$$\min\{f_{\bar{a}b}(z), f_{\bar{a}h}(z)\} \le m+2. \tag{6.111}$$

By (6.105), $\min\{f_{ab}(z), f_{ah}(z)\} \le m+2$. It follows from (6.105), the inclusion $b \in \mathcal{U}_h$, (6.108) and the definition of $\mathcal{U}_h$ (see (6.101), (6.102)) that

$$|f_{\bar{a}b}(z) - f_{\bar{a}h}(z)| \le |f_{ab}(z) - f_{ah}(z)| \le \delta r/16.$$

Thus we have shown that the following property holds:

(C5) If $z \in X$ satisfies (6.111), then $|f_{\bar{a}b}(z) - f_{\bar{a}h}(z)| \le \delta r/16$.

By property (C5) and (6.110),

$$|\inf(f_{\bar{a}h}) - \inf(f_{\bar{a}b})| \le \delta r/16. \tag{6.112}$$

Now assume that $x \in X$ and that

$$f_{\bar{a}b}(x) \le \inf(f_{\bar{a}b}) + \delta r/16. \tag{6.113}$$

Relations (6.113) and (6.109) imply that $f_{\bar{a}b}(x) \le m+2$. Together with the property (C5), (6.113) and (6.112) this inequality implies that

$$f_{\bar{a}h}(x) \le f_{\bar{a}b}(x) + \delta r/16 \le \inf(f_{\bar{a}b}) + \delta r/8 \le \inf(f_{\bar{a}h}) + \delta r/4.$$

Thus $f_{\bar{a}h}(x) \le \inf(f_{\bar{a}h}) + \delta r/4$. In view of this inequality, the relation $h \in Q$ and the definition of $\bar{a}$, A_0 (see (6.106), (6.107)), $\rho(x, A_0) \le \epsilon_0/2$. Together with (6.108) this inequality implies that $\rho(x, A) < \epsilon_0$. Thus we have shown that the following property holds:

For each $b \in \mathcal{B}$ and each $x \in X$ satisfying (6.113) the inequality $\rho(x, A) < \epsilon_0$ is valid.

Together with (6.104) this property implies (H2). This completes the proof of Lemma 6.15.

Theorem 6.12, Lemma 6.15 and Corollary 6.14 imply the following result which was obtained in [127].

Theorem 6.16. *Assume that $\mathcal{B}$ is compact and (H3) ,(H4) and (H5) hold. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to the pair (d_w, d_s) and that for each $a \in \mathcal{F}$ and each $b \in \mathcal{B}$ the minimization problem for f_{ab} on (X, ρ) is well-posed in the generalized sense.*

Remark 6.17. Note that Theorem 6.16 is also valid if $\mathcal{B} = \cup_{i=1}^{\infty} C_i$, the set C_i is compact and (H3), (H4) and (H5) hold with $\mathcal{B} = C_i$ for all natural numbers i.

6.8 Existence results for the problem (P2)

Denote by $C_l(X)$ the set of all lower semicontinuous bounded from below functions $f : X \to R^1 \cup \{\infty\}$ which are not identically ∞. We assume that $\mathcal{A}, \mathcal{B} \subset C_l(X)$ and for any $a \in \mathcal{A}$ and any $b \in \mathcal{B}$,

$$f_{ab}(x) = a(x) + b(x),\ x \in X.$$

Fix $\theta \in X$. For the set $C_l(X)$ we consider the uniformity determined by the following base:

$$U_0(n) = \{(f,g) \in C_l(X) \times C_l(X) :\ |f(x) - g(x)| \le 1/n \text{ for all } x \in X \tag{6.114}$$

$$\text{such that } \rho(x,\theta) \le n \text{ or } \min\{f(x), g(x)\} \le n\},$$

where $n = 1, 2, \ldots$. It is easy to see that the space $C_l(X)$ with this uniformity is metrizable (by a metric d_0). We always assume that $\mathcal{B}$ is equipped with the metric d_0 and with the topology induced by d_0.

For each $f, g \in C_l(X)$ define

$$\tilde{d}_1(f,g) = \sup\{|f(x) - g(x)| :\ x \in X\},$$

$$d_1(f,g) = \tilde{d}_1(f,g)(1 + \tilde{d}_1(f,g))^{-1}.$$

Clearly, the metric space $(C_l(X), d_1)$ is complete.

Denote by $C_v(X)$ the set of all finite-valued functions $g \in C_l(X)$ and by $C(X)$ the set of all finite-valued continuous functions $g \in C_l(X)$.

For any function $f : X \to R^1$ set

$$\mathrm{Lip}(f) = \sup\{|f(x) - f(y)|\rho(x,y)^{-1} :\ x, y \in X,\ x \ne y\}.$$

Denote by $C_L(X)$ the set of all continuous functions $f : X \to R^1$ such that $\mathrm{Lip}(f) < \infty$. For each $f, g \in C_v(X)$ define

$$d_2(f,g) = d_1(f,g) + \mathrm{Lip}(f-g)(\mathrm{Lip}(f-g) + 1)^{-1}.$$

Clearly, the metric space $(C_v(X), d_2)$ is complete, $C(X)$ and $C_L(X)$ are closed subsets of the metric space $(C_v(X), d_2)$ and $C_v(X)$ and $C(X)$ are closed subsets of the metric space $(C_l(X), d_1)$.

Our goal in this section is to prove porosity results for the space $C_l(X)$ and its certain subspaces which were obtained in [127].

Theorem 6.18. *Let $\mathcal{B}$ be a countable union of compact subsets of the metric space $(C_l(X), d_0)$ and let a space $(\mathcal{A}, d_w, d_s)$ be either $(C_l(X), d_1, d_1)$ or $(C_v(X), d_1, d_2)$ or $(C(X), d_1, d_2)$ or $(C_L(X), d_2, d_2)$. Assume that one of the following properties holds:*

1. Each $g \in \mathcal{B}$ is finite-valued; 2. each $g \in \mathcal{A}$ is finite-valued.

Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $g \in \mathcal{F}$ and each $h \in \mathcal{B}$ the problem minimize $g(x) + h(x)$ subject to $x \in X$ on (X, ρ) is well-posed in the generalized sense.

Proof: We may assume without loss of generality that $\mathcal{B}$ is a compact subset of $(C_l(X), d_0)$. In view of Theorem 6.16 we need to show that (H3), (H4) and (H5) hold. Evidently, (H3) and (H4) hold. We show that (H5) holds. Let ϵ be a positive number. Fix

$$\delta \in (0, \min\{1/16, \epsilon/8\}),\ r_0 \in (0, 1]. \tag{6.115}$$

Let $g \in \mathcal{A}$, $r \in (0, r_0]$ and $\{h_1, \ldots, h_q\} \subset \mathcal{B}$ where $q \geq 1$ is an integer. For each $i \in \{1, \ldots, q\}$ choose $x_i \in X$ such that

$$g(x_i) + h_i(x_i) \leq \inf(g + h_i) + \delta r/4. \tag{6.116}$$

Put

$$A = \{x_1, \ldots, x_q\}, \tag{6.117}$$

$$\bar{g}(y) = g(y) + 4^{-1} r \min\{1, \rho(y, A)\},\ y \in X. \tag{6.118}$$

It is easy to see that $\bar{g} \in \mathcal{A}$,

$$d_s(g, \bar{g}) \leq r \tag{6.119}$$

and

$$\sup_{h \in \mathcal{B}} \inf(\bar{g} + h) \leq \sup_{h \in \mathcal{B}} \inf(g + h) + 1. \tag{6.120}$$

Let $i \in \{1, \ldots, q\}$, $y \in X$ and

$$\bar{g}(y) + h_i(y) \leq \inf(\bar{g} + h_i) + \delta r. \tag{6.121}$$

By (6.118), (6.121), (6.117), (6.116) and (6.115),

$$g(y) + h_i(y) + 4^{-1} r \min\{1, \rho(y, A)\} = \bar{g}(y) + h_i(y) \leq \inf(\bar{g} + h_i) + \delta r \leq$$

$$\bar{g}(x_i) + h_i(x_i) + \delta r = g(x_i) + h_i(x_i) + \delta r \leq \inf(g + h_i) +$$

$$4^{-1} \delta r + \delta r \leq g(y) + h_i(y) + 2\delta r,$$

and

$$\min\{1, \rho(y, A)\} \leq 8\delta < \epsilon,\ \rho(y, A) < \epsilon.$$

Now it is easy to see that (H5) holds. This completes the proof of Theorem 6.18.

Now we obtain extensions of Theorem 6.18. We use the convention that $\ln(\infty) = \infty$. Define

$$C_l^+(X) = \{h \in C_l(X) :\ h(z) \geq 0 \text{ for all } z \in X\},\ C_v^+(X) = C_v(X) \cap C_l^+(X),$$

$$C^+(X) = C_l^+(X) \cap C(X),\ C_L^+(X) = C_l^+(X) \cap C_L(X).$$

It is clear that $C_l^+(X)$ is a closed subset of $(C_l(X), d_1)$, $C_v^+(X)$ is a closed subset of $(C_v(X), d_1)$, $C^+(X)$ is a closed subset of $(C(X), d_1)$ and $C_L^+(X)$ is a closed subset of $(C_L(X), d_2)$.

Now we equip the space $C_l^+(X)$ with the metric d_w. For $h, \xi \in C_l^+(X)$ set

$$\tilde{d}_w(h, \xi) = \sup\{|\ln(h(z)+1) - \ln(\xi(z)+1)| : z \in X\},$$

$$d_w(h, \xi) = \tilde{d}_w(h, \xi)(1 + \tilde{d}_w(h, \xi))^{-1}. \tag{6.122}$$

Clearly, the metric space $(C_l^+(X), d_w)$ is complete, $d_w(h, \xi) \le d_1(h, \xi)$ for all $h, \xi \in C_l^+(X)$ and $C_v^+(X)$, $C^+(X)$ are closed subsets of $(C_l^+(X), d_w)$.

It is easy to see that $\tilde{d}_w(h, \xi) \le \epsilon$ where $h, \xi \in C_l^+(X)$ and ϵ is a positive number if and only if

$$(h(z)+1)(\xi(z)+1)^{-1} \in [e^{-\epsilon}, e^{\epsilon}]$$

for all $z \in X$. It means that $h, \xi \in C_l^+(X)$ are close with respect to d_w if and only if the function

$$z \to (h(z)+1)(\xi(z)+1)^{-1},\ z \in X$$

is close to the function which is identically 1 with respect to the topology of the uniform convergence.

Theorem 6.19. *Let $\mathcal{B}$ be a countable union of compact subsets of the metric space $(C_l(X), d_0)$ and let a space $(\mathcal{A}, d_w, d_s)$ be either $(C_l^+(X), d_w, d_1)$ or $(C_v^+(X), d_w, d_2)$ or $(C^+(X), d_w, d_2)$ or $(C_L^+(X), d_2, d_2)$. Assume that one of the following properties hold:*

1. Each $g \in \mathcal{B}$ is finite-valued; 2. each $g \in \mathcal{A}$ is finite-valued.

Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $g \in \mathcal{F}$ and each $h \in \mathcal{B}$ the problem minimize $g(x) + h(x)$ subject to $x \in X$ on (X, ρ) is well-posed in the generalized sense.

Proof: We may assume without loss of generality that $\mathcal{B}$ is a compact subset of the metric space $(C_l(X), d_0)$. In view of Theorem 6.16 we need to show that (H3), (H4) and (H5) hold. Evidently, (H4) holds. Analogously to the proof of Theorem 6.18 we can show that (H5) holds. In order to complete the proof we need to show that (H3) holds.

Let m be a natural number. Since (H4) holds it follows from Lemma 6.13 that there exists a positive number c_0 for which

$$1 + |\inf(g)| < c_0 \text{ for all } g \in \mathcal{B}. \tag{6.123}$$

Fix

$$\alpha \in (0, [4e(m + c_0 + 1)]^{-1}). \tag{6.124}$$

Assume that $g_1, g_2 \in \mathcal{A},\ r \in (0, 1]$,

$$d_w(g_1, g_2) \le \alpha r, \tag{6.125}$$

$h \in \mathcal{B}$, $x \in X$ and

$$\min\{(g_1+h)(x),\ (g_2+h)(x)\} \le m. \tag{6.126}$$

In view of (6.126) and (6.123),

$$\min\{g_1(x), g_2(x)\} \le m + c_0. \tag{6.127}$$

Relations (6.122), (6.125) and (6.124) imply that

$$\tilde{d}_w(g_1,g_2) = d_w(g_1,g_2)(1-d_w(g_1,g_2))^{-1} \le 2d_w(g_1,g_2) \le 2\alpha r,$$

$$(1+g_1(x))(1+g_2(x))^{-1} \in [e^{-2\alpha r}, e^{2\alpha r}]. \tag{6.128}$$

We will show that $|g_1(x)-g_2(x)| \le r$. We may assume without loss of generality that $g_1(x) \ge g_2(x)$. Then (6.128) implies that

$$0 \le g_1(x) - g_2(x) \le e^{2\alpha r}(1+g_2(x)) - 1 - g_2(x) =$$

$$(g_2(x)+1)(e^{2\alpha r}-1) = (g_2(x)+1)2\alpha r e^{\alpha'}$$

where $\alpha' \in [0, 2\alpha r]$. Therefore (6.124) and (6.127) imply that

$$0 \le g_1(x) - g_2(x) \le (g_2(x)+1)2\alpha r e \le 2\alpha r e(m+c_0+1) \le r.$$

Hence (H3) holds. Theorem 6.19 is proved.

Next two theorems are extensions of Theorem 6.18 to classes of convex functions.

Theorem 6.20. *Let X be a nonempty bounded closed convex subset of a Banach space $(E, ||\cdot||)$, $\rho(x,y) = ||x-y||$, $x,y \in X$ and let $\mathcal{A}$ be either $\{f \in C_l(X) :\ f \text{ is convex}\}$ or $\{f \in C_v(X) :\ f \text{ is convex}\}$ or $\{f \in C(X) :\ f \text{ is convex}\}$. Assume that $\mathcal{B}$ is a countable union of compact subsets of the metric space $(C_l(X), d_0)$ and that one of the following properties holds:*

1. Each $g \in \mathcal{B}$ is finite-valued; 2. each $g \in \mathcal{A}$ is finite-valued.

Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_1, d_1) and that for each $g \in \mathcal{F}$ and each $h \in \mathcal{B}$ the problem minimize $g(x)+h(x)$ subject to $x \in X$ on (X,ρ) is well-posed in the generalized sense.

Proof: We may assume without loss of generality that $\mathcal{B}$ is compact. In view of Theorem 6.16 we need to show that (H3), (H4) and (H5) hold. It is easy to see that (H3) and (H4) are true. In order to prove Theorem 6.20 we need to show that (H5) holds. Fix

$$d_0 > \sup\{||x-y|| :\ x,y \in X\}. \tag{6.129}$$

Let ϵ be a positive number. Fix

$$\delta \in (0, (8(d_0+1))^{-1}\epsilon). \tag{6.130}$$

Assume that $g \in \mathcal{A}$, $r \in (0,1]$ and $\{h_1, \dots, h_q\} \subset \mathcal{B}$ where $q \geq 1$ is an integer. For each $i \in \{1, \dots, q\}$ choose $x_i \in X$ such that

$$g(x_i) + h_i(x_i) \leq \inf(g + h_i) + 4^{-1}\delta r. \tag{6.131}$$

Denote by A the convex hull of the set $\{x_1, \dots, x_q\}$ and define

$$\bar{g}(y) = g(y) + (4d_0 + 1)^{-1} r\rho(y, A) \text{ for all } y \in X. \tag{6.132}$$

It is easy to see that A is compact, $\bar{g} \in A$,

$$d_1(\bar{g}, g) \leq r$$

and

$$\sup_{h \in \mathcal{B}} \inf(\bar{g} + h) \leq \sup_{h \in \mathcal{B}} \inf(g + h) + 1.$$

Let $i \in \{1, \dots, q\}$, $y \in X$ and

$$\bar{g}(y) + h_i(y) \leq \inf(\bar{g} + h_i) + \delta r. \tag{6.133}$$

By (6.132), (6.133), (6.131) and (6.130),

$$g(y) + (4d_0 + 1)^{-1} r\rho(y, A) + h_i(y) = \bar{g}(y) + h_i(y) \leq (\bar{g} + h_i)(x_i) + \delta r =$$

$$g(x_i) + h_i(x_i) + \delta r \leq g(y) + h_i(y) + 4^{-1}\delta r + \delta r,$$

$$\rho(y, A) \leq 2\delta(4d_0 + 1) < \epsilon.$$

Thus (H5) is valid. This completes the proof of Theorem 6.20.

Theorem 6.21. *Let X be a nonempty closed convex subset of a Banach space $(E, ||\cdot||)$, $\rho(x, y) = ||x - y||$, $x, y \in X$, $c_0 > 0$ and let $\mathcal{B}$ be a countable union of compact subsets of the metric space $(C_l(X), d_0)$. Assume that a set $\mathcal{A}$ is either $\{f \in C_l(X) :\ f$ is convex and $f(x) \geq c_0||x||, x \in X\}$ or $\{f \in C_v(X) :\ f$ is convex and $f(x) \geq c_0||x||, x \in X\}$ or $\{f \in C(X) :$ f is convex and $f(x) \geq c_0||x||, x \in X\}$ and that one of the following properties holds:*

1. Each $g \in \mathcal{B}$ is finite-valued; 2. each $g \in \mathcal{A}$ is finite-valued.

Then the metric space $(\mathcal{A}, d_w)$ is complete and there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_w) and that for each $g \in \mathcal{F}$ and each $h \in \mathcal{B}$ the problem minimize $g(x) + h(x)$ subject to $x \in X$ on (X, ρ) is well-posed in the generalized sense.

Proof: We may assume without loss of generality that $\mathcal{B}$ is a compact subset of $(C_l(X), d_0)$. In view of Theorem 6.16 we need to show that (H3), (H4) and (H5) hold. It is clear that (H4) holds. Since (H3) holds for the metric space $(C_l^+(X), d_w)$ (see the proof of Theorem 6.19) it is also true for the subspace $(\mathcal{A}, d_w)$. Thus in order to prove Theorem 6.21 we need to show that (H5) holds. Since $\mathcal{B}$ is compact Lemma 6.13 implies that there exists a positive number c_1 such that

$$1 + |\inf(h)| \leq c_1 \text{ for all } h \in \mathcal{B}. \tag{6.134}$$

Let ϵ be a positive number and let m be a natural number. Choose an integer $m_0 \geq 1$ and a number $r_0 > 0$ such that

$$m_0 > (m + c_0 + c_1 + 2)(\min\{c_0, 1\})^{-1}, \; r_0 < (m_0 + 2)^{-1}. \tag{6.135}$$

Put

$$\gamma = \min\{1, c_0\}. \tag{6.136}$$

Fix

$$\delta \in (0, \min\{8^{-1}\gamma\epsilon, 1\}). \tag{6.137}$$

Assume that $g \in \mathcal{A}$ satisfies

$$\inf(g + h) \leq m \text{ for all } h \in \mathcal{B}, \tag{6.138}$$

$r \in (0, r_0]$ and $\{h_1, \dots, h_q\} \subset \mathcal{B}$ where $q \geq 1$ is an integer. For each $i \in \{1, \dots, q\}$ choose $x_i \in X$ such that

$$g(x_i) + h_i(x_i) \leq \inf(g + h_i) + 4^{-1}\delta r. \tag{6.139}$$

Denote by A the convex hull of the set $\{0, x_1, \dots, x_q\}$ and define

$$\bar{g}(y) = g(y) + 4^{-1}r\gamma\rho(y, A) \text{ for all } y \in X. \tag{6.140}$$

It is easy to see that A is compact and $\bar{g} \in A$. We show that $d_w(\bar{g}, g) \leq r$.

It follows from (6.122), (6.140) and (6.136) that

$$d_w(g, \bar{g}) \leq \tilde{d}_w(g, \bar{g}) = \sup\{|\ln(\bar{g}(y) + 1) - \ln(g(y) + 1)| : \; y \in X\} =$$

$$\sup_{y \in X}\{\ln(1 + (\bar{g}(y) - g(y))(g(y) + 1)^{-1})\}$$

$$\leq \sup_{y \in X}\{\ln(1 + 4^{-1}\gamma r\rho(y, A)(1 + c_0||y||)^{-1})\} \leq$$

$$\sup_{y \in X}\{4^{-1}\gamma r\rho(y, A)(1 + c_0||y||)^{-1}\} \leq \sup_{y \in X} 4^{-1}\gamma r||y||(1 + c_0||y||)^{-1} \leq$$

$$4^{-1}\gamma r/c_0 \leq r.$$

Hence

$$d_w(g, \bar{g}) \leq r. \tag{6.141}$$

Now we will estimate $\sup_{h \in \mathcal{B}} \inf(\bar{g} + h)$.

Let $h \in \mathcal{B}$ and $x \in X$. Assume that $||x|| \geq m_0$. Then (6.140), (6.134) and (6.135) imply that

$$\bar{g}(x) + h(x) \geq g(x) + h(x) \geq c_0||x|| - c_1 \geq c_0 m_0 - c_1 > m + 2.$$

Hence

$$\text{if } x \in X,\ ||x|| \geq m_0, \ \text{ then } \bar{g}(x) + h(x) \geq g(x) + h(x) \geq m + 2. \tag{6.142}$$

If $y \in X$ and $||y|| \leq m_0$, then it follows from (6.140), (6.136) and (6.135) that

$$\bar{g}(y) + h(y) = g(y) + h(y) + 4^{-1} r\gamma\rho(y, A) \leq$$

$$g(y) + h(y) + 4^{-1} r\gamma||y|| \leq g(y) + h(y) + 4^{-1} r_0 m_0 \leq$$

$$g(y) + h(y) + 4^{-1}.$$

Thus

$$\inf\{\bar{g}(y) + h(y) :\ y \in X,\ ||y|| \leq m_0\} \leq \inf\{g(y) + h(y) : \tag{6.143}$$

$$y \in X,\ ||y|| \leq m_0\} + 4^{-1}.$$

By (6.142), (6.143) and (6.138),

$$m \geq \inf(g + h) = \inf\{g(y) + h(y) :\ y \in X, ||y|| \leq m_0\},$$

$$\inf(\bar{g} + h) \leq m + 1/4.$$

Thus we have shown that

$$\sup_{h \in \mathcal{B}} \inf(\bar{g} + h) \leq m + 1/4. \tag{6.144}$$

Let $i \in \{1, \ldots, q\}$, $y \in X$ and

$$\bar{g}(y) + h_i(y) \leq \inf(\bar{g} + h_i) + \delta r. \tag{6.145}$$

By (6.140), (6.145), (6.139) and (6.137),

$$g(y) + 4^{-1}\gamma r\rho(y, A) + h_i(y) = \bar{g}(y) + h_i(y) \leq (\bar{g} + h_i)(x_i) + \delta r =$$

$$g(x_i) + h_i(x_i) + \delta r \leq g(y) + h_i(y) + 4^{-1}\delta r + \delta r,$$

$$4^{-1}\gamma\rho(y, A) \leq 2\delta,\ \rho(y, A) \leq 8\delta\gamma^{-1} < \epsilon.$$

Thus $\rho(y, A) < \epsilon$ and (H5) holds (see (6.141), (6.144)). Theorem 6.21 is proved.

6.9 Existence results for the problem (P1)

Let (X, ρ) be a complete metric space. Assume that $\mathcal{B}$ is a Hausdorff topological space and that $\mathcal{B} = \cup_{i=1}^{\infty} \mathcal{B}_i$ where $\mathcal{B}_i$ is a compact subset of $\mathcal{B}$, $i = 1, 2, \dots$. Denote by $\mathcal{M}$ the set of all functions $g : \mathcal{B} \times X \to (-\infty, \infty]$ such that:

(i) For each $b \in \mathcal{B}$ the function $g(b, \cdot) : X \to (-\infty, \infty]$ is lower semicontinuous bounded from below and it is not identically ∞;

(ii) For each $b \in \mathcal{B}$, each natural number n and each positive number ϵ there exists a neighborhood $\mathcal{U}$ of b in $\mathcal{B}$ such that for each $\xi \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{g(b, x), g(\xi, x)\} \le n$ the inequality $|g(b, x) - g(\xi, x)| \le \epsilon$ holds.

For each $f, g \in \mathcal{M}$ set

$$\tilde{d}_s(f, g) = \sup\{|f(b, x) - g(b, x)| : \ (b, x) \in \mathcal{B} \times X\},$$

$$d_s(f, g) = \tilde{d}(f, g)(1 + \tilde{d}(f, g))^{-1}.$$

Clearly, the metric space $(\mathcal{M}, d_s)$ is complete. Denote by $\mathcal{M}_v$ the set of all finite-valued functions $g \in \mathcal{M}$, by $\mathcal{M}_s$ the set of all functions $f \in \mathcal{M}_v$ such that the function $f(b, \cdot) : X \to (-\infty, \infty]$ is continuous for all $b \in \mathcal{B}$, and by $\mathcal{M}_c$ the set of all continuous functions $f \in \mathcal{M}_v$. It is easy to see that $\mathcal{M}_v$, $\mathcal{M}_s$ and $\mathcal{M}_c$ are closed subsets of the metric space $(\mathcal{M}, d_s)$.

Theorem 6.22. *Let $\mathcal{A}$ be either $\mathcal{M}$ or $\mathcal{M}_v$ or $\mathcal{M}_s$ or $\mathcal{M}_c$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $g \in \mathcal{F}$ and each $b \in \mathcal{B}$ the problem minimize $g(b, x)$ subject to $x \in X$ is well-posed in the generalized sense.*

Proof: We may assume without loss of generality that $\mathcal{B}$ is compact. For each $a \in \mathcal{A}$ and each $b \in \mathcal{B}$ set $f_{ab}(x) = a(b, x)$, $x \in X$. In view of Theorem 6.16 we need to show that (H3), (H4) and (H5) hold. It is clear that (H3) holds. (H4) follows from the property (ii). In order to prove Theorem 6.22 it is sufficient to show that (H5) is true.

Let ϵ be a positive number. Fix

$$\delta \in (0, \min\{8^{-1}\epsilon, 1/8\}).$$

Assume that $g \in \mathcal{A}$, $r \in (0, 1]$ and $\{b_1, \dots, b_q\} \subset \mathcal{B}$ where $q \ge 1$ is an integer. For each $i \in \{1, \dots, q\}$ choose $x_i \in X$ such that

$$g(b_i, x_i) \le \inf(g(b_i, \cdot)) + 4^{-1}\delta r. \tag{6.146}$$

Put $A = \{x_1, \dots, x_q\}$ and define

$$\bar{g}(b, x) = g(b, x) + 4^{-1} r \min\{1, \rho(x, A)\} \text{ for all } x \in X.$$

Clearly $\bar{g} \in \mathcal{A}$ and

$$d_s(g, \bar{g}) \leq 4^{-1}r, \ \sup_{b \in \mathcal{B}} \inf(\bar{g}(b, \cdot)) \leq \sup_{b \in \mathcal{B}} \inf(g(b, \cdot)) + 1.$$

Assume that $i \in \{1, \dots, q\}$, $y \in X$ and

$$\bar{g}(b_i, y) \leq \inf(\bar{g}(b_i, \cdot)) + \delta r.$$

By the inequality above, the definition of $\bar{g}$, (6.146) and the choice of δ,

$$g(b_i, y) + 4^{-1} r \min\{1, \rho(y, A)\} = \bar{g}(b_i, y) \leq \bar{g}(b_i, x_i) + \delta r =$$

$$g(b_i, x_i) + \delta r \leq g(b_i, y) + 4^{-1}\delta r + \delta r,$$

$$\min\{1, \rho(y, A)\} < 8\delta < \epsilon, \ \rho(y, A) < \epsilon.$$

Thus (H5) is true. Theorem 6.22 is proved.

Denote by $\mathcal{M}^+$ the set of all nonnegative $g \in \mathcal{M}$. Set $\mathcal{M}_v^+ = \mathcal{M}^+ \cap \mathcal{M}_v$, $\mathcal{M}_c^+ = \mathcal{M}^+ \cap \mathcal{M}_c$ and $\mathcal{M}_s^+ = \mathcal{M}^+ \cap \mathcal{M}_s$. It is easy to see that $\mathcal{M}^+$, $\mathcal{M}_v^+$, $\mathcal{M}_s^+$ and $\mathcal{M}_c^+$ are closed subsets of the complete metric space $(\mathcal{M}, d_s)$.

For each $g, h \in \mathcal{M}^+$ define

$$\tilde{d}_w(g, h) = \sup\{|\ln(g(b, x) + 1) - \ln(h(b, x) + 1)| : \ (b, x) \in \mathcal{B} \times X\},$$

$$d_w(g, h) = \tilde{d}_w(g, h)(1 + \tilde{d}_w(g, h))^{-1}.$$

Clearly, the metric space $(\mathcal{M}^+, d_w)$ is complete, $\mathcal{M}^+$, $\mathcal{M}_s^+$ and $\mathcal{M}_c^+$ are closed subsets of $(\mathcal{M}^+, d_w)$. Evidently, $d_w(g, h) \leq d_s(g, h)$ for all $g, h \in \mathcal{M}^+$.

Theorem 6.23. *Let $\mathcal{A}$ be either $\mathcal{M}^+$ or $\mathcal{M}_v^+$ or $\mathcal{M}_s^+$ or $\mathcal{M}_c^+$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $g \in \mathcal{F}$ and each $b \in \mathcal{B}$ the problem minimize $g(b, x)$ subject to $x \in X$ is well-posed in the generalized sense.*

Proof: We may assume without loss of generality that $\mathcal{B}$ is compact. For each $a \in \mathcal{A}$ and each $b \in \mathcal{B}$ set $f_{ab}(x) = a(b, x)$, $x \in X$. In view of Theorem 6.16 we need to show that (H3), (H4) and (H5) hold. (H4) follows from the property (ii). Analogously to the proof of Theorem 6.22 we can show that (H5) holds. In order to prove Theorem 6.23 it is sufficient to show that (H3) holds.

Let m be a natural number. Fix $\alpha \in (0, (4e(m + 1))^{-1})$. Assume that $r \in (0, 8^{-1}]$, $b \in \mathcal{B}$, $g_1, g_2 \in \mathcal{A}$ and $x \in X$ satisfy

$$d_w(g_1, g_2) \leq \alpha r, \ \min\{g_1(b, x), g_2(b, x)\} \leq m. \tag{6.147}$$

It follows from (6.147) that

$$\tilde{d}_w(g_1, g_2) = d_w(g_1, g_2)(1 - d_w(g_1, g_2))^{-1} \leq \alpha r(1 - \alpha r)^{-1} \leq 2\alpha r,$$

$$(g_1(b, x) + 1)(g_2(b, x) + 1)^{-1} \in [e^{-2\alpha r}, e^{2\alpha r}]. \tag{6.148}$$

We may assume without loss of generality that $g_2(b, x) \geq g_1(b, x)$. By (6.147), (6.148) and the mean value theorem,

$$0 \le g_2(b,x) - g_1(b,x) \le (g_1(b,x)+1)[e^{2\alpha r} - 1] \le (m+1)(e^{2\alpha r} - 1) =$$

$$(m+1)2\alpha r e^{\bar{r}}$$

where $0 \le \bar{r} \le 2\alpha r \le 1/2$. It follows from the choice of α that

$$0 \le g_2(b,x) - g_1(b,x) \le (m+1)2\alpha r e < r.$$

Thus (H3) holds. This completes the proof of Theorem 6.23.

The results of this section were obtained in [127].

6.10 Parametric optimization problems with constraints

Let (X,ρ) be a complete metric space. We study the parametric family of the minimization problems

$$\text{minimize } h(b,x) \text{ subject to } x \in \omega \tag{P}$$

with a parameter $b \in B$, $h(\cdot,\cdot) \in \mathcal{M}$ and $\omega \in S(X)$. Here $\mathcal{B}$ is a Hausdorff topological space which is a countable union of its compact subsets, $\mathcal{M}$ is a complete metric space of functions on $\mathcal{B} \times X$ and $S(X)$ is a complete metric space of nonempty closed subsets of X. We show the existence of a set $\mathcal{F} \subset \mathcal{M} \times S(X)$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M} \times S(X)$ such that for each $(h,\omega) \in \mathcal{F}$ the minimization problem (P) has a solution for all parameters $b \in \mathcal{B}$.

We use the convention that $\infty - \infty = 0$ and $\infty/\infty = 1$.

In this section we usually consider topological spaces with two topologies where one is weaker than the other. (Note that they can coincide.) We refer to them as the weak and the strong topologies, respectively. If (X,d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). Assume that X_1 and X_2 are topological spaces and that each of them is endowed with a weak and a strong topology. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies of X_1 and X_2 and a strong topology which is the product of the strong topologies of X_1 and X_2. If $Y \subset X_1$, then we consider the topological subspace Y with the relative weak and strong topologies (unless other topologies are introduced). If (X_i,d_i), $i=1,2$ are metric spaces with the metrics d_1 and d_2, respectively, then the space $X_1 \times X_2$ is endowed with the metric d defined by

$$d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2),\ (x_i,y_i) \in X \times Y,\ i=1,2.$$

We consider a complete metric space (X,ρ). For each function $g : X \to R^1 \cup \{\infty\}$ we set

$$\inf(g) = \inf\{g(x) :\ x \in X\}.$$

For $x \in X$ and $A \subset X$ set

$$\rho(x, A) = \inf\{\rho(x, y) : \ y \in A\}.$$

Denote by $S(X)$ the collection of all nonempty closed subsets of X. For $A, B \in S(X)$ we define

$$\tilde{H}(A, B) = \max\{\sup\{\rho(x, B) : \ x \in A\}, \ \sup\{\rho(y, A) : \ y \in B\}\},$$

$$H(A, B) = \tilde{H}(A, B)(\tilde{H}(A, B) + 1)^{-1}.$$

It is well known that the metric space $(S(X), H)$ is complete. We equip the set $S(X)$ with the topology induced by the metric H.

Let $h : X \to R^1 \cup \{\infty\}$. Recall that the minimization problem for h on (X, ρ) is well-posed in the generalized sense [40] if $\inf(h)$ is finite, the set $\{x \in X : \ h(x) = \inf(h)\}$ is nonempty and compact and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ satisfying $\lim_{i \to \infty} h(x_i) = \inf(h)$ has a convergent subsequence.

Denote by $\mathcal{L}$ the set of all lower semicontinuous functions $f : X \to R^1$ which are bounded from below and satisfy the following assumption:

(D1) For each $x \in X$, each neighborhood U of x in X and each positive number δ there exists a nonempty open set $V \subset U$ such that $f(y) \le f(x) + \delta$ for all $y \in V$.

Remark 6.24. Let $f : X \to R^1$ be a lower semicontinuous bounded from below function. Then f satisfies (A1) if and only if the epigraph

$$\mathrm{epi}(f) = \{(y, \alpha) \in X \times R^1 : \ \alpha \ge f(y)\}$$

is the closure of its interior.

Let $\mathcal{B}$ be a Hausdorff topological space. We assume that $\mathcal{B}$ is a countable union of its compact subsets. Denote by $\mathcal{M}$ the set of all functions $a : \mathcal{B} \times X \to R^1$ such that for each $b \in \mathcal{B}$ the following properties hold:

(D2) The function $x \to a(b, x)$, $x \in X$ belongs to $\mathcal{L}$;

(D3) For each natural number n and each positive number ϵ there exists a neighborhood U of b in $\mathcal{B}$ such that for each $\xi \in U$ and each $x \in X$ satisfying

$$\min\{a(\xi, x), \ a(b, x)\} \le n$$

the inequality $|a(b, x) - a(\xi, x)| \le \epsilon$ holds.

For $a_1, a_2 \in \mathcal{M}$ we set

$$\tilde{d}_s(a_1, a_2) = \sup\{|a_1(b, x) - a_2(b, x)| : \ b \in \mathcal{B}, \ x \in X\},$$

$$d_s(a_1, a_2) = \tilde{d}_s(a_1, a_2)(1 + \tilde{d}_s(a_1, a_2))^{-1}. \tag{6.149}$$

It is easy to see that the metric space $(\mathcal{M}, d_s)$ is complete.

Fix $\theta \in X$. For the set $\mathcal{M}$ we also consider the uniformity determined by the following base:

$$U_w(n) = \{(a_1, a_2) \in \mathcal{M} \times \mathcal{M} : \ |a_1(b,x) - a_2(b,x)| \leq 1/n \tag{6.150}$$

$$\text{for each } b \in \mathcal{B} \text{ and each } x \in X \text{ such that}$$

$$\rho(x, \theta) \leq n \text{ or } \min\{a_1(b,x), a_2(b,x)\} \leq n\},$$

where $n = 1, 2, \ldots$. Clearly the space $\mathcal{M}$ with this uniformity is metrizable (by a metric d_w). For the space $\mathcal{M}$ we consider the strong and weak topologies induced by the metrics d_s and d_w, respectively.

Denote by $\mathcal{M}_s$ the set of all functions $a \in \mathcal{M}$ such that the function $x \to a(b,x)$, $x \in X$ is continuous for all $b \in \mathcal{B}$ and denote by $\mathcal{M}_c$ the set of all continuous functions $a \in \mathcal{M}$. It is easy to see that $\mathcal{M}_s$ and $\mathcal{M}_c$ are closed subsets of the metric space $(\mathcal{M}, d_s)$.

Define $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ where $\mathcal{A}_2 = S(X)$ and $\mathcal{A}_1$ is either $\mathcal{M}$ or $\mathcal{M}_s$ or $\mathcal{M}_c$. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ we consider the problem

$$\text{minimize } a_1(b,x) \text{ subject to } x \in a_2$$

where a parameter $b \in \mathcal{B}$.

We prove the following theorem obtained in [119].

Theorem 6.25. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywherere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $a = (a_1, a_2) \in \mathcal{F}$ and each $b \in \mathcal{B}$ the problem minimize $a_1(b,x)$ subject to $x \in a_2$ is well-posed in the generalized sense.*

We prove this result as a realization of the variational principle which was established in Section 6.1 (Theorem 6.1).

6.11 Proof of Theorem 6.25

It is clear that $\mathcal{B} = \cup_{i=1}^{\infty} \mathcal{B}_i$ where $\mathcal{B}_i$, $i = 1, 2, \ldots$ is a compact subset of $\mathcal{B}$. In order to prove Theorem 6.25 it is sufficient to show that for each natural number i there exists a set $\mathcal{F}_i \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that the following property holds:

For each $a = (a_1, a_2) \in \mathcal{F}_i$ $(a = (a_1, a_2))$ and each $b \in \mathcal{B}_i$ the minimization problem $a_1(b,x) \to \min$, $x \in a_2$ is well-posed in the generalized sense.

Fix a natural number s. For each $a = (a_1, a_2) \in \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ and each $b \in \mathcal{B}_s$ define

$$f_{ab}(x) = a_1(b,x) \text{ if } x \in a_2, \ \ \text{otherwise } f_{ab}(x) = \infty. \tag{6.151}$$

Evidently, the function $f_{ab} : X \to R^1 \cup \{\infty\}$ is lower semicontinuous for any $a \in \mathcal{A}$ and any $b \in \mathcal{B}_s$. In order to prove Theorem 6.25 it is sufficient to show that there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that the following property holds:

For each $a = (a_1, a_2) \in \mathcal{F}$ and each $b \in \mathcal{B}_s$ the minimization problem $a_1(b, x) \to \min$, $x \in a_2$ is well-posed in the generalized sense.

In view of Theorem 6.1 to meet this goal we need to show that (H) holds with $\mathcal{B} = \mathcal{B}_s$.

Proposition 6.4 and (D3) imply the following auxiliary result.

Lemma 6.26. *Let $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Then the function*

$$b \to \inf\{a_1(b, x) : \ x \in a_2\}, \ b \in \mathcal{B}$$

is continuous.

Let $a = (a_1, a_2) \in \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, $\epsilon \in (0, 1)$. We construct $\bar{a} = (\bar{a}_1, \bar{a}_2) \in \mathcal{A}$ (see hypothesis (H) in Section 6.1).

Since the set $\mathcal{B}_s$ is compact, Lemma 6.26 implies that there exists an integer $n_0 \geq 4$ such that

$$|\inf\{a_1(b, u) : \ u \in a_2\}| < n_0 \text{ for all } b \in \mathcal{B}_s. \tag{6.152}$$

Fix positive numbers

$$\epsilon_0 < \epsilon/4, \ \eta < \epsilon\epsilon_0/16. \tag{6.153}$$

By the definition of $\mathcal{M}$ (see (D3)) for each $b \in \mathcal{B}_s$ there exists an open neighborhood $\mathcal{U}_b$ of $b \in \mathcal{B}$ such that:

(P1) For each $\xi \in \mathcal{U}_b$ and each $x \in X$ satisfying

$$\min\{a_1(\xi, x), a_1(b, x)\} \leq 4n_0$$

the inequality $|a_1(\xi, x) - a_1(b, x)| \leq \eta/16$ holds.

Since $\mathcal{B}_s$ is compact and

$$\mathcal{B}_s \subset \cup_{b \in \mathcal{B}_s} \mathcal{U}_b$$

there exists a finite set $\{b_1, \dots, b_q\} \subset \mathcal{B}_s$ where $q \geq 1$ is an integer such that

$$\mathcal{B}_s \subset \cup_{i=1}^{q} \mathcal{U}_{b_i}. \tag{6.154}$$

Put

$$\Omega = \{x \in X : \ \rho(x, a_2) < \epsilon_0\}. \tag{6.155}$$

Let $i \in \{1, \dots, q\}$. It follows from (D2) and (D1) that there exists a nonempty open set $E_i \subset X$ such that

$$E_i \subset \Omega, \ a_1(b_i, z) \leq \inf\{a_1(b, u) : \ u \in \Omega\} + 16^{-1}\eta \text{ for all } z \in E_i. \tag{6.156}$$

Let

$$\bar{x}_i \in E_i \tag{6.157}$$

and define

$$\bar{a}_2 = a_2 \cup \{\bar{x}_i : \ i = 1, \dots, q\}, \tag{6.158}$$

$$\bar{a}_1(b,x) = a_1(b,x) + 4^{-1}\epsilon_0 \min\{1, \rho(x, \{\bar{x}_i : \ i = 1, \dots, q\})\}, \tag{6.159}$$

$$b \in \mathcal{B}, \ x \in X.$$

Put

$$\bar{a} = (\bar{a}_1, \bar{a}_2).$$

Clearly, $\bar{a}_1 \in \mathcal{A}_1$, $\bar{a}_2 \in \mathcal{A}_2$. By (6.153), (6.155) and (6.157)–(6.159),

$$H(a_2, \bar{a}_2) \le \tilde{H}(a_2, \bar{a}_2) < \epsilon_0 < \epsilon/4 \tag{6.160}$$

and

$$d_s(a_1, \bar{a}_1) \le \tilde{d}_s(a_1, \bar{a}_1) \le 4^{-1}\epsilon_0 < \epsilon/4. \tag{6.161}$$

We construct a neighborhood $\mathcal{V}$ of $\bar{a}$ in $\mathcal{A}$ with the weak topology (see hypothesis (H) in Section 6.1).

Fix an integer $n_1 \ge 1$ such that

$$n_1 > 16 + 2n_0 + 32\epsilon_0^{-2} + 16[\epsilon_0 - \max\{\rho(\bar{x}_i, a_2) : i = 1, \dots, q\}]^{-1}, \tag{6.162}$$

$$\{z \in X : \ \rho(z, \bar{x}_i) \le 8n_1^{-1}\} \subset E_i, \ i = 1, \dots, q. \tag{6.163}$$

Define

$$\mathcal{V}_1 = \{\xi \in \mathcal{A}_1 : \ (\xi, \bar{a}_1) \in U_w(n_1)\}, \tag{6.164}$$

$$\mathcal{V}_2 = \{h \in \mathcal{A}_2 : \ H(h, \bar{a}_2) < 1/n_1\}, \tag{6.165}$$

$$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2. \tag{6.166}$$

It is easy to see that $\mathcal{V}$ is a neighborhood of $(\bar{a}_1, \bar{a}_2)$ in $\mathcal{A}$ with the weak topology.

We show that hypothesis (H) holds with $\bar{a}$, $\mathcal{V}$, $x_i = \bar{x}_i$, $i = 1, \dots, q$ and $\delta = \eta$. The inequalities (6.160) and (6.161) imply part (i) of hypothesis (H).

Assume that

$$b \in \mathcal{B}_s, \ \xi = (\xi_1, \xi_2) \in \mathcal{V}. \tag{6.167}$$

In view of (6.154) there exists $j \in \{1, \dots, q\}$ such that

$$b \in \mathcal{U}_{b_j}. \tag{6.168}$$

By (6.167) and (6.165),

$$\tilde{H}(\xi_2, \bar{a}_2) = H(\xi_2, \bar{a}_2)(1 - H(\xi_2, \bar{a}_2))^{-1} \le 2H(\xi_2, \bar{a}_2) < 2/n_1. \tag{6.169}$$

Therefore (see (6.158)) there exists $\bar{y} \in X$ such that

$$\bar{y} \in \xi_2, \ \rho(\bar{y}, \bar{x}_j) < 2/n_1. \tag{6.170}$$

It follows from (6.170), (6.163) and (6.156) that

$$\bar{y} \in E_j \subset \Omega. \quad (6.171)$$

By (6.171), (6.157) and (6.156),

$$\max\{a_1(b_j, \bar{y}), a_1(b_j \bar{x}_j)\} \le \inf\{a_1(b_j, u) : \; u \in \Omega\} + 16^{-1}\eta. \quad (6.172)$$

We show that $\xi_1(b, \bar{y}) \le n_0 + 1$. In view of (6.159), (6.169), (6.172), (6.171) and (6.157),

$$\bar{a}_1(b_j, \bar{y}) \le a_1(b_j, \bar{y}) + 2^{-1}\epsilon_0 n_1^{-1} \le a_1(b_j, \bar{x}_j) + 16^{-1}\eta + 2^{-1}\epsilon_0 n_1^{-1}. \quad (6.173)$$

Relations (6.172), (6.152) and (6.155) imply that

$$a_1(b_j, \bar{y}) \le n_0 + 16^{-1}\eta. \quad (6.174)$$

By (6.174), (6.168) and the property (P1) (with $b = b_j$, $\xi = b$, $x = \bar{y}$),

$$|a_1(b_j, \bar{y}) - a_1(b, \bar{y})| \le \eta/16. \quad (6.175)$$

In view of (6.174) and (6.175),

$$a_1(b, \bar{y}) \le a_1(b_j, \bar{y}) + \eta/16 \le n_0 + \eta/8. \quad (6.176)$$

By (6.176), (6.159) and (6.153),

$$\bar{a}_1(b, \bar{y}) \le a_1(b, \bar{y}) + 4^{-1}\epsilon_0 \le n_0 + \eta/8 + \epsilon/4 \le n_0 + 1/2. \quad (6.177)$$

It follows from (6.177), (6.167), (6.164), (6.166), (6.150) and (6.162) that

$$|\bar{a}_1(b, \bar{y}) - \xi_1(b, \bar{y})| \le n_1^{-1}.$$

Combined with (6.177) and (6.162) this inequality implies that

$$\xi_1(b, \bar{y}) \le \bar{a}_1(b, \bar{y}) + 1/n_1 \le n_0 + 1/2 + 1/n_1 \le n_0 + 1$$

and

$$\xi_1(b, \bar{y}) \le n_0 + 1. \quad (6.178)$$

By (6.170) and (6.178),

$$\inf\{\xi_1(b, u) : \; u \in \xi_2\} \le n_0 + 1. \quad (6.179)$$

Note that (6.179) holds for any $b \in \mathcal{B}_s$ and any $\xi = (\xi_1, \xi_2) \in \mathcal{V}$. Therefore we have

$$\inf\{\bar{a}_1(b, u) : \; u \in \xi_2\} \le n_0 + 1. \quad (6.180)$$

Let

$$h_1, h_2 \in \mathcal{A}_1, \; \{h_1, h_2\} = \{\xi_1, \bar{a}_1\}. \quad (6.181)$$

Assume that

$$z \in \xi_2,\ h_1(b, z) \le \inf\{h_1(b, u) :\ u \in \xi_2\} + 1. \tag{6.182}$$

It follows from (6.179), (6.180), (6.181) and (6.182) that $h_1(b, z) \le n_0 + 2$. Together with (6.181), (6.164), (6.166), (6.150) and (6.162) this inequality implies that

$$|h_2(b, z) - h_1(b, z)| \le 1/n_1. \tag{6.183}$$

Hence we have shown that the following property holds:

(P2) If $z \in X$ satisfies (6.182), then (6.183) holds.

By property (P2), (6.182), (6.183) and (6.181),

$$\inf\{h_2(b, u) :\ u \in \xi_2\} \le \inf\{h_2(b, u) :\ u \in \xi_2 \text{ and}$$

$$h_1(b, u) \le \inf\{h_1(b, v) :\ v \in \xi_2\} + 1\} \le \inf\{h_1(b, u) + 1/n_1 :$$

$$u \in \xi_2 \text{ and } h_1(b, u) \le \inf\{h_1(b, v) :\ v \in \xi_2\} + 1\} =$$

$$1/n_1 + \inf\{h_1(b, u) :\ u \in \xi_2 \text{ and } h_1(b, u) \le$$

$$\inf\{h_1(b, v) :\ v \in \xi_2\} + 1\} = 1/n_1 + \inf\{h_1(b, u) :\ u \in \xi_2\}$$

and

$$|\inf\{\bar{a}_1(b, u) :\ u \in \xi_2\} - \inf\{\xi_1(b, u) :\ u \in \xi_2\}| \le 1/n_1. \tag{6.184}$$

Let

$$p_1, p_2 \in \mathcal{B}_s,\ \{p_1, p_2\} = \{b, b_j\}. \tag{6.185}$$

Assume that

$$z \in \xi_2,\ \bar{a}_1(p_1, z) \le \inf\{\bar{a}_1(p_1, u) :\ u \in \xi_2\} + 1. \tag{6.186}$$

By (6.159), (6.180) (recall that (6.180) holds for any $b \in \mathcal{B}_s$),

$$a_1(p_1, z) \le \bar{a}_1(p_1, z) \le n_0 + 2. \tag{6.187}$$

Property (P1) (with $b = b_j$, $\xi = b$), (6.187), (6.159), (6.168) and (6.185) imply that

$$|\bar{a}_1(p_1, z) - \bar{a}_1(p_2, z)| = |a_1(p_1, z) - a_1(p_2, z)| \le \eta/16. \tag{6.188}$$

We have shown that the following property holds:

(P3) If $z \in X$ satisfies (6.186), then (6.188) holds.

By property (P3), (6.186), (6.188) and (6.185),

$$\inf\{\bar{a}_1(p_2, z) :\ z \in \xi_2\} \le$$

$$\inf\{\bar{a}_1(p_2, z) :\ z \in \xi_2 \text{ and } \bar{a}_1(p_1, z) \le \inf\{\bar{a}_1(p_1, u) :\ u \in \xi_2\} + 1\} \le$$

$$\inf\{\bar{a}_1(p_1, z) + \eta/16 :\ z \in \xi_2 \text{ and } \bar{a}_1(p_1, z) \le \inf\{\bar{a}_1(p_1, u) :\ u \in \xi_2\} + 1\} =$$

$$\eta/16 + \inf\{\bar{a}_1(p_1, z) :\ z \in \xi_2 \text{ and } \bar{a}_1(p_1, z) \le \inf\{\bar{a}_1(p_1, u) :\ u \in \xi_2\} + 1\} =$$

$$\eta/16 + \inf\{\bar{a}_1(p_1, z) :\ z \in \xi_2\}$$

and

$$|\inf\{\bar{a}_1(b, z) :\ z \in \xi_2\} - \inf\{\bar{a}_1(b_j, z) :\ z \in \xi_2\}| \le \eta/16. \tag{6.189}$$

Now we show that part (ii) of hypothesis (H) holds. Assume that

$$z \in \xi_2 \text{ and } \xi_1(b, z) \le \inf\{\xi_1(b, u) :\ u \in \xi_2\} + \eta. \tag{6.190}$$

By (6.190) and (6.169),

$$\rho(z, \bar{a}_2) \le \tilde{H}(\xi_2, \bar{a}_2) < 2/n_1. \tag{6.191}$$

Property (P2), (6.182), (6.183), (6.181), (6.190) and (6.153) imply that

$$|\xi_1(b, z) - \bar{a}_1(b, z)| \le 1/n_1. \tag{6.192}$$

In view of (6.192), (6.190) and (6.184),

$$\bar{a}_1(b, z) \le \xi_1(b, z) + 1/n_1 \le \inf\{\xi_1(b, u) : u \in \xi_2\} + \eta + \tag{6.193}$$
$$1/n_1 \le \inf\{\bar{a}_1(b, u) :\ u \in \xi_2\} + \eta + 2/n_1.$$

By (6.193), the property (P3), (6.190), (6.186), (6.188), (6.185), (6.153) and (6.162),

$$|\bar{a}_1(b, z) - \bar{a}_1(b_j, z)| \le \eta/16. \tag{6.194}$$

It follows from (6.194), (6.193) and (6.189) that

$$\bar{a}_1(b_j, z) \le \bar{a}_1(b, z) + \eta/16 \le \tag{6.195}$$
$$\inf\{\bar{a}_1(b, u) :\ u \in \xi_2\} + \eta + 2/n_1 + \eta/16 \le$$
$$\inf\{\bar{a}_1(b_j, u) : u \in \xi_2\} + \eta/8 + 2/n_1 + \eta.$$

In view of (6.191), (6.158), (6.163), (6.155), (6.156), (6.162) and (6.163),

$$z \in \Omega. \tag{6.196}$$

By (6.196), (6.157) and (6.156),

$$a_1(b_j, z) \ge a_1(b_j, \bar{x}_j) - 16^{-1}\eta. \tag{6.197}$$

By (6.159), (6.195), (6.170), (6.196), (6.162), (6.153) and (6.172),

$$a_1(b_j, z) + 4^{-1}\epsilon_0 \min\{1, \rho(z, \{\bar{x}_i :\ i = 1, \dots, q\})\} =$$
$$\bar{a}_1(b_j, z) \le \inf\{\bar{a}_1(b_j, u) :\ u \in \xi_2\} + \eta/8 + \eta + 2/n_1 \le$$
$$\bar{a}_1(b_j, \bar{y}) + \eta/8 + \eta + 2/n_1 \le$$
$$2^{-1}\epsilon_0 n_1^{-1} + a_1(b_j, \bar{y}) + \eta/8 + \eta + 2/n_1 \le$$
$$a_1(b_j, z) + 8^{-1}\eta + 2^{-1}\epsilon_0 n_1^{-1} + 16^{-1}\eta + 2/n_1 <$$
$$a_1(b_j, z) + 2\eta + 4n_1^{-1}$$

and

$$\min\{1, \rho(z, \{\bar{x}_i :\ i = 1, \dots, q\})\} \le (8n + 16n_1^{-1})\epsilon_0^{-1} < \epsilon.$$

Thus we have shown that (H) holds with $\mathcal{B} = \mathcal{B}_s$. This completes the proof of Theorem 6.25.

6.12 Comments

In this chapter we study a parametric family of the problems minimize $f(b, x)$ subject to $x \in X$ on a complete metric space X with a parameter b which belongs to a Hausdorff compact space $\mathcal{B}$. Here $f(\cdot, \cdot)$ belongs to a space of functions on $\mathcal{B} \times X$ endowed with an appropriate uniform structure. Using the generic approach and the porosity notion we show that for most functions $f(\cdot, \cdot)$ the minimization problem has a solution for all parameters $b \in \mathcal{B}$. These results and their extensions are obtained as realizations of abstract variational principles. The chapter is based on the works [106, 109, 113, 119, 127].

7

Optimization with Increasing Objective Functions

7.1 Preliminaries

Let K be a nonempty closed subset of a Banach ordered space $(X, ||\cdot||, \geq)$. A function $f : K \to R^1 \cup \{+\infty\}$ is called increasing if

$$f(x) \leq f(y) \text{ for all } x, y \in K \text{ such that } x \leq y.$$

Increasing functions are considered in many models of mathematical economics. As a rule both utility and production functions are increasing with respect to natural order relations.

We study the existence of a solution of the minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in K \tag{P1}$$

where $f : K \to R^1 \cup \{+\infty\}$ is an increasing lower semicontinuous function. In [88] it was established the generic existence of solutions of the problem (P1) for certain classes of increasing lower semicontinuous functions f which satisfy the growth condition

$$f(x) \geq \phi(||x||) - a \text{ for all } x \in K,$$

where a is a given real number and $\phi : [0, \infty) \to [0, \infty)$ is a given increasing function such that $\phi(t) \to +\infty$ as $t \to +\infty$. Note that the perturbations which are usually used to obtain a generic existence result are not suitable for these classes since they break the monotonicity. In [88] we proposed the new kind of perturbations which allowed us to established the generic existence of solutions for certain classes of increasing lower semicontinuous functions satisfying the growth condition.

In the present chapter we prove significant improvements of the results of [88]. We establish a general variational principle which is an extension of the variational principles obtained in Section 4. Our generic existence results are obtained as realizations of this principle.

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In contrast with [88], where the growth condition was used in the main results, in the present chapter we also establish generic existence results for spaces of increasing functions assuming only that the set K is bounded from below.

We equip the space of functions with weak and strong topologies. We construct a subset of this space which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets such that for each function the corresponding minimization problem has a solution. This approach allows us to enlarge the set of functions, for which we can guarantee that (P1) has a unique solution.

In the present chapter we also consider the following minimization problem:

$$\text{minimize } f(x) \text{ subject to } x \in A, \tag{P2}$$

where A is a nonempty closed subset of K and $f : K \to R^1$ is an increasing continuous function. We show that for a generic pair (f, A) the minimization problem (P2) has a solution.

The chapter is based on the works [114, 115].

7.2 A variational principle

We use the following notations and definitions. Let $(X, ||\cdot||, \geq)$ be a Banach ordered space and $X_+ = \{x \in X : \ x \geq 0\}$ be the cone of its positive elements. Assume that X_+ is a closed convex cone such that $||x|| \leq ||y||$ for each $x, y \in X_+$ satisfying $x \leq y$. We assume that the cone X_+ has the following property (A):

Property (A): If $\{x_i\}_{i=1}^{\infty} \subset X$, $x_{i+1} \leq x_i$ for all integers $i \geq 1$ and $\sup\{||x_i|| : i = 1, 2, \dots\} < \infty$, then the sequence $\{x_i\}_{i=1}^{\infty}$ converges.

The property (A) is well-known in the theory of ordered Banach spaces (see, for example, [60, 88, 90]). Recall that the cone X_+ has the property (A) if the space X is reflexive. The property (A) also holds for the cone of nonnegative functions (with respect to usual order relation) in the space L_1 of all integrable on a measure space functions.

Assume that K is a closed subset of X. For each function $f : Y \to [-\infty, \infty]$, where Y is a nonempty set we define

$$\mathrm{dom}(f) = \{y \in Y : |f(y)| < \infty\},\ \inf(f) = \inf\{f(y) : y \in Y\}$$

and

$$\mathrm{epi}(f) = \{(y, \alpha) \in Y \times R^1 :\ \alpha \geq f(y)\}.$$

We use the convention that $\infty - \infty = 0$.

We consider a complete metric space $(\mathcal{A}, d)$ which is called the data space. We consider the space $\mathcal{A}$ with the topology τ_s induced by the metric d. The topology τ_s is called the strong topology. The space $\mathcal{A}$ is also equipped with a topology τ_w which is weaker than τ_s. The topology τ_w is called the weak topology. We assume that the topology τ_w is induced by a metric $d_w : \mathcal{A}\times\mathcal{A} \to R^1$.

For each $a \in \mathcal{A}$ and each number $r > 0$ set

$$B(a,r) = \{b \in \mathcal{A} : d(a,b) < r\},\ B_w(a,r) = \{b \in \mathcal{A} : d_w(a,b) < r\}. \tag{7.1}$$

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function $f_a : K \to [-\infty, \infty]$ is associated and f_a is not identically ∞ for all $a \in \mathcal{A}$. The following is the basic hypotheses about the functions.

(H) For each $a \in \mathcal{A}$ and each pair of positive numbers ϵ and γ there exist $\bar{a} \in \mathcal{A}$, $x \in K$ and $r, \eta, c > 0$ such that $d(a, \bar{a}) < \epsilon$, $-\infty < f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + \epsilon$ and for each $b \in \mathcal{A}$ satisfying $d_w(\bar{a}, b) < r$ the following properties hold:

$\inf(f_b) > -\infty$;

if $z \in K$ satisfies $f_b(z) \le \inf(f_b) + \eta$, then $||z|| \le c$, $|f_b(z) - f_{\bar{a}}(x)| \le \gamma$ and there is $u \in X$ such that $z \le x + u$ and $||u|| < \epsilon$;

there is $y \in K$ such that $||y - x|| \le \epsilon$ and $f_b(y) \le \inf(f_b) + \epsilon$.

The hypothesis (H) is of technical nature. It turns out that this hypothesis is valid in many natural situations.

The following theorem was proved in [114].

Theorem 7.1. *Assume that (H) holds. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{F}$ the following assertions hold:*

1. $\inf(f_a)$ is finite and attained at a point $x^{(a)} \in K$ such that for each $x \in K$ satisfying $f_a(x) = \inf(f_a)$ the inequality $x \le x_a$ holds.

2. For any positive number ϵ there exist $\delta > 0$ and a neighborhood U of a in $\mathcal{A}$ with the weak topology such that for each $b \in U$, $\inf(f_b)$ is finite, and if $x \in K$ satisfies $f_b(x) \le \inf(f_b) + \delta$, then $|f_a(x^{(a)}) - f_b(x)| < \epsilon$ and there is $u \in X$ such that $||u|| < \epsilon$ and $x \le x^{(a)} + u$.

Proof: It follows from (H) that for each $a \in \mathcal{A}$ and each natural number n there exist $\bar{b}(a,n) \in \mathcal{A}$, $x(a,n) \in K$ and positive numbers $r(a,n)$, $c(a,n)$, $\eta(a,n)$ such that

$$d(a, \bar{b}(a,n)) < 2^{-n},\ -\infty < f_{\bar{b}(a,n)}(x(a,n)) \le \inf(f_{\bar{b}(a,n)}) + 2^{-n} \tag{7.2}$$

and for each $b \in B_w(\bar{b}(a,n), r(a,n))$ the following properties hold:

$$\inf(f_b) > -\infty; \tag{7.3}$$

there exists $y_b \in K$ such that

$$||y_b - x(a,n)|| \le 2^{-n},\ f_b(y_b) \le \inf(f_b) + 2^{-n}; \tag{7.4}$$

if $z \in K$ satisfies $f_b(z) \le \inf(f_b) + \eta(a,n)$, then

$$||z|| \le c(a,n),\ |f_b(z) - f_{\bar{b}(a,n)}(x(a,n))| \le 2^{-n}, \tag{7.5}$$

and there is $u \in X$ such that

$$z \le x(a,n) + u \text{ and } ||u|| < 2^{-n}. \tag{7.6}$$

We may assume without loss of generality that

$$\eta(a,n),\ r(a,n) < 4^{-n-1} \text{ for all } a \in \mathcal{A} \text{ and all natural numbers } n. \tag{7.7}$$

Define

$$\mathcal{F} = \cap_{q=1}^{\infty} \cup \{B_w(\bar{b}(a,n), r(a,n)) :\ a \in \mathcal{A},\ n \ge q\}. \tag{7.8}$$

Relations (7.1) and (7.2) imply that $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$.

Assume that $a \in \mathcal{F}$. (7.8) implies that there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$ and a strictly increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that

$$a \in B_w(\bar{b}(a_n,k_n), r(a_n,k_n)),\ n = 1,2,\dots. \tag{7.9}$$

It follows from (7.9) and (7.7) that we may assume without loss of generality that

$$B_w(\bar{b}(a_{n+1},k_{n+1}), r(a_{n+1},k_{n+1})) \subset B_w(\bar{b}(a_n,k_n), r(a_n,k_n)),\ n = 1,2,\dots \tag{7.10}$$

and

$$4 \cdot 2^{-k_{n+1}} < \eta(a_n,k_n),\ n = 1,2,\dots. \tag{7.11}$$

In view of (7.11) and (7.7),

$$\eta(a_{n+1},k_{n+1}) < \eta(a_n,k_n),\ n = 1,2,\dots. \tag{7.12}$$

Let n be a natural number. By (7.2),

$$-\infty < f_{\bar{b}(a_{n+1},k_{n+1})}(x(a_{n+1},k_{n+1})) \le \inf(f_{\bar{b}(a_{n+1},k_{n+1})}) + 2^{-k_{n+1}}. \tag{7.13}$$

By (7.10)–(7.13) and the definition of $\bar{b}(a_n,k_n)$, $x(a_n,k_n)$, $r(a_n,k_n)$, $c(a_n,k_n)$, $\eta(a_n,k_n)$ (see (7.4)–(7.6)),

$$||x(a_{n+1},k_{n+1})|| \le c(a_1,k_1), \tag{7.14}$$

and that there exists $u_n \in X$ for which

$$||u_n|| < 2^{-k_n},\ x(a_{n+1},k_{n+1}) \le x(a_n,k_n) + u_n. \tag{7.15}$$

For each natural number n put

$$y_n = x(a_n,k_n) + \sum_{i=n}^{\infty} u_i. \tag{7.16}$$

Clearly the sequence $\{y_n\}_{n=1}^{\infty}$ is well-defined and for each natural number n,

$$y_{n+1} - y_n = x(a_{n+1}, k_{n+1}) - x(a_n, k_n) - u_n \le 0. \tag{7.17}$$

By (7.14), (7.16) and (7.15), the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded. In view of (7.17) and the property (A) there exists $x^{(a)} = \lim_{n\to\infty} y_n$. Together with (7.16) and (7.15) this equality implies that

$$x^{(a)} = \lim_{n\to\infty} x(a_n, k_n). \tag{7.18}$$

In view of (7.9) and the definition of $\bar{b}(a_n, k_n)$, $x(a_n, k_n)$, $r(a_n, k_n)$ (see (7.4)), for each natural number n there exists $z_n \in K$ such that

$$||z_n - x(a_n, k_n)|| \le 2^{-k_n}, \; f_a(z_n) \le \inf(f_a) + 2^{-k_n}.$$

These inequalities, (7.18) and lower semicontinuity of f_a imply that $f_a(x^{(a)}) = \inf(f_a)$.

Assume now that $x \in K$ and $f_a(x) = \inf(f_a)$. It follows from (7.9) and the definition of $\bar{b}(a_n, k_n)$, $x(a_n, k_n)$, $r(a_n, k_n)$ (see (7.5) and (7.6)) that for each natural number n there exists $v_n \in X$ such that

$$||v_n|| \le 2^{-k_n}, \; x \le x(a_n, k_n) + v_n.$$

In view of these inequalities and (7.18), $x \le x^{(a)}$. Hence the first assertion of Theorem 7.1 is true.

We turn now to the second assertion. Assume that $\epsilon > 0$. Choose a natural number m for which

$$||x^{(a)} - x(a_m, k_m)|| < 4^{-1}\epsilon \text{ and } 2^{-k_m} < 4^{-1}\epsilon. \tag{7.19}$$

Assume that

$$b \in B_w(\bar{b}(a_m, k_m), r(a_m, k_m)), \; x \in K \text{ and } f_b(x) \le \inf(f_b) + \eta(a_m, k_m). \tag{7.20}$$

It follows from (7.20) and the definition of $\bar{b}(a_m, k_m)$, $r(a_m, k_m)$, $\eta(a_m, k_m)$ (see (7.5), (7.6)) that

$$|f_b(x) - f_{\bar{b}(a_m,k_m)}(x(a_m, k_m))| \le 2^{-k_m} \tag{7.21}$$

and there exists $v \in X$ such that

$$||v|| < 2^{-k_m}, \; x \le x(a_m, k_m) + v. \tag{7.22}$$

Since (7.21) and (7.22) hold for any b and x satisfying (7.20) we have

$$|f_a(x^{(a)}) - f_{\bar{b}(a_m,k_m)}(x(a_m, k_m))| \le 2^{-k_m}. \tag{7.23}$$

By (7.22) and (7.19),

$$x \le x^{(a)} + (x(a_m, k_m) - x^{(a)} + v),$$

$$||x(a_m, k_m) - x^{(a)} + v|| \le ||v|| + ||x(a_m, k_m) - x^{(a)}|| < 2^{-k_m} + 4^{-1}\epsilon < 2^{-1}\epsilon$$

and

$$|f_a(x^{(a)}) - f_b(x)| \le 2^{1-k_m} < \epsilon.$$

Theorem 7.1 is proved.

An element $x \in K$ is called minimal if for each $y \in K$ satisfying $y \le x$ the equality $x = y$ holds.

We also use the following hypotheses.

(H1) For each $a \in \mathcal{A}$, each pair of positive numbers ϵ and γ there exist $\bar{a} \in \mathcal{A}$, a minimal element $x \in K$ and numbers $r, \eta, c > 0$ such that $d(a, \bar{a}) < \epsilon$, $-\infty < f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + \epsilon$ and for each $b \in \mathcal{A}$ satisfying $d_w(\bar{a}, b) < r$ the following properties hold:

$\inf(f_b) > -\infty$;

if $z \in K$ satisfies $f_b(z) \le \inf(f_b) + \eta$, then $||z|| \le c$, $|f_b(z) - f_{\bar{a}}(x)| \le \gamma$ and there exists $u \in X$ such that $z \le x + u$ and $||u|| < \epsilon$;

there exists $y \in K$ such that $||y - x|| \le \epsilon$ and $f_b(y) \le \inf(f_b) + \epsilon$.

The hypothesis (H1) is of technical nature. It turns out that this hypothesis is valid in many natural situations.

Theorem 7.2. *Assume that (H1) holds and that the set of all minimal elements of K is a closed subset of X. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{F}$ the following assertions hold:*

1. $\inf(f_a)$ *is finite and attained at a unique point* $x^{(a)} \in K$.

2. For any positive number ϵ there exist a positive number δ and a neighborhood U of a in $\mathcal{A}$ with the weak topology such that for each $b \in U$, $\inf(f_b)$ is finite, and if $x \in K$ satisfies $f_b(x) \le \inf(f_b) + \delta$, then $|f_a(x^{(a)}) - f_b(x)| < \epsilon$ and there exists $u \in X$ such that $||u|| < \epsilon$ and $x \le x^{(a)} + u$.

The proof of Theorem 7.2 is analogous to the proof of Theorem 7.1. Note that in view of (H1) we can assume that $x(a, n)$ is a minimal element of K. Then x_a constructed in the proof of Theorem 7.1 is also a minimal element of K. This implies the uniqueness of a minimizer of the function f_a for all $a \in \mathcal{F}$.

In this chapter we use the following functional $\lambda : X \to R^1$ defined by

$$\lambda(x) = \inf\{||y|| : y \ge x\},\ x \in X. \tag{7.24}$$

The function λ introduced in [88] has the following properties.

(i) The function λ is sublinear. Namely,

$$\lambda(\alpha x) = \alpha\lambda(x) \text{ for all } \alpha \ge 0 \text{ and all } x \in X,$$

$$\lambda(x_1 + x_2) \le \lambda(x_1) + \lambda(x_2) \text{ for all } x_1, x_2 \in X.$$

(ii) $\lambda(x) = 0$ if $x \le 0$.
(iii) If $x_1, x_2 \in X$ and $x_1 \le x_2$, then $\lambda(x_1) \le \lambda(x_2)$.
(iv) $0 \le \lambda(x) \le \|x\|$ for all $x \in X$.
It is easy to see that for each $x, y \in X$,

$$|\lambda(x) - \lambda(y)| \le ||x - y||. \tag{7.25}$$

7.3 Spaces of increasing coercive functions

Denote by $\mathcal{M}$ the set of all increasing lower semicontinuous bounded from below functions $f : K \to R^1 \cup \{+\infty\}$ which are not identically $+\infty$. Recall that a function $f : K \to R^1 \cup \{+\infty\}$ is called increasing if

$$f(x) \le f(y) \text{ for all } x, y \in X \text{ such that } x \le y.$$

We equip the set $\mathcal{M}$ with strong and weak topologies. For the space $\mathcal{M}$ we consider the uniformity determined by the following base:

$$E_s(n) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(x) - g(x)| \le n^{-1} \text{ for all } x \in K\} \tag{7.26}$$

where n is a natural number. It is easy to see that this uniform space $\mathcal{M}$ is metrizable (by a metric d) and complete. Denote by τ_s the topology in $\mathcal{M}$ induced by this uniformity. The topology τ_s is called the strong topology.

For each $(x, \alpha) \in X \times R^1$ put $||(x, \alpha)|| = ||x|| + |\alpha|$. For each lower semicontinuous bounded from below function $f : K \to R^1 \cup \{+\infty\}$ with a nonempty epigraph epi(f) define a function $\Delta_f : K \times R^1 \to R^1$ by

$$\Delta_f(x, \alpha) = \inf\{||x - y|| + |\alpha - \beta| : \ (y, \beta) \in \text{ epi}(f)\}, \ (x, \alpha) \in K \times R^1. \tag{7.27}$$

For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_w(n) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \ |\Delta_f(x, \alpha) - \Delta_g(x, \alpha)| \le n^{-1} \tag{7.28}$$
$$\text{for all } (x, \alpha) \in K \times R^1 \text{ satisfying } ||x|| + |\alpha| \le n\}$$

where $n = 1, 2, \dots$. This uniform space $\mathcal{M}$ is metrizable (by a metric d_w). Denote by τ_w the epi-distance topology in $\mathcal{M}$ induced by this uniformity (see [2]). The topology τ_w is called the weak topology. It is clear that the topology τ_w is weaker than τ_s.

Assume that $\phi : K \to R^1 \cup \{+\infty\}$ satisfies

$$\phi(x) \to +\infty \text{ as } ||x|| \to +\infty. \tag{7.29}$$

Denote by $\mathcal{M}(\phi)$ the set of all $f \in \mathcal{M}$ such that

$$f(x) \ge \phi(x) \text{ for all } x \in K \tag{7.30}$$

and by $\mathcal{M}_c(\phi)$ the set of all continuous finite-valued functions $f \in \mathcal{M}(\phi)$.

Evidently, $\mathcal{M}_c(\phi)$ and $\mathcal{M}(\phi)$ are closed subsets of $\mathcal{M}$ with the strong topology. We consider the topological subspaces $\mathcal{M}_c(\phi)$, $\mathcal{M}(\phi) \subset \mathcal{M}$ with the relative weak and strong topologies.

We say that the set K has property (P) if for each $x \in K$ there is a minimal element $y \in K$ such that $y \leq x$ and the set of all minimal elements of K is a closed subset of X.

Remark 7.3. It follows from Zorn's lemma that property (P) holds if the Banach space X is reflexive, the set K is bounded from below and convex, and the set of all minimal elements of K is a closed subset of X.

The following theorem was proved in [114].

Theorem 7.4. *Assume that $\mathcal{A}$ is either $\mathcal{M}(\phi)$ or $\mathcal{M}_c(\phi)$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the following assertions hold:*

1. $\inf(f)$ *is finite and attained at a point $x^{(f)} \in K$ such that for each $x \in K$ satisfying $f(x) = \inf(f)$ the inequality $x \leq x_f$ holds.*

2. For each positive number ϵ there exist a positive number δ and a neighborhood U of f in $\mathcal{A}$ with the weak topology such that for each $g \in U$, $\inf(g)$ *is finite, and if $x \in K$ satisfies $g(x) \leq \inf(g) + \delta$, then $|f(x^{(f)}) - g(x)| < \epsilon$ and there exists $u \in X$ such that $||u|| < \epsilon$ and $x \leq x^{(f)} + u$. Moreover if K has the property (P), then* $\inf(f_a)$ *is attained at the unique point $x^{(f)}$.*

7.4 Proof of Theorem 7.4

Lemma 7.5. *Let $f \in \mathcal{A}$. Then there exist a neighborhood U of f in $\mathcal{A}$ with the weak topology and a real number c such that for each $g \in U$ the inequality $g(x) \geq c$ holds for all $x \in K$.*

Proof: There exists $c_0 \in R^1$ such that $f(x) \geq c_0$ for all $x \in K$. Relation (7.29) implies that there exists a positive number c_1 such that

$$\phi(x) \geq 4(|c_0| + 2) \text{ for all } x \in K \text{ satisfying } ||x|| \geq c_1. \tag{7.31}$$

Fix a number $c < 0$ and an integer $n \geq 1$ such that

$$c < c_0 - 3, \tag{7.32}$$

$$n > |c| + c_1. \tag{7.33}$$

Put

$$U = \{g \in \mathcal{A} : (f, g) \in \mathcal{E}_w(n)\}. \tag{7.34}$$

Let $g \in U$ and $x \in K$. We will show that

$$g(x) \geq c. \tag{7.35}$$

It follows from (7.31), (7.32), (7.29) and (7.30) that we may assume that

$$||x|| < c_1. \tag{7.36}$$

Assume that (7.35) does not hold. Then $g(x) < c$ and

$$(x, c) \in \text{ epi}(g). \tag{7.37}$$

It follows from (7.36) and (7.33) that $||x|| + c < |c| + c_1 < n$. In view of this inequality, (7.37), (7.34) and (7.28), $\Delta_f(x, c) \leq n^{-1}$ and there exists $(y, \beta) \in$ epi(f) such that $||x-y||+|\beta-c| \leq 2$. Thus $\beta \geq f(y)$, $\beta \leq 2+c$ and $f(y) \leq 2+c$. Combined with (7.32) this implies that $f(y) \leq c_0 - 1$, a contradiction. Thus (7.35) holds. This completes the proof of Lemma 7.5.

We can easily prove the following auxiliary result.

Lemma 7.6. *Assume that $f \in \mathcal{A}$ and δ is a positive number. Then there exists a neighborhood U of f in $\mathcal{A}$ with the weak topology such that* $\inf(g) \leq \inf(f)+\delta$ *for each $g \in U$.*

By using Lemmas 7.5, 7.6, the growth condition (7.29) and the definition of the epi-distance topology we can prove in a straightforward manner the following result.

Lemma 7.7. *Let $f \in \mathcal{A}$ and γ be a positive number. Then there exists a neighborhood U of f in $\mathcal{A}$ with the weak topology such that* $\inf(g) > \inf(f) - \gamma$ *for each $g \in U$.*

Denote by $\mathcal{B}$ the set of all $f \in \mathcal{A}$ for which there is $x_f \in K$ such that $f(x_f) = \inf(f)$. If K has the property (P) we may assume that x_f is a minimal element of K for each $f \in \mathcal{B}$.

Lemma 7.8. *Let $f \in \mathcal{A}$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{B}$ such that $f_n(x) \geq f(x)$ for all $x \in K$ and all natural numbers n and $f_n \to f$ as $n \to \infty$ in the space $\mathcal{A}$ with the strong topology.*

Proof: For each natural number n there exists $x_n \in K$ such that $f(x_n) \leq \inf(f) + n^{-1}$. For $n = 1, 2, \ldots$ define

$$f_n(x) = \max\{f(x), f(x_n)\} \text{ for all } x \in K.$$

Evidently, $f_n \in \mathcal{B}$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} f_n = f$ in the strong topology. This completes the proof of Lemma 7.8.

Lemma 7.9. *Let $f \in \mathcal{B}$, $x_f \in K$,*

$$f(x_f) = \inf(f),\ \epsilon \in (0, 1) \textit{ and } \delta \in (0, 16^{-1}\epsilon^2). \tag{7.38}$$

Define a function $\bar{f} \in \mathcal{A}$ *by*

$$\bar{f}(x) = f(x) + 2^{-1}\epsilon \min\{\lambda(x - x_f), 1\} \text{ for all } x \in K. \tag{7.39}$$

Then there exist a neighborhood U *of* $\bar{f}$ *in* $\mathcal{A}$ *with the weak topology and a number* $c > 0$ *such that the following assertions are true:*

1. For each $g \in U$ *and each* $x \in K$ *satisfying* $g(x) \leq \inf(g) + \delta$ *the inequalities*

$$|g(x) - f(x_f)| \leq \epsilon, \ ||x|| \leq c \tag{7.40}$$

hold and there is $v \in X$ *such that*

$$||v|| \leq \epsilon, \ x \leq x_f + v. \tag{7.41}$$

2. For each $g \in U$ *there exists* $z \in K$ *such that*

$$||z - x_f|| \leq \epsilon, \ |g(z) - f(x_f)| \leq \epsilon \text{ and } g(z) \leq \inf(g) + \epsilon. \tag{7.42}$$

Proof: It follows from Lemmas 7.6 and 7.7 that there exists a neighborhood U_0 of $\bar{f}$ in $\mathcal{A}$ with the weak topology such that

$$|\inf(\bar{f}) - \inf(g)| \leq 2^{-1}\delta \text{ for all } g \in U_0. \tag{7.43}$$

In view of (7.29) there exists a number $c > 1$ such that

$$\phi(x) \geq |\inf(\bar{f})| + 4 \text{ for all } x \in K \text{ satisfying } ||x|| \geq c. \tag{7.44}$$

Fix an integer

$$n > |\inf(\bar{f})| + 4 + 4\delta^{-1} + c + ||x_f|| \tag{7.45}$$

and put

$$U = U_0 \cap \{g \in \mathcal{A} : \ (\bar{f}, g) \in E_w(n)\}. \tag{7.46}$$

Assume that

$$g \in U, \ x \in K \text{ and } g(x) \leq \inf(g) + \delta. \tag{7.47}$$

In view of (7.47) and (7.43),

$$g(x) \leq \inf(\bar{f}) + 2\delta. \tag{7.48}$$

It follows from (7.48), (7.44) and (7.30) that

$$||x|| \leq c. \tag{7.49}$$

By (7.49), (7.48), (7.46), (7.28) and (7.45), there exist $(y, \alpha) \in \text{epi}(\bar{f})$ such that

$$||x - y|| \leq 2n^{-1} \text{ and } |\alpha - \inf(\bar{f}) + 2\delta| \leq 2/n. \tag{7.50}$$

By (7.50), (7.39), (7.38) and (7.45),

$$f(y) + 2^{-1}\epsilon \min\{\lambda(y - x_f), 1\} = \bar{f}(y) \leq \alpha \leq \inf(\bar{f}) + 2\delta + 2/n \leq$$

$$\inf(\bar{f}) + 3\delta = f(x_f) + 3\delta \le f(y) + 3\delta,$$
$$\min\{\lambda(y - x_f), 1\} \le 6\delta\epsilon^{-1}$$

and

$$\lambda(y - x_f) \le 6\epsilon/16. \tag{7.51}$$

It follows from the definition of λ (see (7.24)) that there exists $u \in X$ such that $y \le x_f + u$, $||u|| \le 2^{-1}\epsilon$. Combined with (7.50) and (7.45) this implies that

$$x \le x_f + u + x - y, \ ||x - y + u|| \le \epsilon. \tag{7.52}$$

(7.40) and (7.41) now follow from (7.52), (7.49), (7.47) and (7.43). Since $(x_f, f(x_f)) \in \text{epi}(\bar{f})$ relations (7.46), (7.28) and (7.47) imply that there exists $(z, \beta) \in \text{epi}(g)$ such that

$$||z - x_f|| + |\beta - f(x_f)| \le 2/n < 2^{-1}\delta. \tag{7.53}$$

It follows from (7.53) and (7.43) that

$$f(x_f) - \delta \le \inf(g) \le g(z) \le \beta \le f(x_f) + \delta/2 \le \inf(g) + \delta.$$

Lemma 7.9 is proved.

For each $a \in \mathcal{A}$ set $f_a = a$. The hypotheses (H) follow from Lemmas 7.8 and 7.9. If K has the property (P), then these lemmas imply (H1). Theorem 7.4 now follows from Theorems 7.1 and 7.2.

7.5 Spaces of increasing noncoercive functions

We consider the space $\mathcal{M}$ with the complete metrizable (by the metric d) uniformity which is determined by the base $E_s(n)$, $n = 1, 2, \dots$ (see (7.26)) and with the the strong topology τ_s induced by this uniformity.

For the set $\mathcal{M}$ we consider the uniformity determined by the following base:

$$\widehat{\mathcal{E}}_w(n) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \ \sup\{\Delta_f(x, \alpha) : \ (x, \alpha) \in \ \text{epi}(g)\} \le 1/n, \tag{7.54}$$
$$\sup\{\Delta_g(x, \alpha) : \ (x, \alpha) \in \ \text{epi}(f)\} \le 1/n\},$$

where $n = 1, 2, \dots$. The space $\mathcal{M}$ with this uniformity is metrizable (by a metric d_w). The topology induced by this uniformity it is called the weak topology. Denote by $\mathcal{M}_c$ the set of all continuous finite-valued functions $f \in \mathcal{M}$. Clearly $\mathcal{M}_c$ is a closed subset of $\mathcal{M}$ with the strong topology. We consider the topological subspace $\mathcal{M}_c \subset \mathcal{M}$ with the relative weak and strong topologies.

In this section we use the property (P) introduced in Section 7.3. The following theorem was obtained in [114].

Theorem 7.10. *Assume that there is $\bar{z} \in X$ such that $\bar{z} \leq x$ for all $x \in K$ and assume that $\mathcal{A}$ is either $\mathcal{M}$ or $\mathcal{M}_c$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the following assertions hold:*

1. $\inf(f)$ *is finite and attained at a point $x^{(f)} \in K$ such that for each $x \in K$ satisfying $f(x) = \inf(f)$ the inequality $x \leq x_f$ holds.*

2. For any positive number ϵ there exist a positive number δ and a neighborhood U of f in $\mathcal{A}$ with the weak topology such that for each $g \in U$, $\inf(g)$ *is finite, and if $x \in K$ satisfies $g(x) \leq \inf(g) + \delta$, then $|f(x^{(f)}) - g(x)| < \epsilon$ and there is $u \in X$ such that $||u|| < \epsilon$ and $x \leq x^{(f)} + u$.*

Moreover if K has the property (P), then $\inf(f_a)$ *is attained at the unique point $x^{(f)}$.*

7.6 Proof of Theorem 7.10

We can easily prove the following auxiliary result.

Lemma 7.11. *Let $f \in \mathcal{A}$ and let ϵ be a positive number. Then there exists a neighborhood U of f in $\mathcal{A}$ with the weak topology such that $|\inf(g) - \inf(f)| \leq \epsilon$ for all $g \in U$.*

Denote by $\mathcal{B}$ the set of all $f \in \mathcal{A}$ for which there is $x_f \in K$ such that $f(x_f) = \inf(f)$. If K has the property (P), we may assume that x_f is a minimal element of K for each $f \in \mathcal{B}$.

Analogously to Lemma 7.8 we can prove the following lemma.

Lemma 7.12. *The set $\mathcal{B}$ is everywhere dense in $\mathcal{A}$ with the strong topology.*

Lemma 7.13. *Let $f \in \mathcal{B}$, $x_f \in K$,*

$$f(x_f) = \inf(f),\ \epsilon \in (0,1),\ \delta \in (0, 16^{-1}\epsilon^2). \tag{7.55}$$

Define a function $\bar{f} \in \mathcal{A}$ by

$$\bar{f}(x) = f(x) + 2^{-1}\epsilon \min\{\lambda(x - x_f), 1\} \text{ for all } x \in K. \tag{7.56}$$

Then there exist a neighborhood U of $\bar{f}$ in $\mathcal{A}$ with the weak topology and a number $c > 0$ such that for each $g \in U$ the following properties hold:

If $x \in K$ satisfies $g(x) \leq \inf(g) + \delta$, then there is $v \in X$ such that

$$||v|| \leq \epsilon,\ x \leq x_f + v \text{ and } |g(x) - f(x_f)| \leq \epsilon,\ ||x|| \leq c; \tag{7.57}$$

for each $g \in U$ there is $z \in K$ such that

$$||z - x_f|| \leq \epsilon,\ |g(z) - f(x_f)| \leq \epsilon \text{ and } g(z) \leq \inf(g) + \epsilon. \tag{7.58}$$

Proof: There exists $\bar{z} \in X$ such that

$$\bar{z} \le x \text{ for all } x \in K. \tag{7.59}$$

Lemma 7.11 implies that there exists a neighborhood U_0 of f in $\mathcal{A}$ with the weak topology such that

$$|\inf(\bar{f}) - \inf(g)| \le 2^{-1}\delta \text{ for all } g \in U_0. \tag{7.60}$$

Fix

$$c > 4(||\bar{z}|| + ||x_f|| + 1), \tag{7.61}$$

an integer

$$n > |\inf(\bar{f})| + 4 + 4\delta^{-1} \tag{7.62}$$

and put

$$U = U_0 \cap \{g \in \mathcal{A} :\ (f, g) \in \widehat{\mathcal{E}}_w(n)\}. \tag{7.63}$$

Assume that

$$g \in U,\ x \in K,\ g(x) \le \inf(g) + \delta. \tag{7.64}$$

In view of (7.60), (7.55) and (7.56),

$$|g(x) - f(x_f)| = |g(x) - \inf(\bar{f})| \le 2\delta. \tag{7.65}$$

Thus $(x, \inf(\bar{f}) + 2\delta) \in \mathrm{epi}(g)$. It follows from (7.65), (7.63) and (7.54) that there exists $(y, \alpha) \in \mathrm{epi}(\bar{f})$ such that

$$||y - x|| \le 2/n,\ |\alpha - (\inf(\bar{f}) + 2\delta)| \le 2/n. \tag{7.66}$$

By (7.56), (7.66), (7.62) and (7.55),

$$f(y) + 2^{-1}\epsilon \min\{\lambda(y - x_f), 1\} = \bar{f}(y) \le \alpha \le$$

$$\inf(\bar{f}) + 2\delta + 2/n \le f(x_f) + 3\delta \le f(y) + 3\delta,$$

$$\min\{\lambda(y - x_f), 1\} \le 6\delta\epsilon^{-1} \le 6 \cdot 16^{-1}\epsilon,$$

and

$$\lambda(y - x_f) \le 6 \cdot 16^{-1}\epsilon.$$

It follows from the definition of λ (see (7.24)) that there exists $u \in X$ such that $||u|| \le 2^{-1}\epsilon$, $y \le x_f + u$. Combined with (7.66) and (7.62) this implies that

$$x \le x_f + x - y + u,\ ||x - y + u|| \le \epsilon. \tag{7.67}$$

It follows from (7.67), (7.59) and (7.61) that

$$||x|| \le ||x - \bar{z}|| + ||\bar{z}|| \le ||\bar{z}|| + ||x_f + x - y + u - \bar{z}|| \le \tag{7.68}$$

$$2||\bar{z}|| + ||x_f|| + 1 < c.$$

(7.57) now follows from (7.65), (7.67) and (7.68). Since $(x_f, f(x_f)) \in \mathrm{epi}(\bar{f})$ relations (7.63), (7.54) and (7.62) imply that there exists $(z, \beta) \in \mathrm{epi}(g)$ such that

$$||z - x_f|| + |\beta - f(x_f)| \leq 2/n < \delta/2.$$

This relation and (7.60) imply that

$$f(x_f) - \delta \leq \inf(g) \leq g(z) \leq \beta \leq f(x_f) + \delta/2 \leq \inf(g) + \delta.$$

Lemma 7.13 is proved.

For each $a \in \mathcal{A}$ set $f_a = a$. The hypotheses (H) follow from Lemmas 7.12 and 7.13. If K has the property (P), then these lemmas imply (H1). Theorem 7.10 now follows from Theorems 7.1 and 7.2.

7.7 Spaces of increasing quasiconvex functions

Assume that the set K is convex. We consider the space $\mathcal{M}$ with the weak and strong topologies (uniformities) introduced in Section 7.3 (see (7.26) and (7.28)). A function $f \in \mathcal{M}$ is called quasiconvex if for each $\alpha \in R^1$ the set $\{x \in K : f(x) \leq \alpha\}$ is convex.

Denote by $\mathcal{M}_q$ the set of all quasiconvex functions $f \in \mathcal{M}$ and by $\mathcal{M}_{qc}$ the set of all finite-valued continuous quasiconvex functions $f \in \mathcal{M}$. It is clear that $\mathcal{M}_q$ and $\mathcal{M}_{qc}$ are closed subsets of $\mathcal{M}$ with the strong topology.

Let $\phi : K \to R^1 \cup \{+\infty\}$ satisfy

$$\phi(x) \to +\infty \text{ as } ||x|| \to +\infty. \tag{7.69}$$

Consider the spaces $\mathcal{M}_c(\phi)$ and $\mathcal{M}(\phi)$ defined in Section 7.3 and set

$$\mathcal{M}_q(\phi) = \mathcal{M}_q \cap \mathcal{M}(\phi),\ \mathcal{M}_{qc}(\phi) = \mathcal{M}_{qc} \cap \mathcal{M}(\phi). \tag{7.70}$$

Evidently, $\mathcal{M}_q(\phi)$, $\mathcal{M}_{qc}(\phi)$ are closed subsets of $\mathcal{M}$ with the strong topology. We consider the topological subspaces $\mathcal{M}_q$, $\mathcal{M}_{qc}$, $\mathcal{M}_q(\phi)$, $\mathcal{M}_{qc}(\phi) \subset \mathcal{M}$ with the relative strong and weak topologies.

The following theorem was obtained in [114].

Theorem 7.14. *Assume that one of the following cases holds:*

1. $\mathcal{A}$ is either $\mathcal{M}_q(\phi)$ or $\mathcal{M}_{qc}(\phi)$;

2. there is $\bar{z} \in X$ such that $\bar{z} \leq x$ for all $x \in K$ and $\mathcal{A}$ is either $\mathcal{M}_q$ or $\mathcal{M}_{qc}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the following assertions hold:

1. $\inf(f)$ *is finite and attained at a point $x^{(f)} \in K$ such that for each $x \in K$ satisfying $f(x) = \inf(f)$ the inequality $x \leq x_f$ holds.*

2. For any positive number ϵ there exist a positive number δ and a neighborhood U of f in $\mathcal{A}$ with the weak topology such that for each $g \in U$, $\inf(g)$

is finite, and if $x \in K$ *satisfies* $g(x) \le \inf(g) + \delta$*, then* $|f(x^{(f)}) - g(x)| < \epsilon$ *and there exists* $u \in X$ *such that* $||u|| < \epsilon$ *and* $x \le x^{(f)} + u$*.*

Moreover if K *has the property (P), then* $\inf(f)$ *is attained at the unique point* $x^{(f)}$*.*

7.8 Proof of Theorem 7.14

Lemma 7.15. *Let* $f \in \mathcal{M}(\phi)$ *and* ϵ *be a positive number. Then there exist a neighborhood* U *of* f *in* $\mathcal{M}(\phi)$ *with the weak topology and a positive number* c *such that* $|\inf(f) - \inf(g)| \le \epsilon$ *for all* $g \in U$ *and for each* $g \in U$ *and each* $z \in K$ *satisfying* $g(z) \le \inf(g) + 4$ *the inequality* $||z|| \le c$ *is true.*

Lemma 7.14 follows from Lemmas 7.6 and 7.7 and the growth condition (7.69).

Denote by $\mathcal{B}$ the set of all $f \in \mathcal{A}$ for which there exists $x_f \in K$ such that $f(x_f) = \inf(f)$. If K has property (P), we may assume that x_f is a minimal element of K for all $f \in \mathcal{B}$. Analogously to Lemma 7.8 we can prove the following auxiliary result.

Lemma 7.16. *The set* $\mathcal{B}$ *is everywhere dense in the space* $\mathcal{A}$ *with the strong topology.*

It is easy to prove in a straightforward manner the following result.

Lemma 7.17. *Let* $f \in \mathcal{B}$*,* $x_f \in K$ *and* $f(x_f) = \inf(f)$*. For each* $\delta \in (0, 1)$ *define* $f_\delta : K \to R^1 \cup \{\infty\}$ *by*

$$f_\delta(x) = \max\{f(x), f(x_f) + \delta \min\{\lambda(x - x_f), 1\}\} \text{ for all } x \in K. \quad (7.71)$$

Then $f_\delta \in \mathcal{A}$ *for each* $\delta \in (0, 1)$ *and* $f_\delta \to f$ *as* $\delta \to 0$ *in* $\mathcal{A}$ *with the strong topology.*

Lemma 7.18. *Assume that* $\bar{z} \in X$*,*

$$\bar{z} \le x \text{ for all } x \in K, \; f \in \mathcal{M}_q, \; x_f \in K, \; f(x_f) = \inf(f), \quad (7.72)$$

$\delta \in (0, 1)$ *and* f_δ *is defined by (7.71). Then there exist a neighborhood* U *of* f_δ *in* $\mathcal{M}_q$ *with the weak topology and a positive number* c *such that for each* $g \in U$ *and each* $z \in K$ *satisfying* $||z|| \ge c$ *the inequality* $g(z) \ge f(x_f) + 64^{-1}\delta^2$ *holds.*

Proof: Let $y \in K$. Then $y \ge \bar{z}$ and there is $v \in X$ such that

$$y - x_f \le v, \; ||v|| \le \lambda(y - x_f) + 1.$$

We have

$$||y|| \le ||y - \bar{z}|| + ||\bar{z}|| \le ||\bar{z}|| + ||x_f + v - \bar{z}|| \le$$

$$2||\bar{z}|| + ||x_f|| + ||v|| \le 2||\bar{z}|| + ||x_f|| + \lambda(y - x_f) + 1.$$

Hence

$$||y|| \le \lambda(y - x_f) + 2||\bar{z}|| + ||x_f|| + 1 \text{ for all } y \in K. \tag{7.73}$$

Fix

$$c \ge 4(2||\bar{z}|| + ||x_f|| + 1), \tag{7.74}$$

an integer $n \ge 1$ satisfying

$$n > ||x_f|| + 4c + |f(x_f)| + 8 + 128\delta^{-2} \tag{7.75}$$

and put

$$U = \{g \in \mathcal{M}_q : \ (f_\delta, g) \in \mathcal{E}_w(n)\} \tag{7.76}$$

(see (7.28)). Assume that $g \in U$,

$$x \in K, \ \lambda(x - x_f) = 4^{-1}c. \tag{7.77}$$

Relations (7.77), (7.73) and (7.74) imply that

$$||x|| \le c. \tag{7.78}$$

We show that

$$g(x) > f(x_f) + 64^{-1}\delta^2. \tag{7.79}$$

Let us assume the contrary. Then $(x, f_\delta(x_f) + \delta^2/64) \in \mathrm{epi}(g)$ and it follows from (7.76), (7.75), (7.78) and (7.28) that

$$\Delta_{f_\delta}(x, f_\delta(x_f) + \delta^2/64) \le 1/n.$$

Therefore there exists $(y, \beta) \in \mathrm{epi}(f_\delta)$ such that

$$||y - x||, \ |\beta - f_\delta(x_f) - 64^{-1}\delta^2| < 2/n. \tag{7.80}$$

Combined with (7.75) this implies that

$$||y - x|| < 64^{-1}\delta^2 \tag{7.81}$$

and

$$f_\delta(y) \le \beta \le f_\delta(x_f) + \delta^2/32 = f(x_f) + \delta^2/32.$$

Combined with (7.71) this inequality implies that

$$f(x_f) + \delta^2/32 \ge f_\delta(y) = \max\{f(y), f(x_f) + \delta \min\{\lambda(y - x_f), 1\}\},$$

$$\min\{\lambda(y - x_f), 1\} \le \delta/32 \text{ and } \lambda(y - x_f) \le \delta/32.$$

It follows from (7.81) that $\lambda(x - x_f) < 1/2$. This is contradictory to (7.77). The contradiction we have reached proves that (7.79) holds for each $x \in K$ satisfying (7.77).

Assume that $g \in U$,

$$z \in K,\ \lambda(z - x_f) > 4^{-1}c. \tag{7.82}$$

We show that

$$g(z) \geq f(x_f) + 64^{-1}\delta^2. \tag{7.83}$$

Let us assume the contrary. Then

$$g(z) < f(x_f) + 64^{-1}\delta^2. \tag{7.84}$$

Since $(x_f, f(x_f)) \in \text{epi}(f_\delta)$ relations (7.75), (7.76) and (7.28) imply that there exists $(y, \beta) \in \text{epi}(g)$ such that

$$||y - x_f|| \leq 2/n < 64^{-1}\delta^2,\ |\beta - f(x_f)| < 2/n < 64^{-1}\delta^2.$$

Thus

$$\lambda(y - x_f) \leq ||y - x_f|| < 64^{-1}\delta^2,\ g(y) \leq \beta < f(x_f) + 64^{-1}\delta^2. \tag{7.85}$$

It follows from (7.85), (7.82) and (7.75) that the line segment joining z and y contains a point h with $\lambda(h - x_f) = 4^{-1}c$. By (7.79) which holds with $x = h$, (7.84) and (7.85),

$$g(h) > f(x_f) + 64^{-1}\delta^2 \geq \max\{g(z), g(y)\},$$

a contradiction. Since the inequality $||z|| \geq c$ implies $\lambda(z - x_f) > 4^{-1}c$ (see (7.73), (7.74)), the obtained contradiction completes the proof of Lemma 7.18.

Lemma 7.19. *Assume that $\bar{z} \in X$, $\bar{z} \leq x$ for all $x \in K$, $f \in \mathcal{M}_q$, $x_f \in K$, $f(x_f) = \inf(f)$, $\delta \in (0,1)$, f_δ is defined by (7.71) and $\epsilon \in (0,1)$. Then there exist a neighborhood U of f_δ in $\mathcal{M}_q$ with the weak topology and a positive number c such that $|\inf(f_\delta) - \inf(g)| \leq \epsilon$ for all $g \in U$, and for each $g \in U$ and each $z \in K$ satisfying $g(z) \leq \inf(g) + 128^{-1}\delta^2$ the inequality $||z|| \leq c$ holds.*

Proof: We may assume that $\epsilon < (128 \cdot 4)^{-1}\delta^2$. Lemma 7.18 implies that there exist a neighborhood U_0 of f_δ in $\mathcal{M}_q$ with the weak topology and $c > 0$ such that for each $g \in U_0$ and each $z \in K$ satisfying $||z|| \geq c$ the inequality

$$g(z) \geq f(x_f) + 64^{-1}\delta^2 \tag{7.86}$$

holds. Fix an integer

$$n > 4(||x_f|| + |f(x_f)| + 1 + c) + 8\epsilon^{-1} \tag{7.87}$$

and put

$$U = U_0 \cap \{g \in \mathcal{M}_q :\ (f_\delta, g) \in \mathcal{E}_w(n)\}. \tag{7.88}$$

Assume that $g \in U$. It follows from (7.88), (7.28) and (7.87) that

$$\Delta_g(x_f, f(x_f)) \le n^{-1}$$

and that there exists $(y, \beta) \in \mathrm{epi}(g)$ such that

$$||x_f - y|| < 2/n, \; |\beta - f(x_f)| < 2/n, \tag{7.89}$$

$$\inf(g) \le g(y) \le \beta < f(x_f) + 2/n < f(x_f) + \epsilon/2 = \inf(f_\delta) + \epsilon/2.$$

Now we show that $\inf(f_\delta) \le \inf(g) + \epsilon$. There is $z \in K$ such that

$$g(z) \le \inf(g) + \epsilon/4. \tag{7.90}$$

It follows from (7.90), (7.89) and the inequality $\epsilon < (128 \cdot 4)^{-1}\delta^2$ that

$$g(z) \le f(x_f) + 3/4\epsilon < f(x_f) + (128 \cdot 4)^{-1}\delta^2,$$

and in view of the definition of U_0,

$$||z|| < c_0. \tag{7.91}$$

Assume that

$$\inf(f_\delta) > \inf(g) + \epsilon. \tag{7.92}$$

Relations (7.92) and (7.90) imply that $g(z) < f(x_f) - \epsilon/2$ and

$$(z, f(x_f) - \epsilon/2) \in \; \mathrm{epi}(g). \tag{7.93}$$

By (7.93), (7.91), (7.87) and (7.88),

$$\Delta_{f_\delta}(z, f(x_f) - \epsilon/2) \le 1/n$$

and there exists $(\bar{z}, \bar{\gamma}) \in \mathrm{epi}(f_\delta)$ such that

$$||z - \bar{z}|| \le 2/n, \; |\bar{\gamma} - f(x_f) + \epsilon/2| < 2/n.$$

Hence

$$f_\delta(\bar{z}) \le \bar{\gamma} \le f(x_f) - \epsilon/2 + 2/n < f(x_f),$$

a contradiction (see (7.87)).

The contradiction we have reached proves that $\inf(f_\delta) \le \inf(g) + \epsilon$. Together with (7.89) this implies that

$$|\inf(g) - \inf(f_\delta)| \le \epsilon \text{ for all } g \in U. \tag{7.94}$$

Assume that $g \in U$,

$$z \in K, \; g(z) \le \inf(g) + 128^{-1}\delta^2.$$

Since $\epsilon < (128 \cdot 4)^{-1}\delta^2$ relation (7.94) implies that $g(z) < f(x_f) + 64^{-1}\delta^2$. In view of this inequality and the definition of U_0, $||z|| < c$. This completes the proof of Lemma 7.19.

Lemma 7.20. *Let $f \in \mathcal{B}$, $x_f \in K$, $f(x_f) = \inf(f)$, $\delta \in (0,1)$ f_δ is defined by (7.71),*

$$\epsilon \in (0,1),\ \gamma \in (0, (8^{-1}\epsilon\delta)^2 128^{-1}). \tag{7.95}$$

Then there exist a neighborhood U of f_δ in $\mathcal{A}$ with the weak topology and $c > 0$ such that for each $g \in U$ the following assertions hold:

For each $x \in K$ satisfying $g(x) \le \inf(g) + \gamma$,

$$|g(x) - f(x_f)| \le \epsilon,\ ||x|| \le c \ \text{ and there is } v \in X \text{ such that } ||v|| \le \epsilon,\ x \le x_f + v; \tag{7.96}$$

there is $z \in K$ such that

$$||z - x_f|| \le \epsilon,\ g(z) \le \inf(g) + \epsilon. \tag{7.97}$$

Proof: Lemmas 7.19 and 7.15 imply that there exist a neighborhood U_0 of f_δ in $\mathcal{A}$ with the weak topology and a positive number c such that

$$|\inf(g) - \inf(f_\delta)| \le 8^{-1}\gamma \text{ for all } g \in U_0 \tag{7.98}$$

and that for each $g \in U_0$ and each $x \in K$ satisfying $g(x) \le \inf(g) + 128^{-1}\delta^2$,

$$||x|| \le c. \tag{7.99}$$

Fix an integer

$$n > 4(||x_f|| + |f(x_f)| + 1 + c) + 8^{-1}\gamma \tag{7.100}$$

and put

$$U = U_0 \cap \{g \in \mathcal{A} : (f_\delta, g) \in \mathcal{E}_w(n)\} \tag{7.101}$$

(see (7.28)). Assume that $g \in U$, $x \in K$ and

$$g(x) \le \inf(g) + \gamma. \tag{7.102}$$

It follows from (7.95), (7.102) and the definition of U_0 that inequality (7.99) holds. Relations (7.102) and (7.98) imply that

$$|g(x)| \le \gamma + |\inf(g)| \le \gamma + |f(x_f)| + 8^{-1}\gamma. \tag{7.103}$$

It follows from (7.103), (7.99), (7.100) and (7.101) that $\Delta_{f_\delta}(x, g(x)) \le n^{-1}$. Therefore there exists $(y, \alpha) \in \mathrm{epi}(f_\delta)$ such that

$$||x - y|| < 2/n,\ |g(x) - \alpha| < 2/n.$$

Combined with (7.100) this implies that

$$||x - y|| < 4^{-1}\gamma,\ |g(x) - \alpha| < 4^{-1}\gamma. \tag{7.104}$$

In view of (7.104), (7.102) and (7.98),

$$f(x_f) \le f_\delta(y) \le \alpha \le g(x) + 4^{-1}\gamma \le \inf(g) + \gamma + 4^{-1}\gamma \le \tag{7.105}$$

$$f(x_f) + \gamma + 4^{-1}\gamma + 8^{-1}\gamma.$$

By (7.71), (7.105) and (7.95),

$$f(x_f) + \delta \min\{\lambda(y - x_f), 1\} \le f_\delta(y) \le f(x_f) + 3/2\gamma,$$

$$\min\{\lambda(y - x_f), 1\} \le \delta^{-1}(3/2)\gamma \le 3(2\delta)^{-1}(8^{-1}\epsilon\delta)^2 128^{-1}$$

and

$$\lambda(y - x_f) \le 2^{-14} 3\epsilon^2\delta. \tag{7.106}$$

In view of (7.106) and (7.104),

$$\lambda(x - x_f) < \epsilon/2.$$

It follows from this inequality and the definition of λ that there exists $v \in X$ such that $||v|| < \epsilon$ and $x \le x_f + v$. Clearly, (7.105) and (7.99) imply (7.96). It follows from (7.101) and (7.100) that $\Delta_g(x_f, f(x_f)) \le 1/n$ and there exists $(z, \beta) \in \text{epi}(g)$ for which

$$||z - x_f|| < 2/n, \ |\beta - f(x_f)| < 2/n. \tag{7.107}$$

By (7.107), (7.100), (7.95) and (7.98), $||z - x_f|| < \epsilon$ and

$$g(z) \le \beta \le f(x_f) + 2/n \le \inf(g) + 2/n + \gamma/8 < \inf(g) + \gamma/2 <$$

$$\inf(g) + \epsilon.$$

Therefore (7.107) holds. This completes the proof of Lemma 7.20.

For each $a \in \mathcal{A}$ set $f_a = a$. The hypotheses (H) follow from Lemmas 7.20, 7.16 and 7.17. If K has the property (P), then these lemmas imply (H1). Theorem 7.14 now follows from Theorems 7.1 and 7.2.

7.9 Spaces of increasing convex functions

Assume that the set K is convex. Denote by $\mathcal{M}^{(co)}$ the set of all increasing lower semicontinuous convex functions $f : K \to R^1 \cup \{+\infty\}$ which are not identically $+\infty$ and which are bounded from below on bounded subsets of K. Denote by $\mathcal{M}_c^{(co)}$ the set of all finite-valued continuous functions $f \in \mathcal{M}^{(co)}$. We equip the set $\mathcal{M}^{(co)}$ with weak and strong topologies.

For the set $\mathcal{M}^{(co)}$ we consider the uniformity determined by the following base:

$$\widehat{E}_s(n) = \{(f, g) \in \mathcal{M}^{(co)} \times \mathcal{M}^{(co)} : \ |f(x) - g(x)| \le 1/n \tag{7.108}$$

$$\text{for all } x \in K \text{ such that } ||x|| \le n\},$$

where $n = 1, 2, \ldots$. It is easy to see that this uniform space is metrizable and complete. Denote by τ_s the topology in $\mathcal{M}^{(co)}$ induced by this uniformity. The topology τ_s is called the strong topology. Evidently, $\mathcal{M}_c^{(co)}$ is a closed subset of $\mathcal{M}^{(co)}$ with the strong topology. We equip the set $\mathcal{M}^{(co)}$ with a weak topology which is defined as in Section 7.3.

For each function $f : K \to R^1 \cup \{+\infty\}$ with a nonempty epigraph epi(f) define a function $\Delta_f : K \times R^1 \to R^1$ by

$$\Delta_f(x, \alpha) = \inf\{||x - y|| + |\alpha - \beta| : \ (y, \beta) \in \text{ epi}(f)\}, \ (x, \alpha) \in K \times R^1.$$

For the set $\mathcal{M}^{(co)}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_w(n) = \{(f, g) \in \mathcal{M}^{(co)} \times \mathcal{M}^{(co)} : \ |\Delta_f(x, \alpha) - \Delta_g(x, \alpha)| \le 1/n \tag{7.109}$$

$$\text{for all } (x, \alpha) \in K \times R^1 \text{ satisfying } ||x|| + |\alpha| \le n\},$$

where $n = 1, 2, \ldots$. This uniform space is metrizable. We endow the space $\mathcal{M}^{(co)}$ with the topology τ_w induced by this uniformity. It is clear that τ_w is weaker than τ_s. The topology τ_w is called the weak topology.

Let $\phi : K \to R^1 \cup \{+\infty\}$ satisfy

$$\phi(x) \to +\infty \text{ as } ||x|| \to +\infty. \tag{7.110}$$

Put

$$\mathcal{M}^{(co)}(\phi) = \{f \in \mathcal{M}^{(co)} : \ f(x) \ge \phi(x) \text{ for all } x \in K\},$$
$$\mathcal{M}_c^{(co)}(\phi) = \mathcal{M}^{(co)}(\phi) \cap \mathcal{M}_c^{(co)}.$$

It is easy to see that $\mathcal{M}^{(co)}(\phi)$ and $\mathcal{M}_c^{(co)}(\phi)$ are closed subsets of $\mathcal{M}^{(co)}$ with the strong topology. We consider the topological subspaces $\mathcal{M}^{(co)}$, $\mathcal{M}_c^{(co)}$, $\mathcal{M}^{(co)}(\phi)$, $\mathcal{M}_c^{(co)}(\phi) \subset \mathcal{M}^{(co)}$ with the relative weak and strong topologies. For each $a \in \mathcal{M}^{(co)}$ put $f_a = a$.

The following theorem was obtained in [114].

Theorem 7.21. *Assume that one of the following cases holds:*

1. $\mathcal{A}$ is either $\mathcal{M}^{(co)}(\phi)$ or $\mathcal{M}_c^{(co)}(\phi)$;

2. there exists $\bar{z} \in X$ such that $\bar{z} \le x$ for all $x \in K$ and $\mathcal{A}$ is either $\mathcal{M}^{(co)}$ or $\mathcal{M}_c^{(co)}$.

Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $f \in \mathcal{F}$ the following assertions hold:

1. inf(f) *is finite and attained at a point $x^{(f)} \in K$ such that for each $x \in K$ satisfying $f(x) = \inf(f)$ the inequality $x \le x_f$ holds.*

2. For any positive number ϵ there exist a positive number δ and a neighborhood U of f in $\mathcal{A}$ with the weak topology such that for each $g \in U$, inf(g) *is finite, and if $x \in K$ satisfies $g(x) \le \inf(g) + \delta$, then $|f(x^{(f)}) - g(x)| < \epsilon$ and there exists $u \in X$ such that $||u|| < \epsilon$ and $x \le x^{(f)} + u$.*

Moreover if K has the property (P), then inf(f) *is attained at the unique point $x^{(f)}$.*

7.10 Proof of Theorem 7.21

Since $\mathcal{M}^{(co)}(\phi) \subset \mathcal{M}(\phi)$ Lemmas 7.6, 7.7 and the growth condition (7.110) imply the following auxiliary result.

Lemma 7.22. *Let $f \in \mathcal{M}^{(co)}(\phi)$ and ϵ be a positive number. Then there exist a neighborhood U of f in $\mathcal{M}^{(co)}(\phi)$ with the weak topology and a number $c > 0$ such that*

$$|\inf(f) - \inf(g)| \leq \epsilon \text{ for all } g \in U,$$

and for each $g \in U$ and each $x \in K$ satisfying $g(z) \leq \inf(g) + 4$ the inequality $||z|| \leq c$ is true.

Denote by $\mathcal{B}$ the set of all $f \in \mathcal{A}$ for which there exists $x_f \in K$ such that $f(x_f) = \inf(f)$. If K has property (P), then we assume that x_f is a minimal element of K for all $f \in \mathcal{B}$. Analogously to Lemma 7.8 we can prove the following lemma.

Lemma 7.23. *The set $\mathcal{B}$ is everywhere dense in the space $\mathcal{A}$ with the strong topology.*

We can easily prove in a straightforward manner the following auxiliary result.

Lemma 7.24. *Let $f \in \mathcal{B}$, $x_f \in K$ and $f(x_f) = \inf(f)$. For each $\delta \in (0, 1)$ define $f_\delta : K \to R^1 \cup \{\infty\}$ by*

$$f_\delta(x) = \max\{f(x), f(x_f) + \delta\lambda(x - x_f)\} \text{ for all } x \in K.$$

Then $f_\delta \in \mathcal{A}$ for each $\delta \in (0, 1)$ and $f_\delta \to f$ as $\delta \to 0$ in $\mathcal{A}$ with the strong topology.

Lemma 7.25. *Assume that $\bar{z} \in X$,*

$$\bar{z} \leq x \text{ for all } x \in K,\ f \in \mathcal{M}^{(co)},\ x_f \in K,\ f(x_f) = \inf(f),\ \delta \in (0, 1)$$

and f_δ is defined by

$$f_\delta(x) = \max\{f(x), f(x_f) + \delta\lambda(x - x_f)\} \text{ for all } x \in K.$$

Then there exist a neighborhood U of f_δ in $\mathcal{M}^{(co)}$ with the weak topology and a positive constant c such that for each $g \in U$ and each $z \in K$ satisfying $||z|| \geq c$ the inequality $g(z) \geq f(x_f) + 64^{-1}\delta^2$ holds.

Lemma 7.25 is proved analogously to the proof of Lemma 7.18. Analogously to the proof of Lemma 7.19 we can prove the following lemma.

Lemma 7.26. *Assume that* $\bar{z} \in X$, $\bar{z} \le x$ *for all* $x \in K$, $f \in \mathcal{M}^{(co)}$, $x_f \in K$, $f(x_f) = \inf(f)$, $\delta \in (0,1)$ *and* f_δ *is defined by*

$$f_\delta(x) = \max\{f(x), f(x_f) + \delta\lambda(x - x_f)\} \text{ for all } x \in K.$$

Then there exist a neighborhood U *of* f_δ *in* $\mathcal{M}^{(co)}$ *with the weak topology and a positive constant* c *such that the following assertions are true:*

1. $\inf(g)$ *is finite and* $|\inf(f_\delta) - \inf(g)| \le \epsilon$ *for all* $g \in U$;

2. for each $g \in U$ *and each* $z \in K$ *satisfying* $g(z) \le \inf(g) + 128^{-1}\delta^2$ *the inequality* $||z|| \le c$ *holds.*

Analogously to Lemma 7.20 we can prove the following auxiliary result.

Lemma 7.27. *Let* $f \in \mathcal{B}$, $x_f \in K$, $f(x_f) = \inf(f)$, $\delta \in (0,1)$, f_δ *is defined by*

$$f_\delta(x) = \max\{f(x), f(x_f) + \delta\lambda(x - x_f)\} \text{ for all } x \in K,$$

$\epsilon \in (0,1)$ *and* $\gamma \in (0, (8^{-1}\epsilon\delta)^2 128^{-1})$. *Then there exist a neighborhood* U *of* f_δ *in* $\mathcal{A}$ *with the weak topology and a positive constant* c *such that for each* $g \in U$ *the following assertions hold:*

1. $\inf(g)$ *is finite and there exists* $z \in K$ *such that* $||z - x_f|| \le \epsilon$ *and* $g(z) \le \inf(g) + \epsilon$;

2. if $x \in K$ *satisfies* $g(x) \le \inf(g) + \gamma$, *then* $|g(x) - f(x_f)| \le \epsilon$, $||x|| \le c$ *and there is* $v \in X$ *such that* $||v|| \le \epsilon$ *and* $x \le x_f + v$.

The hypotheses (H) follow from Lemmas 7.27, 7.23 and 7.24. If K has property (P), then these lemmas imply (H1). Theorem 7.21 follows from Theorems 7.1 and 7.2.

7.11 The generic existence result for the minimization problem (P2)

Denote by $\mathcal{L}$ the set of all increasing lower semicontinuous bounded from below functions $f : K \to R^1$ which satisfy the following assumption:

(A) For each $x \in K$, each neighborhood U of x in K and each positive number δ there exists an open nonempty set $V \subset U$ such that $f(y) \le f(x) + \delta$ for all $y \in V$.

Denote by $\mathcal{L}_c$ the set of all finite-valued continuous functions $f \in \mathcal{L}$.

Remark 7.28. Let $f : K \to R^1$ be a lower semicontinuous bounded from below function. Then f satisfies assumption (A) if and only if $\mathrm{epi}(f) = \{(y, \alpha) \in K \times R^1 : \alpha \ge f(y)\}$ is the closure of its interior in the space $K \times R^1$.

For $x \in K$ and $A \subset K$ set

$$\rho(x, A) = \inf\{||x - y|| : y \in A\}.$$

For each positive number r put

$$B(r) = \{x \in K : \ ||x|| \le r\}.$$

Denote by $S(K)$ the collection of all closed nonempty sets in K. We equip the set $S(K)$ with a weak and a strong topology.

For the set $S(K)$ we consider the uniformity determined by the following base:

$$\mathcal{G}_s(n) = \{(A, B) \in S(K) \times S(K) : \ \rho(x, B) \le n^{-1} \text{ for all } x \in A \quad (7.111)$$
$$\text{and } \rho(y, A) \le n^{-1} \text{ for all } y \in B\}, \ n = 1, 2, \ldots.$$

It is well known that the space $S(K)$ with this uniformity is metrizable (by a metric H_s) and complete. The Hausdorff topology induced by this uniformity is a strong topology in the space $S(K)$.

Also for the set $S(K)$ we consider the uniformity determined by the following base:

$$\mathcal{G}_w(n) = \{(A, B) \in S(K) \times S(K) : \ |\rho(x, A) - \rho(x, B)| \le n^{-1} \quad (7.112)$$
$$\text{for all } x \in K \text{ satisfying } ||x|| \le n\},$$

where $n = 1, 2, \ldots$. It is well known that the set $S(K)$ with this uniformity is metrizable (by a metric H_w). The bounded Hausdorff (Attouch–Wets) topology induced by this uniformity is a weak topology in the space $S(K)$.

For the space $\mathcal{L}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_{\mathcal{L}w}(n) = \{(f, g) \in \mathcal{L} \times \mathcal{L} : \ |f(x) - g(x)| \le n^{-1} \text{ for all } x \in B(n)\}, \quad (7.113)$$

where $n = 1, 2, \ldots$. It is clear that this uniform space is metrizable (by a metric h_w). This uniformity induces the weak topology in the space $\mathcal{L}$.

For the set $\mathcal{L}$ we consider the uniformity determined by the following base:

$$\mathcal{E}_{\mathcal{L}c}(n) = \{(f, g) \in \mathcal{L} \times \mathcal{L} : \ |f(x) - g(x)| \le 1/n \text{ for all } x \in K\} \quad (7.114)$$

where $n = 1, 2, \ldots$. It is easy to see that this uniform space is metrizable (by a metric h_s) and complete. This uniformity induces a strong topology in the space $\mathcal{L}$.

Let a function $\phi : \ K \to R^1$ satisfy

$$\phi(x) \to \infty \text{ as } ||x|| \to \infty. \quad (7.115)$$

Denote by $\mathcal{L}(\phi)$ the set of all $f \in \mathcal{L}$ such that

$$f(x) \ge \phi(x) \text{ for all } x \in K \quad (7.116)$$

and set $\mathcal{L}_c(\phi) = \mathcal{L}_c \cap \mathcal{L}(\phi)$. Evidently, $\mathcal{L}_c(\phi)$, $\mathcal{L}(\phi)$ and $\mathcal{L}_c$ are closed subsets of $\mathcal{L}$ with the strong topology.

For the space $\mathcal{L} \times S(K)$ we consider the product topology induced by the metric

$$d((f,A),(g,B)) = h_s(f,g) + H_s(A,B),\ (f,A),(g,B) \in \mathcal{L} \times S(K) \quad (7.117)$$

which is called a strong topology and consider the product topology induced by the metric

$$d_w((f,A),(g,B)) = h_w(f,g) + H_w(A,B),\ (f,A),(g,B) \in \mathcal{L} \times S(K) \quad (7.118)$$

which is called a weak topology.

For each $a = (g,B) \in \mathcal{L} \times S(K)$ we define a lower semicontinuous function $f_a : K \to R^1 \cup \{\infty\}$ by

$$f_a(x) = g(x) \text{ for all } x \in B \text{ and } f_a(x) = \infty \text{ for all } x \in K \setminus B. \quad (7.119)$$

The following two theorems ware obtained in [114].

Theorem 7.29. *Assume that $\mathcal{A}$ is either $\mathcal{L}(\phi) \times S(K)$ or $\mathcal{L}_c(\phi) \times S(K)$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) sets such that for each $a \in \mathcal{F}$ the following assertions hold:*

1. $\inf(f_a)$ is finite and attained at a point $x^{(a)} \in K$ such that for each $x \in K$ satisfying $f_a(x) = \inf(f_a)$ the inequality $x \le x_a$ holds.

2. For any positive number ϵ there exist a positive number δ and a neighborhood U of a in $\mathcal{A}$ with the weak topology such that for each $b \in U$, $\inf(f_b)$ is finite, and if $x \in K$ satisfies $f_b(x) \le \inf(f_b) + \delta$, then $|f_a(x^{(a)}) - f_b(x)| < \epsilon$ and there exists $u \in X$ such that $||u|| < \epsilon$ and $x \le x^{(a)} + u$.

Theorem 7.30. *Assume that there exists $\bar{z} \in X$ such that $\bar{z} \le x$ for all $x \in K$ and that $\mathcal{A}$ is either $\mathcal{L} \times S(K)$ or $\mathcal{L}_c \times S(K)$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{F}$ the following assertions hold:*

1. $\inf(f_a)$ is finite and attained at a point $x^{(a)} \in K$ such that for each $x \in K$ satisfying $f_a(x) = \inf(f_a)$ the inequality $x \le x_a$ holds.

2. For any positive number ϵ there exist a positive δ and a neighborhood U of a in $\mathcal{A}$ with the strong topology such that for each $b \in U$, $\inf(f_b)$ is finite, and if $x \in K$ satisfies $f_b(x) \le \inf(f_b) + \delta$, then $|f_a(x^{(a)}) - f_b(x)| < \epsilon$ and there exists $u \in X$ such that $||u|| < \epsilon$ and $x \le x^{(a)} + u$.

7.12 Proofs of Theorems 7.29 and 7.30

We consider the topological subspaces $\mathcal{L}(\phi) \subset \mathcal{L}$ with the relative weak and strong topologies.

Proposition 7.31. *Let $f \in \mathcal{L}(\phi)$, $A \in S(K)$ and $\epsilon, \gamma \in (0,1)$. Then there exist $\widehat{A} \in S(X)$, $\widehat{f} \in \mathcal{L}(\phi)$, $\widehat{x} \in K$, $\eta > 0$, a neighborhood U of $(\widehat{f}, \widehat{A})$ in $\mathcal{L}(\phi) \times S(K)$ with the weak topology and a positive constant c_1 such that*

$$\widehat{A} = A \cup \{\widehat{x}\},\ \rho(\widehat{x}, A) < \epsilon_0, \tag{7.120}$$

$$\widehat{f}(x) = f(x) + \epsilon_0 \min\{1, \lambda(x - \widehat{x})\},\ x \in K \tag{7.121}$$

with $\epsilon_0 \in (0, \epsilon)$ and for each $(g, D) \in U$ the following assertions hold:
For each $z \in D$ satisfying

$$g(z) \le \inf\{g(y) :\ y \in D\} + \eta \tag{7.122}$$

there exists $v \in X$ such that

$$||v|| \le \gamma,\ z \le \widehat{x} + v \text{ and } |g(z) - f(\widehat{x})| \le \gamma,\ ||z|| \le c_1; \tag{7.123}$$

there exists $y \in D$ such that

$$||y - \widehat{x}|| \le \gamma,\ g(y) \le \inf\{g(z) :\ z \in D\} + \gamma. \tag{7.124}$$

Proof: Fix

$$\epsilon_0 \in (0, 4^{-1} \min\{\epsilon, \gamma\}),\ \epsilon_1 \in (0, 4^{-1}\epsilon_0),\ \eta \in (0, 16^{-1}\epsilon_1\epsilon_0). \tag{7.125}$$

Put

$$\Omega = \{x \in K :\ \rho(x, A) < \epsilon_0\}. \tag{7.126}$$

Assumption (A) implies that there exists an open nonempty subset E of K such that

$$E \subset \Omega,\ f(z) \le \inf\{f(u) :\ u \in \Omega\} + 16^{-1}\eta \text{ for all } z \in E. \tag{7.127}$$

Let $\widehat{x} \in E$ and define $\widehat{A}$, $\widehat{f}$ by (7.120) and (7.121). Fix an integer

$$n_1 > 8(1 + |f(\widehat{x})|). \tag{7.128}$$

It follows from (7.113), (7.115) and (7.116) that there exist a neighborhood V_1 of $\widehat{f}$ in the space $\mathcal{L}(\phi)$ with the weak topology and $c_0 \in R^1$ and $c_1 > 1$ such that

$$\inf\{g(x) :\ x \in K\} \ge c_0 \text{ for all } g \in V_1 \tag{7.129}$$

and

$$g(x) \ge 2n_1 + 4 \text{ for each } g \in V_1 \text{ and each } x \in K \text{ satisfying } ||x|| \ge c_1. \tag{7.130}$$

Fix an integer

$$n_2 > 2c_1 + 2|c_0| + 2n_1 + 16(\epsilon_0 - \rho(\widehat{x}, A))^{-1} + 8||\widehat{x}|| + 8\eta^{-1} + 8 \tag{7.131}$$

such that

$$\{z \in K : \ ||z - \widehat{x}|| \le 8n_2^{-1}\} \subset E. \tag{7.132}$$

Set

$$U = \{(g, B) \in V_1 \times S(K) : \ (\widehat{f}, g) \in \mathcal{E}_{\mathcal{L}w}(n_2), \ (\widehat{A}, B) \in \mathcal{G}_w(n_2)\}. \tag{7.133}$$

We show that for all $(g, D) \in U$

$$\inf\{g(u) : \ u \in D\} = \inf\{g(u) : \ u \in D \cap B(n_2/2 - 2)\}. \tag{7.134}$$

Let $(g, D) \in U$. Relations (7.133) and (7.131) imply that there exists $y \in D$ for which

$$||y - \widehat{x}|| \le 2n_2^{-1}. \tag{7.135}$$

In view of (7.135), (7.132) and (7.127),

$$y \in E \subset \Omega, \ |f(y) - f(\widehat{x})| \le 8^{-1}\eta. \tag{7.136}$$

Combined with (7.121) and (5.135) this implies that

$$|\widehat{f}(y) - \widehat{f}(\widehat{x})| \le 8^{-1}\eta + 2\epsilon_0 n_2^{-1}. \tag{7.137}$$

It follows from (7.137), (7.135), (7.131), (7.133) and (7.128) that $|g(y) - \widehat{f}(y)| \le n_2^{-1}$ and

$$\inf\{g(u) : \ u \in D\} \le g(y) \le \widehat{f}(\widehat{x}) + n_2^{-1} + 8^{-1}\eta + 2\epsilon_0 n_2^{-1} < n_1. \tag{7.138}$$

Relations (7.133), (7.130) and (7.131) imply that

$$\inf\{g(u) : \ u \in D\} = \inf\{g(u) : \ u \in D \text{ and } g(u) \le n_1 + 2\} \tag{7.139}$$

$$= \inf\{g(u) : \ u \in D \cap B(c_1)\} = \inf\{g(u) : \ u \in D \cap B(n_2/2 - 2)\}$$

and (7.134) holds. We have shown (see (7.138), (7.128)) that for all $(g, D) \in U$

$$\inf\{g(u) : \ u \in D\} \le f(\widehat{x}) + 2^{-1}\eta < n_1. \tag{7.140}$$

Assume that $(g, D) \in U$, $z \in D$ and (7.122) holds. By (7.140), (7.122), (7.130) and (7.131),

$$||z|| \le c_1 < n_2/2 - 1. \tag{7.141}$$

It follows from (7.141) and (7.133) that

$$|g(z) - \widehat{f}(z)| \le n_2^{-1}, \ \rho(z, \widehat{A}) \le n_2^{-1}. \tag{7.142}$$

By (7.142), (7.121), (7.126), (7.131) and (7.127),

$$z \in \Omega, \ f(z) \ge f(\widehat{x}) - 16^{-1}\eta. \tag{7.143}$$

It follows from (7.121), (7.142), (7.140), (7.122), (7.143) and (7.127) that

$$f(z) + \epsilon_0 \min\{1, \lambda(z - \widehat{x})\} = \widehat{f}(z) \le g(z) + n_2^{-1} \le \tag{7.144}$$

$$\le \inf\{g(y) : y \in D\} + \eta + n_2^{-1} \le f(\widehat{x}) + \eta/2 + \eta + n_2^{-1} \le$$

$$f(z) + 16^{-1}\eta + \eta + 2^{-1}\eta + n_2^{-1} \le f(z) + 2\eta.$$

By (7.144) and (7.125),

$$\min\{1, \lambda(z - \widehat{x})\} \le 2\eta\epsilon_0^{-1}, \ \lambda(z - \widehat{x}) \le 2\eta\epsilon_0^{-1}$$

and there exists $v \in X$ such that

$$||v|| \le 4\eta\epsilon_0^{-1}, \ z \le \widehat{x} + v. \tag{7.145}$$

It follows from (7.125), (7.131), (7.143), (7.144), (7.140), (7.122) and (7.125) that

$$-\gamma < -n_2^{-1} - 16^{-1}\eta < \widehat{f}(z) - f(\widehat{x}) - n_2^{-1} \le g(z) - f(\widehat{x}) \le \eta + 2^{-1}\eta < \gamma. \tag{7.146}$$

In view of (7.146), (7.145) and (7.141), relation (7.123) holds. We have shown that (7.38) is true and there exists $y \in D$ such that (7.135) holds. Since (7.146) is valid for all $z \in D$ satisfying (7.122) we conclude that

$$|\inf\{g(u) : u \in D\} - f(\widehat{x})| \le \eta + \eta/2. \tag{7.147}$$

Combined with (7.138), (7.131) and (7.125) this relation implies that

$$g(y) \le \inf\{g(u) : u \in D\} + 1/n_2 + \eta/8 + 2\epsilon_0/n_2 + 3\eta/2 \le$$

$$\inf\{g(u) : u \in D\} + \gamma. \tag{7.148}$$

Together with (7.135), (7.131) this inequality implies (7.124). This completes the proof of the proposition.

The hypotheses (H) follow from Proposition 7.31. Theorem 7.29 follows from Theorem 7.1.

Theorem 7.30 is proved analogously to Theorem 7.29. The proof of Theorem 7.30 is based on a modification of Proposition 7.31. In this modification $f, \widehat{f} \in \mathcal{L}$, U is a neighborhood of $(\widehat{f}, \widehat{A})$ in $\mathcal{L} \times S(K)$ with the strong topology and V_1 is a neighborhood of $\widehat{f}$ in the space $\mathcal{L}$ with the strong topology such that (7.129) is true. In the definition of U we use the uniformity defined by (7.114). The existence of a positive number c_1 follows from the inequalities

$$||z|| \le ||\bar{z}|| + ||z - \bar{z}|| \le ||\bar{z}|| + ||\widehat{x} + v - \bar{z}|| \le$$

$$2||\bar{z}|| + ||\widehat{x}|| + ||v|| \le c_1$$

where $c_1 = 2||\bar{z}|| + ||\widehat{x}|| + 1$.

7.13 Well-posedness of optimization problems with increasing cost functions

We use the convention that $\infty - \infty = 0$, $\infty/\infty = 1$ and $\ln(\infty) = \infty$.

Assume that $\mathcal{A}$ is a nonempty set and $d_w, d_s : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are two metrics which satisfy $d_w(a,b) \le d_s(a,b)$ for all $a, b \in \mathcal{A}$. We assume that the metric space $(\mathcal{A}, d_s)$ is complete. The topology induced in $\mathcal{A}$ by the metric d_s is called the strong topology and the topology induced in $\mathcal{A}$ by the metric d_w is called the weak topology.

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function $f_a : K \to [-\infty, +\infty]$ is associated and f_a is not identically ∞ for all $a \in \mathcal{A}$.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$ if the following assertions hold:

1. $\inf(f_a)$ is finite and attained at a point $x^{(a)} \in K$ such that for each $x \in K$ satisfying $f_a(x) = \inf(f_a)$ the inequality $x \le x^{(a)}$ holds.

2. For any positive number ϵ there exist a positive number δ and a neighborhood U of a in $\mathcal{A}$ with the weak topology such that for each $b \in U$, $\inf(f_b)$ is finite, and if $x \in K$ satisfies $f_b(x) \le \inf(f_b) + \delta$, then $|f_a(x^{(a)}) - f_b(x)| < \epsilon$ and there exists $u \in X$ such that $||u|| < \epsilon$ and $x \le x^{(a)} + u$.

Note that if $X_+ = \{0\}$, then our definition reduces to those in Section 4.1.

For each natural number n denote by $\mathcal{A}_n$ the set of all $a \in \mathcal{A}$ which have the following property:

(P1) There exist $x \in K$ and $r > 0, \eta > 0, c > 0$ such that

$$-\infty < f_a(x) < \inf(f_a) + 1/n \tag{7.149}$$

and for each $b \in \mathcal{A}$ satisfying $d_w(a,b) < r$, $\inf(f_b)$ is finite, and if $z \in K$ satisfies $f_b(z) \le \inf(f_b) + \eta$, then

$$||z|| \le c, \ |f_b(z) - f_a(x)| \le 1/n \tag{7.150}$$

and there exists $u \in X$ such that $||u|| \le 1/n$ and $z \le x + u$.

The following result was obtained in [115].

Proposition 7.32. *Assume that $a \in \cap_{n=1}^{\infty} \mathcal{A}_n$. Then the minimization problem for f_a on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$.*

Proof: It follows from (P1) that for each natural number n there exist $x_n \in K$, $r_n > 0$ and positive numbers η_n and c_n such that

$$-\infty < f_a(x_n) < \inf(f_a) + 2^{-n} \tag{7.151}$$

and the following property holds:

(P2) For each $b \in \mathcal{A}$ satisfying $d_w(a,b) < r_n$, $\inf(f_b)$ is finite and if $z \in K$ satisfies $f_b(z) \le \inf(f_b) + \eta_n$, then there exists $u \in X$ such that

$$||u|| \le 2^{-n}, \ z \le x_n + u \text{ and } ||z|| \le c_n, \ |f_b(z) - f_a(x_n)| \le 2^{-n}. \tag{7.152}$$

We may assume without loss of generality that for all natural numbers n,

$$\eta_n,\ r_n < 4^{-n-1} \text{ and } \eta_n < \eta_1. \tag{7.153}$$

There exists a strictly increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that

$$4 \cdot 2^{-k_{n+1}} < \eta(k_n) \text{ for all integers } n \geq 1. \tag{7.154}$$

Let n be a natural number. In view of (7.151),

$$-\infty < f_a(x_{k_{n+1}}) < \inf(f_a) + 2^{-k_{n+1}} < \inf(f_a) + \eta(k_n). \tag{7.155}$$

It follows from (7.155), (7.153) and the definition of c_1 that

$$||x_{k_{n+1}}|| \leq c_1. \tag{7.156}$$

By (7.155), (P2) (see (7.152)) and the definition of x_{k_n} and η_{k_n}, there exists $u_n \in X$ such that

$$||u_n|| \leq 2^{-k_n},\ x_{k_{n+1}} \leq x_{k_n} + u_n \tag{7.157}$$

and

$$|f_a(x_{k_{n+1}}) - f_a(x_{k_n})| \leq 2^{-k_n}. \tag{7.158}$$

Put

$$y_n = x_{k_n} + \sum_{i=n}^{\infty} u_i. \tag{7.159}$$

It is easy to see that the sequence $\{y_n\}_{n=1}^{\infty}$ is well-defined. Relations (7.159) and (7.157) imply that for each natural number n,

$$y_{n+1} - y_n = x_{k_{n+1}} + \sum_{i=n+1}^{\infty} u_i - (x_{k_n} + \sum_{i=n}^{\infty} u_i) = \tag{7.160}$$

$$x_{k_{n+1}} - x_{k_n} - u_n \leq 0.$$

In view of (7.159), (7.157) and (7.156),

$$\sup\{||y_n||:\ n = 1, 2, \dots\} < \infty. \tag{7.161}$$

By (7.161), (7.160) and property (A), there exists $x^{(a)} = \lim_{n\to\infty} y_n$. Together with (7.159) and (7.157) this equality implies that

$$x^{(a)} = \lim_{n\to\infty} x_{k_n}. \tag{7.162}$$

It follows from (7.162), (7.151) and the lower semicontinuity of f_a that

$$f_a(x^{(a)}) = \inf(f_a). \tag{7.163}$$

Assume now that $x \in K$ and $f_a(x) = \inf(f_a)$. In view of the definition of x_{k_n}, $n = 1, 2, \dots$ (see the property (P2)) for each natural number n there exists $v_n \in X$ such that

$$||v_n|| \le 2^{-k_n},\ x \le x_{k_n} + v_n.$$

These inequalities and (7.162) imply that $x \le x^{(a)}$.

It follows from (7.163) and property (P2) that for all natural numbers n

$$|f_a(x^{(a)}) - f_a(x_n)| \le 2^{-n}. \tag{7.164}$$

Let ϵ be a positive number. Fix an integer $m \ge 1$ such that

$$||x^{(a)} - x_{k_m}|| < 4^{-1}\epsilon,\ 2^{-k_m} < 4^{-1}\epsilon. \tag{7.165}$$

Assume that $b \in B_w(a, r_{k_m}/2)$, $x \in K$ and

$$f_b(x) \le \inf(f_b) + \eta_{k_m}. \tag{7.166}$$

It follows from (7.166) and the definition of η_{k_m}, r_{k_m}, x_{k_m} (see property (P2)) that

$$|f_b(x) - f_a(x_{k_m})| \le 2^{-k_m} \tag{7.167}$$

and that there exists $v \in X$ for which

$$||v|| < 2^{-k_m},\ x \le x_{k_m} + v. \tag{7.168}$$

By (7.168) and (7.165),

$$x \le x_{k_m} + v = x^{(a)} + (x_{k_m} - x^{(a)} + v),$$

$$||x_{k_m} - x^{(a)} + v|| \le ||x^{(a)} - x_{k_m}|| + ||v|| < \epsilon/2. \tag{7.169}$$

In view of (7.167), (7.164) and (7.165),

$$|f_a(x^{(a)}) - f_b(x)| \le |f_a(x^{(a)}) - f_a(x_{k_m})| + |f_a(x_{k_m}) - f_b(x)| \le$$

$$2^{-k_m} + 2^{-k_m} < \epsilon/2.$$

Proposition 7.32 is proved.

Recall that an element $x \in K$ is called minimal if for each $y \in K$ satisfying $y \le x$ the equality $x = y$ is true. Denote by K_{min} the set of all minimal elements of K.

For each natural number n denote by $\tilde{\mathcal{A}}_n$ the set of all $a \in \mathcal{A}$ which have the property (P1) with $x \in K_{min}$.

Analogously to the proof of Proposition 7.32 we can prove the following result.

Proposition 7.33. *Assume that the set K_{min} is a closed subset of the Banach space X and $a \in \cap_{n=1}^{\infty} \tilde{\mathcal{A}}_n$. Then the minimization problem for f_a on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$ and* $\inf(f_a)$ *is attained at a unique point.*

In the proof of Proposition 7.33 we choose $x_n \in K_{min}$ $(n = 1, 2, \dots)$. This implies that $\inf(f_a)$ is attained at the unique point $x^{(a)} \in K_{min}$ (see (7.161)).

Remark 7.34. Note that the assertion 1 in the definition of a strongly well-posed minimization problem for f_a can be represented in the following way:

$\inf(f_a)$ is finite and the set

$$\mathrm{argmin}_{x \in K} f_a = \{x \in K : \ f_a(x) = \inf(f_a)\}$$

has the largest element.

We construct an example of an increasing function h for which the set $\mathrm{argmin}(h)$ is not a singleton and has the largest element. Define a continuous increasing function $\psi : [0, \infty) \to R^1$ by

$$\psi(t) = 0, \ t \in [0, 1/2], \ \psi(t) = 2t - 1, \ t \in (1/2, \infty). \tag{7.170}$$

Let $n \geq 1$ be an integer and consider the Euclidean space R^n. Let $K = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}$. Define a function $h : K \to R^1$ by

$$h(x) = \psi(\max\{x_i : \ i = 1, \dots, n\}), \ x \in K.$$

Clearly, h is a continuous increasing function,

$$\inf\{h(x) : \ x \in K\} = 0$$

and the set

$$\{x \in K : \ h(x) = 0\} = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \in [0, 1/2], \ i = 1, \dots, n\}$$

is not a singleton and has the largest element $(1/2, \dots, 1/2)$.

Remark 7.35. The following example shows that in some cases the sets $\mathcal{A}_n$ can be empty. Let $\mathcal{A} = K = R^1$. For each $a \in R^1$ consider the function $f_a : K \to R^1$, where $f_a = 0$ for any $x \leq a$ and $f_a(x) > 0$ for any $x > a$. Clearly, the set $\mathcal{A}_n$ is empty for any integer $n \geq 1$.

7.14 Variational principles

We use the notations and definitions introduced in Section 7.13. The following are the basic hypotheses about the functions.

(H1) For each $a \in \mathcal{A}$, $\inf(f_a)$ is finite.

(H2) For each positive number ϵ and each natural number m there exist $\delta > 0$, $r_0 > 0$ such that the following property holds:

(P3) For each $a \in \mathcal{A}$ satisfying $\inf(f_a) \leq m$ and each $r \in (0, r_0]$ there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K$ and $\bar{d} > 0$ such that

$$d_s(a, \bar{a}) \le r, \ \inf(f_{\bar{a}}) \le m+1, \ f_{\bar{a}}(\bar{x}) \le \inf(f_{\bar{a}}) + \epsilon \tag{7.171}$$

and if $x \in K$ satisfies

$$f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + \delta r, \tag{7.172}$$

then $||x|| \le \bar{d}$ and there exists $u \in X$ for which $||u|| \le \epsilon$, $x \le \bar{x} + u$.

(H3) For each natural number m there exist $\alpha \in (0,1)$ and a positive number r_0 such that for each $r \in (0, r_0]$, each $a_1, a_2 \in \mathcal{A}$ satisfying $d_w(a_1, a_2) \le \alpha r$ and each $x \in K$ satisfying $\min\{f_{a_1}(x), f_{a_2}(x)\} \le m$ the inequality $|f_{a_1}(x) - f_{a_2}(x)| \le r$ holds.

The following result was obtained in [115].

Theorem 7.36. *Assume that (H1), (H2) and (H3) hold. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and for each $a \in \mathcal{F}$ the minimization problem for f_a on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$.*

Proof: Recall that for each natural number n, $\mathcal{A}_n$ is the set of all $a \in \mathcal{A}$ which have the property (P1). Proposition 7.32 implies that in order to prove the theorem it is sufficient to show that the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) for any natural number n. Then the theorem holds with $\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{A}_n$.

Let n be a natural number. We will show that the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). To meet this goal it is sufficient to show that for each natural number m the set

$$\Omega_{nm} := \{a \in \mathcal{A} \setminus \mathcal{A}_n : \ \inf(f_a) \le m\} \tag{7.173}$$

is porous in $\mathcal{A}$ with respect to (d_w, d_s).

Let m be a natural number. It follows from (H3) that there exist

$$\alpha_1 \in (0,1), \ r_1 \in (0, 1/2) \tag{7.174}$$

such that for each $r \in (0, r_1]$, each $a_1, a_2 \in \mathcal{A}$ satisfying $d_w(a_1, a_2) \le \alpha_1 r$ and each $x \in K$ satisfying

$$\min\{f_{a_1}(x), f_{a_2}(x)\} \le m+4 \tag{7.175}$$

the inequality $|f_{a_1}(x) - f_{a_2}(x)| \le r$ holds.

It follows from (H2) that there exist $\alpha_2, r_2 \in (0,1)$ such that the following property holds:

(P4) For each $a \in \mathcal{A}$ satisfying $\inf(f_a) \le m+2$ and each $r \in (0, r_2]$ there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K$ and $\bar{d} > 0$ such that

$$d_s(a, \bar{a}) \le r, \ \inf(f_{\bar{a}}) \le m+3, \ f_{\bar{a}}(\bar{x}) \le \inf(f_{\bar{a}}) + (2n)^{-1}$$

and if $x \in K$ satisfies $f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + 4r\alpha_2$, then $||x|| \le \bar{d}$ and there exists $u \in X$ for which $||u|| \le (2n)^{-1}$ and $x \le \bar{x} + u$.

Fix

$$\bar{\alpha} \in (0, \alpha_1\alpha_2/16),\ \bar{r} \in (0, r_1 r_2 \bar{\alpha}/n). \tag{7.176}$$

Let $a \in \mathcal{A}$ and $r \in (0, \bar{r}]$. There are two cases:

$$B_{d_s}(a, r/4) \cap \{\xi \in \mathcal{A} : \inf(f_\xi) \le m + 2\} = \emptyset; \tag{7.177}$$

$$B_{d_s}(a, r/4) \cap \{\xi \in \mathcal{A} : \inf(f_\xi) \le m + 2\} \ne \emptyset; \tag{7.178}$$

Assume that (7.177) holds. We show that for each $\xi \in B_{d_w}(a, \bar{r})$ the inequality $\inf(f_\xi) > m$ is valid. Let us assume the contrary. Then there exists $\xi \in \mathcal{A}$ such that

$$d_w(\xi, a) \le \bar{r},\ \inf(f_\xi) \le m. \tag{7.179}$$

There exists $y \in K$ such that

$$f_\xi(y) \le m + 1/2. \tag{7.180}$$

By the definition of α_1, r_1 (see (7.174), (7.175)), (7.180), (7.179) and (7.176),

$$|f_a(y) - f_\xi(y)| \le \alpha_1^{-1}\bar{r} \le 1/4.$$

In view of this inequality and (7.180),

$$\inf(f_a) \le f_a(y) \le f_\xi(y) + 1/4 \le m + 1,$$

a contradiction (see (7.177)). Therefore

$$B_{d_w}(a, \bar{r}) \subset \{\xi \in \mathcal{A} : \inf(f_\xi) > m\}$$

and (7.173) implies that

$$B_{d_w}(a, \bar{r}) \cap \Omega_{nm} = \emptyset. \tag{7.181}$$

Thus we have shown that (7.177) implies (7.181).

Assume that (7.178) holds. Then there exists $a_1 \in \mathcal{A}$ such that

$$d_s(a, a_1) \le r/4,\ \inf(f_{a_1}) \le m + 2. \tag{7.182}$$

It follows from the definition of α_2, r_2, the property (P4), (7.182) and (7.176) that there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K$, $\bar{d} > 0$ such that

$$d_s(a_1, \bar{a}) \le r/4,\ \inf(f_{\bar{a}}) \le m + 3,\ f_{\bar{a}}(\bar{x}) \le \inf(f_{\bar{a}}) + (2n)^{-1} \tag{7.183}$$

and that the following property holds:

(P5) If $x \in K$ satisfies $f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + r\alpha_2$, then $||x|| \le \bar{d}$ and there exists $u \in X$ for which $||u|| \le (2n)^{-1}$ and $x \le \bar{x} + u$.

By (7.183) and (7.182),

$$d_s(a, \bar{a}) \le r/2. \tag{7.184}$$

Assume that

$$\xi \in B_{d_w}(\bar{a}, \bar{\alpha}r). \tag{7.185}$$

In view of (7.183),

$$\inf(f_{\bar{a}}) = \inf\{f_{\bar{a}}(z) : \ z \in K, \ f_{\bar{a}}(z) \le m + 7/2\}. \tag{7.186}$$

Let $x \in K$ satisfy

$$f_{\bar{a}}(x) \le m + 7/2. \tag{7.187}$$

By (7.185), (7.187), (7.176) and the definition of α_1, r_1 (see (7.174), (7.175)),

$$|f_{\bar{a}}(x) - f_\xi(x)| \le \bar{\alpha}r\alpha_1^{-1} \le \alpha_2 r/16.$$

Since these inequalities hold for any $x \in K$ satisfying (7.187) it follows from (7.186) that

$$\inf(f_\xi) \le \inf\{f_\xi(x) : \ x \in K, \ f_{\bar{a}}(x) \le m + 7/2\} \le$$
$$\inf\{f_{\bar{a}}(x) + \alpha_2 r/16 : \ x \in K, \ f_{\bar{a}}(x) \le m + 7/2\} =$$
$$\alpha_2 r/16 + \inf(f_{\bar{a}}).$$

Moreover since (7.187) holds with $x = \bar{x}$ (see (7.183)), we obtain that $|f_{\bar{a}}(\bar{x}) - f_\xi(\bar{x})| \le \alpha_2 r/16$. Hence

$$\inf(f_\xi) \le \inf(f_{\bar{a}}) + \alpha_2 r/16, \ |f_{\bar{a}}(\bar{x}) - f_\xi(\bar{x})| \le \alpha_2 r/16. \tag{7.188}$$

Let $x \in K$ satisfy

$$f_\xi(x) \le \inf(f_\xi) + 1/4. \tag{7.189}$$

By (7.189), (7.188), (7.183) and (7.176), $f_\xi(x) \le m + 7/2$. This inequality, (7.185), (7.176) and the definition of α_1, r_1 (see (7.174), (7.175)) imply that

$$|f_{\bar{a}}(x) - f_\xi(x)| \le \bar{\alpha}r\alpha_1^{-1} \le \alpha_2 r/16.$$

Thus the following property holds:

(P6) If $x \in K$ satisfies (7.189), then $|f_{\bar{a}}(x) - f_\xi(x)| \le \alpha_2 r/16$.

By (P6),

$$\inf(f_{\bar{a}}) \le \inf\{f_{\bar{a}}(x) : \ x \in K, \ f_\xi(x) \le \inf(f_\xi) + 1/4\} \le$$
$$\inf\{f_\xi(x) + \alpha_2 r/16 : \ x \in K, \ f_\xi(x) \le \inf(f_\xi) + 1/4\} =$$
$$\alpha_2 r/16 + \inf(f_\xi).$$

Hence

$$\inf(f_{\bar{a}}) \le \inf(f_\xi) + \alpha_2 r/16.$$

Together with (7.188) and (7.183) this inequality implies that

$$|\inf(f_{\bar{a}}) - \inf(f_\xi)| \le \alpha_2 r/16, \ f_\xi(\bar{x}) \le \inf(f_\xi) + \alpha_2 r/8 + (2n)^{-1}. \tag{7.190}$$

Assume that $x \in K$ and

$$f_s(x) \le \inf(f_\xi) + \alpha_2 r/16. \tag{7.191}$$

In view of (P6),

$$|f_{\bar{a}}(x) - f_\xi(x)| \le \alpha_2 r/16. \tag{7.192}$$

By (7.191), (7.190), (7.183) and (7.176),

$$|f_\xi(x) - f_{\bar{a}}(\bar{x})| \le |f_\xi(x) - \inf(f_\xi)| + |\inf(f_\xi) - \inf(f_{\bar{a}})| +$$

$$|\inf(f_{\bar{a}}) - f_{\bar{a}}(\bar{x})| \le \alpha_2 r/16 + \alpha_2 r/16 + (2n)^{-1} < n^{-1}$$

and

$$|f_\xi(x) - f_{\bar{a}}(\bar{x})| < n^{-1}. \tag{7.193}$$

By (7.192), (7.191) and (7.190),

$$f_{\bar{a}}(x) \le f_\xi(x) + r\alpha_2/16 \le \inf(f_\xi) + \alpha_2 r/8 \le \inf(f_{\bar{a}}) + 3\alpha_2 r/16,$$

$$f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + 3\alpha_2 r/16. \tag{7.194}$$

In view of (7.194) and the property (P5), $||x|| \le \bar{d}$ and there exists $u \in X$ such that $||u|| \le (2n)^{-1}$ and $x \le u + \bar{x}$. Thus if $x \in K$ satisfies (7.191), then (7.193) holds, $||x|| \le \bar{d}$ and there exists $u \in X$ such that

$$||u|| \le (2n)^{-1} \text{ and } x \le u + \bar{x}. \tag{7.195}$$

Thus we have shown that for each $\xi \in B_{d_w}(\bar{a}, \bar{\alpha} r)$ the inequalities (7.190) hold and if $x \in K$ satisfies (7.191), then (7.193) is valid, $||x|| \le \bar{d}$ and there exists $u \in X$ such that $||u|| \le (2n)^{-1}$ and $x \le u + \bar{x}$.

It follows from the definition of $\mathcal{A}_n$ and (7.173) that

$$B_{d_w}(\bar{a}, \bar{\alpha} r/2) \subset \mathcal{A}_n \subset \mathcal{A} \setminus \Omega_{nm}. \tag{7.196}$$

Since (7.187) implies (7.181) we conclude that in both cases

$$B_{d_w}(\bar{a}, \bar{\alpha} r/2) \cap \Omega_{nm} = \emptyset \tag{7.197}$$

with $\bar{a} \in \mathcal{A}$ satisfying (7.184). (Note that if (7.177) holds, then $\bar{a} = a$.) Therefore the set Ω_{nm} is porous in $\mathcal{A}$ with respect to (d_w, d_s). This implies that $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) for all natural numbers n. Therefore $\mathcal{A} \setminus (\cap_{n=1}^{\infty} \mathcal{A}_n)$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). Theorem 7.36 is proved.

We also use the following hypotheses about the functions.

(H4) For each positive number ϵ and each natural number m there exist positive numbers δ, r_0 such that the following property holds:

(P7) For each $a \in \mathcal{A}$ satisfying $\inf(f_a) \le m$ and each $r \in (0, r_0]$ there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K_{min}$ and $\bar{d} > 0$ such that (7.171) holds and if $x \in K$ satisfies (7.172), then $||x|| \le \bar{d}$ and there exists $u \in X$ for which

$$||u|| \le \epsilon, \ x \le \bar{x} + u. \tag{7.198}$$

Theorem 7.37. *Assume that (H1), (H3) and (H4) hold and K_{min} is a closed subset of the Banach space X. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $a \in \mathcal{F}$ the following assertions hold:*

1. The minimization problem for f_a on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$.

2. $\inf(f_a)$ is attained at a unique point.

We can prove Theorem 7.37 analogously to the proof of Theorem 7.36. Recall that for each natural number n, $\tilde{\mathcal{A}}_n$ is the set of all $a \in \mathcal{A}$ which have the property (P1) with $x \in K_{min}$. Set

$$\mathcal{F} = \cap_{n=1}^{\infty} \tilde{\mathcal{A}}_n. \tag{7.199}$$

Proposition 7.33 implies that for each $a \in \mathcal{F}$ assertions 1 and 2 hold. Thus in order to prove Theorem 7.37 it is sufficient to show that for each natural number n the set $\mathcal{A} \setminus \tilde{\mathcal{A}}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). We can show this fact analogously to the proof of Theorem 7.36.

Note that the results of this section were obtained in [115].

7.15 Spaces of increasing functions

In the section we use the functional $\lambda : X \to R^1$ introduced in Section 7.2 (see (7.24)).

Denote by $\mathcal{M}$ the set of all increasing lower semicontinuos bounded from below functions $f : K \to R^1 \cup \{+\infty\}$ which are not identically $+\infty$. For each $f, g \in \mathcal{M}$ set

$$\tilde{d}_s(f, g) = \sup\{|f(x) - g(x)| : x \in X\}, \tag{7.200}$$

$$d_s(f, g) = \tilde{d}_s(f, g)(1 + \tilde{d}_s(f, g))^{-1}. \tag{7.201}$$

Clearly, the metric space $(\mathcal{M}, d_s)$ is complete. Denote by $\mathcal{M}_v$ the set of all finite-valued functions $f \in \mathcal{M}$ and by $\mathcal{M}_c$ the set of all finite-valued continuous functions $f \in \mathcal{M}$. It is easy to see that $\mathcal{M}_v$ and $\mathcal{M}_c$ are closed subsets of the metric space $(\mathcal{M}, d_s)$.

We say that the set K has property (C) if K_{min} is a closed subset of K and for each $x \in K$ there is $y \in K_{min}$ such that $y \leq x$.

Denote by $\mathcal{M}_g$ the set of all $f \in \mathcal{M}$ such that $f(x) \to \infty$ as $||x|| \to \infty$. Clearly $\mathcal{M}_g$ is a closed subset of the metric space $(\mathcal{M}, d_s)$. Set $\mathcal{M}_{gc} = \mathcal{M}_g \cap \mathcal{M}_c$, $\mathcal{M}_{gv} = \mathcal{M}_g \cap \mathcal{M}_v$.

Clearly,

$$\mathcal{M}_c \subset \mathcal{M}_v \subset \mathcal{M} \text{ and } \mathcal{M}_{gc} \subset \mathcal{M}_{gv} \subset \mathcal{M}_g \subset \mathcal{M}. \tag{7.202}$$

Remark 7.38. Let $K = X_+$ and define

$$f_1(x) = ||x||,\ x \in K,\ f_2(x) = ||x||,\ x \in K \setminus \{0\},\ f_2(0) = -1,$$

$$f_3(x) = ||x|| \text{ if } x \in K \text{ and } ||x|| \leq 1,\ f_3(x) = +\infty \text{ if } x \in K \text{ and } ||x|| > 1.$$

It is easy to see that

$$f_1 \in \mathcal{M}_{gc},\ f_2 \in \mathcal{M}_{gv} \setminus \mathcal{M}_{gc},\ f_3 \in \mathcal{M}_g \setminus \mathcal{M}_{gv}.$$

Theorem 7.39. *Assume that $\mathcal{A}$ is either $\mathcal{M}_g$ or $\mathcal{M}_{gv}$ or $\mathcal{M}_{gc}$ and that $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{A}, d_s)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is attained at a unique point.*

Proof: Theorems 7.36 and 7.37 imply that we need to show that (H1), (H2) and (H3) hold and that the property (C) implies (H4). It is easy to see that (H1) holds. For each $f, g \in \mathcal{M}$,

$$\tilde{d}_s(f,g) = d_s(f,g)(1 - d_s(f,g))^{-1}, \tag{7.203}$$

$$\text{if } d_s(f,g) \leq 1/2, \text{ then } \tilde{d}_s(f,g) \leq 2d_s(f,g).$$

These inequalities imply (H3).

We show that (H2) holds and that the property (C) implies (H4).

Let $f \in \mathcal{A}$, $\epsilon \in (0,1)$ and $r \in (0,1]$. Fix $\bar{x} \in K$ for which

$$f(\bar{x}) \leq \inf(f) + \epsilon r/8. \tag{7.204}$$

If K has the property (C), then we assume that $\bar{x}$ is a minimal element of K. Define

$$\bar{f}(x) = f(x) + 2^{-1} r \min\{1, \lambda(x - \bar{x})\} \text{ for all } x \in K. \tag{7.205}$$

It is clear that $\bar{f} \in \mathcal{A}$, $d_s(f, \bar{f}) \leq \tilde{d}_s(f, \bar{f}) \leq r/2$ and

$$\inf(\bar{f}) \leq \bar{f}(\bar{x}) = f(\bar{x}) \leq \inf(f) + \epsilon r/8.$$

Let $x \in K$ and $\bar{f}(x) \leq \inf(\bar{f}) + \epsilon r/8$. Then (7.205) and (7.204) imply that

$$f(x) + 2^{-1} r \min\{1, \lambda(x - \bar{x})\} = \bar{f}(x) \leq \inf(\bar{f}) + \epsilon r/8 \leq \bar{f}(\bar{x}) + \epsilon r/8 =$$

$$f(\bar{x}) + \epsilon r/8 \leq f(x) + \epsilon r/4,\ \min\{1, \lambda(x - \bar{x})\} \leq \epsilon/2$$

and

$$\lambda(x - \bar{x}) \leq \epsilon/2.$$

It follows from the definition of λ that there exists $u \in X$ such that $x \leq \bar{x} + u$ and $||u|| < \epsilon$. Since $\bar{f}(y) \to \infty$ as $||y|| \to \infty$ we obtain that $||x|| \leq \bar{d}$ where $\bar{d}$ is a positive constant which depends only on $\bar{f}$. Thus (H2) holds and if K has the property (C), then (H4) holds. This completes the proof of Theorem 7.39.

Theorem 7.40. *Assume that there exists $\bar{z} \in X$ such that $\bar{z} \leq x$ for all $x \in K$, a space $\mathcal{A}$ is either $\mathcal{M}$ or $\mathcal{M}_v$ or $\mathcal{M}_c$ and that $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{A}, d_s)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is attained at a unique point.*

Proof: We can prove Theorem 7.40 analogously to the proof of Theorem 7.39. The existence of a constant $\bar{d}$ is obtained in the following manner. Let $x \in K$, $u \in X$, $x \leq \bar{x}+u$ and $||u|| < \epsilon$. Then $||x|| \leq ||x-\bar{z}||+||\bar{z}|| \leq ||\bar{z}||+||\bar{x}+u-\bar{z}|| \leq 2|\bar{z}|| + ||\bar{x}|| + \epsilon$ and $||x|| \leq \bar{d}$ where $\bar{d} = 2||\bar{z}|| + ||\bar{x}|| + \epsilon$.

Denote by $\mathcal{M}^+$ the set of all $f \in \mathcal{M}$ such that $f(x) \geq 0$ for all $x \in K$. It is easy to see that $\mathcal{M}^+$ is a closed subset of the metric space $(\mathcal{M}, d_s)$. Define $\mathcal{M}_v^+ = \mathcal{M}^+ \cap \mathcal{M}_v$, $\mathcal{M}_c^+ = \mathcal{M}^+ \cap \mathcal{M}_c$, $\mathcal{M}_g^+ = \mathcal{M}^+ \cap \mathcal{M}_g$, $\mathcal{M}_{gv}^+ = \mathcal{M}^+ \cap \mathcal{M}_{gv}$ and $\mathcal{M}_{gc}^+ = \mathcal{M}^+ \cap \mathcal{M}_{gc}$.

For each $f, g \in \mathcal{M}^+$ set

$$\tilde{d}_w(f, g) = \sup\{|\ln(f(z)+1) - \ln(g(z)+1)| : \ z \in K\}, \tag{7.206}$$

$$d_w(f, g) = \tilde{d}_w(f, g)(1 + \tilde{d}_w(f, g))^{-1}.$$

Clearly, the metric space $(\mathcal{M}^+, d_w)$ is complete and $\mathcal{M}_v^+$, $\mathcal{M}_c^+$, $\mathcal{M}_g^+$, $\mathcal{M}_{gv}^+$ and $\mathcal{M}_{gc}^+$ are closed subsets of $(\mathcal{M}^+, d_w)$. It is easy to see that $d_w(f, g) \leq d_s(f, g)$ for all $f, g \in \mathcal{M}^+$.

Theorem 7.41. *Assume that one of the following cases holds:*

1. $\mathcal{A}$ is either $\mathcal{M}_g^+$ or $\mathcal{M}_{gv}^+$ or $\mathcal{M}_{gc}^+$;

2. there is $\bar{z} \in X$ such that $\bar{z} \leq x$ for all $x \in K$ and that $\mathcal{A}$ is either $\mathcal{M}^+$ or $\mathcal{M}_v^+$ or $\mathcal{M}_c^+$.

Let $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$. If K has the property (C), then for each $f \in \mathcal{F}$, $inf(f)$ is attained at a unique point.

Proof: Theorems 7.36 and 7.37 imply that we need to show that (H1), (H2) and (H3) hold and that the property (C) implies (H4). It is easy to see that (H1) holds. Analogously to the proofs of Theorems 7.39 and 7.40 we can show that (H2) is valid. Thus in order to prove Theorem 7.41 it is sufficient to show that (H3) is true.

Let m be a natural number. Fix $\alpha \in (0, 1)$ for which

$$\alpha < 4^{-1}(2e(m+1))^{-1}. \tag{7.207}$$

Let $r \in (0, 1]$, $x \in K$, $f_1, f_2 \in \mathcal{A}$,

$$d_w(f_1, f_2) \leq \alpha r \ \text{and} \ \min\{f_1(x), f_2(x)\} \leq m.$$

We may assume without loss of generality that $f_1(x) \le f_2(x)$. Then

$$\tilde{d}_w(f_1, f_2) \le 2d_w(f_1, f_2) \le 2\alpha r,$$

$$\ln(f_2(x)+1) - \ln(f_1(x)+1) \le 2\alpha r,\ f_2(x)+1 \le (f_1(x)+1)e^{2\alpha r}$$

and

$$|f_2(x) - f_1(x)| \le (f_1(x)+1)(e^{2\alpha r}-1) \le (m+1)(e^{2\alpha r}-1) =$$

$$2\alpha r(m+1)e^{r_1}$$

with $r_1 \in [0, 2\alpha r]$. Then (7.207) implies that

$$|f_2(x) - f_1(x)| \le 2(m+1)\alpha r e^{2\alpha r} \le 2\alpha e(m+1)r < r.$$

Hence (H3) holds. This completes the proof of Theorem 7.41.

Fix a positive constant c_0 and denote by $\mathcal{M}^{(co)}$ the set of all convex functions $f \in \mathcal{M}$ such that $f(x) \ge c_0||x||$ for all $x \in K$. It is easy to see that $\mathcal{M}^{(co)}$ is a closed subset of the metric space $(\mathcal{M}^+, d_w)$. Put $\mathcal{M}_v^{(co)} = \mathcal{M}^{(co)} \cap \mathcal{M}_v^+$, $\mathcal{M}_c^{(co)} = \mathcal{M}^{(co)} \cap \mathcal{M}_c^+$. It is easy to see that $\mathcal{M}_v^{(co)}$ and $\mathcal{M}_c^{(co)}$ are closed subsets of the metric space $(\mathcal{M}^+, d_w)$.

Theorem 7.42. *Assume that a space $\mathcal{A}$ is either $\mathcal{M}^{co}$ or $\mathcal{M}_v^{(co)}$ or $\mathcal{M}_c^{(co)}$ and that $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_w) and that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{A}, d_w)$. If the set K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is obtained at a unique point.*

Proof: It follows from Theorems 7.36 and 7.37 that we need to show that (H1), (H2) and (H3) hold and that the property (C) implies (H4). Evidently, (H1) is true. Analogously to the proof of Theorem 7.41 we can show that (H3) is valid.

We show that (H2) holds and that the property (C) implies (H4).

Let $f \in \mathcal{A}$, $\epsilon \in (0,1)$ and m be a natural number. Fix an integer $m_0 \ge 1$ and a positive number r_0 such that

$$m_0 > (m + c_0 + 2)(\min\{c_0, 1\})^{-1},\ r_0 < (m_0+2)^{-1}. \tag{7.208}$$

Fix

$$\delta \in (0, 8^{-1}\epsilon \min\{1, c_0\}). \tag{7.209}$$

Assume that $f \in \mathcal{A}$ and

$$r \in (0, r_0],\ \inf(f) \le m. \tag{7.210}$$

There exists $\bar{x} \in K$ such that

$$f(\bar{x}) \le \inf(f) + 4^{-1}\delta r(m+1)^{-1}. \tag{7.211}$$

If K has the property (C) we assume that $\bar{x}$ is a minimal element of K. Define

$$\bar{f}(x) = f(x) + 4^{-1} r \min\{1, c_0\}(m+1)^{-1}\lambda(x - \bar{x}),\ x \in K. \tag{7.212}$$

Clearly, $\bar{f} \in \mathcal{A}$. By (7.211) and (7.210),

$$c_0||\bar{x}|| \le f(\bar{x}) \le \inf(f) + 1 \le m+1,\ ||\bar{x}|| \le (m+1)c_0^{-1}. \tag{7.213}$$

Relations (7.206) and (7.212) imply that

$$d_w(f, \bar{f}) \le \tilde{d}_w(f, \bar{f}) = \sup\{|\ln(\bar{f}(x)+1) - \ln(f(x)+1)| :\ x \in K\} =$$

$$\sup_{x \in K}\{\ln(1 + (\bar{f}(x) - f(x))(f(x)+1)^{-1})\} \le$$

$$\sup_{x \in K}\{\ln(1 + (4^{-1} r \min\{1, c_0\}\lambda(x - \bar{x}))((1 + c_0||x||)(m+1))^{-1})\} \le$$

$$\le \sup_{x \in K}\{4^{-1} r \min\{1, c_0\}\lambda(x - \bar{x})((1 + c_0||x||)(m+1))^{-1}\} \le$$

$$4^{-1} r \sup_{x \in K}\{\min\{1, c_0\}||x - \bar{x}||((1 + c_0||x||)(m+1))^{-1}\} \le$$

$$4^{-1} r \sup_{x \in K}\{c_0||x||(1 + c_0||x||)^{-1} + ||\bar{x}|| \min\{1, c_0\}(m+1)^{-1}\} \le 2^{-1} r.$$

Hence

$$d_w(f, \bar{f}) \le 2^{-1} r. \tag{7.214}$$

By (7.212) and (7.211),

$$\bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + 4^{-1}\delta r(m+1)^{-1} \le \inf(\bar{f}) + 4^{-1}\delta r(m+1)^{-1}. \tag{7.215}$$

It follows from (7.210) that

$$\inf(\bar{f}) \le \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + 4^{-1}\delta r(m+1)^{-1} \le m+1. \tag{7.216}$$

Let $x \in K$ and

$$\bar{f}(x) \le \inf(\bar{f}) + 4^{-1}\delta r(m+1)^{-1}. \tag{7.217}$$

In view of (7.217) and (7.216),

$$c_0||x|| \le m+2,\ ||x|| \le c_0^{-1}(m+2). \tag{7.218}$$

It follows from (7.212), (7.217), (7.211) and (7.209) that

$$f(x) + 4^{-1} r(m+1)^{-1}\{1, c_0\}\lambda(x - \bar{x}) = \bar{f}(x) \le \bar{f}(\bar{x}) + 4^{-1}\delta r(m+1)^{-1} =$$

$$f(\bar{x}) + 4^{-1}\delta r(m+1)^{-1} \le f(x) + 2^{-1}\delta r(m+1)^{-1},$$

$$\min\{1, c_0\}\lambda(x - \bar{x}) \le 2\delta,$$

$$\lambda(x - \bar{x}) \leq 2\delta(\min\{1, c_0\})^{-1} < 4^{-1}\epsilon.$$

Therefore there exists $u \in X$ such that $||u|| < 4^{-1}\epsilon$ and $x \leq \bar{x} + u$. Thus (H2) holds and if K has the property (C), then (H4) holds. Theorem 7.42 is proved.

A function $f : K \to R^1 \cup \{\infty\}$ is called quasiconvex if the set

$$\{x \in K : \; f(x) \leq \alpha\}$$

is convex for any $\alpha \in R^1$. Denote by $\mathcal{M}^{(qu)}$ the set of all quasiconvex functions $f \in \mathcal{M}$. It is easy to see that $\mathcal{M}^{(qu)}$ is a closed subset of the metric space $(\mathcal{M}, d_s)$. Put $\mathcal{M}_v^{(qu)} = \mathcal{M}^{(qu)} \cap \mathcal{M}_v$, $\mathcal{M}_c^{(qu)} = \mathcal{M}^{(qu)} \cap \mathcal{M}_c$, $\mathcal{M}_g^{(qu)} = \mathcal{M}^{(qu)} \cap \mathcal{M}_g$, $\mathcal{M}_{gv}^{(qu)} = \mathcal{M}^{(qu)} \cap \mathcal{M}_{gv}$ and $\mathcal{M}_{gc}^{(qu)} = \mathcal{M}^{(qu)} \cap \mathcal{M}_{gc}$.

Theorem 7.43. *Assume that one of the following cases holds:*

1. A space $\mathcal{A}$ is either $\mathcal{M}_g^{(qu)}$ or $\mathcal{M}_{qv}^{(gu)}$ or $\mathcal{M}_{gc}^{(qu)}$;

2. there exists $\bar{z} \in X$ such that $\bar{z} \leq x$ for all $x \in K$ and a space $\mathcal{A}$ is either $\mathcal{M}^{(qu)}$ or $\mathcal{M}_v^{(qu)}$ or $\mathcal{M}_c^{(qu)}$.

Let $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{A}, d_s)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ *is attained at a unique point.*

Proof: In view of Theorems 7.36 and 7.37 we need to show that (H1), (H2) and (H3) hold and that the property (C) implies (H4). It is easy to see that (H1) is true. Analogously to the proof of Theorem 7.39 we can how that (H3) holds. Now we show that (H2) holds and that the property (C) implies (H4).

Let $f \in \mathcal{A}$, $\epsilon \in (0, 1]$ and $r \in (0, 1]$. Fix $\bar{x} \in K$ such that

$$f(\bar{x}) \leq \inf(f) + \epsilon r/8. \tag{7.219}$$

If K has the property (C), then we assume that $\bar{x}$ is a minimil element of K. Define

$$\bar{f}(x) = \max\{f(x), f(\bar{x}) + 2^{-1} r \min\{\lambda(x - \bar{x}), 1\}\}, \; x \in K. \tag{7.220}$$

It is easy to see that $\bar{f} \in \mathcal{A}$,

$$d_s(f, \bar{f}) \leq \tilde{d}_s(f, \bar{f}) \leq r. \tag{7.221}$$

$$\bar{f}(\bar{x}) = f(\bar{x}) \leq \inf(f) + \epsilon r/8 \leq \inf(\bar{f}) + \epsilon r/8 \tag{7.222}$$

and

$$\inf(\bar{f}) \leq \bar{f}(\bar{x}) = f(\bar{x}) \leq \inf(f) + 1.$$

Let $x \in K$ and

$$\bar{f}(x) \leq \inf(\bar{f}) + \epsilon r/8. \tag{7.223}$$

It follows from (7.220) and (7.223) that

$$2^{-1}r\min\{\lambda(x-\bar{x}),1\}+f(\bar{x})\leq\bar{f}(x)\leq\bar{f}(\bar{x})+\epsilon r/8=$$

$$f(\bar{x})+\epsilon r/8,\ \min\{\lambda(x-\bar{x}),1\}\leq\epsilon/4,\ \lambda(x-\bar{x})\leq\epsilon/4$$

and there exists $u\in X$ such that

$$||u||<\epsilon/2,\ x\leq\bar{x}+u. \tag{7.224}$$

In the first case we choose $\bar{d}>0$ such that $\bar{f}(y)>\inf(\bar{f})+1$ for all $y\in K$ satisfying $||y||>\bar{d}$, and obtain that $||x||\leq\bar{d}$. In the second case

$$\bar{z}\leq x\leq\bar{x}+u,\ ||x||\leq||\bar{z}||+||x-\bar{z}||\leq$$

$$||\bar{z}||+||\bar{x}+u-\bar{z}||\leq 2||\bar{z}||+||\bar{x}||+||u||\leq 2||\bar{z}|+||\bar{x}||+1$$

and $||x||\leq\bar{d}:=2||\bar{z}||+||\bar{x}||+1$. Therefore in both cases (H2) holds and if the set K has the property (C), then (H4) is true. This completes the proof of Theorem 7.43.

Theorem 7.44. *Let $\mathcal{A}$ be defined as in Theorem 7.43 and let $\mathcal{A}^+$ be the set of all $f\in\mathcal{A}$ such that $f(x)\geq 0$ for all $x\in K$. Assume that $f_a=a$ for all $a\in\mathcal{A}^+$. Then the metric spaces $(\mathcal{A}^+,d_s)$ and $(\mathcal{A}^+,d_w)$ are complete and there exists a set $\mathcal{F}\subset\mathcal{A}^+$ such that the complement $\mathcal{A}^+\setminus\mathcal{F}$ is σ-porous in $\mathcal{A}^+$ with respect to (d_w,d_s) and that for each $f\in\mathcal{F}$ the minimization problem for f on K is strongly well-posed with respect to $(\mathcal{A}^+,d_w)$. If the set K has the property (C), then for each $f\in\mathcal{F}$, $\inf(f)$ is attained at a unique point.*

Proof: It follows from Theorems 7.36 and 7.37 that we need to show that (H1), (H2) and (H3) are true and that the property (C) implies (H4). It is easy to see that (H1) is true. Analogously to the proof of Theorem 7.43 we can show that (H2) holds and that the property (C) implies (H4). We can prove (H3) as in the proof of Theorem 7.41.

The results of this section were obtained in [115].

7.16 Comments

In this chapter we consider the problem minimize $f(x)$ subject to $x\in K$ where K is a closed subset of an ordered Banach space X and f belongs to a space of increasing lower semicontinuous functions on K. We show that for most functions f in this space the corresponding minimization problem has a unique solution. Instead of considering the existence of solutions for a single cost function we study it for a space of all such cost functions equipped with an appropriate complete uniformity. Using the generic approach and the porosity notion we show that a solution exists for most functions. These results and their extensions are obtained as realizations of abstract variational principles.

8

Generic Well-Posedness of Minimization Problems with Constraints

8.1 Variational principles

In this chapter we usually consider topological spaces with two topologies where one is weaker than the other. (Note that they can coincide). We refer to them as weak and strong topologies, respectively. If (X, d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). Assume that X_1 and X_2 are topological spaces and that each of them is endowed with a weak and a strong topology. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies of X_1 and X_2 and a strong topology which is the product of the strong topologies of X_1 and X_2. If $Y \subset X_1$, then we consider the topological subspace Y with the relative weak and strong topologies (unless other topologies are introduced). If (X_i, d_i), $i = 1, 2$ are metric spaces with the metric d_1 and d_2, respectively, then the space $X_1 \times X_2$ is endowed with the metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2),\ (x_i, y_i) \in X \times Y,\ i = 1, 2.$$

For each function $f : Y \to [-\infty, \infty]$ where Y is nonempty, we set $\inf(f) = \inf\{f(y) : y \in Y\}$.

We consider a metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology and denoted by τ_s. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology and denoted by τ_w. (Note that these topologies can coincide.) We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$. In our study we use the following basic hypotheses about the functions introduced in Section 4.1.

A.J. Zaslavski, *Optimization on Metric and Normed Spaces*, Springer Optimization and Its Applications 44, DOI 10.1007/978-0-387-88621-3_8,

(H1) For each $a \in \mathcal{A}$ and each pair of positive numbers ϵ and γ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A} : \ d(a,b) < \epsilon\} \neq \emptyset$$

and for any $b \in \mathcal{W}$

(i) $\inf(f_b)$ is finite;

(ii) if $z \in X$ is such that $f_b(z) \le \inf(f_b) + \eta$, then $\rho(z,x) \le \gamma$ and $|f_b(z) - \alpha| \le \gamma$.

(H2) if $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in X.

Let $a \in \mathcal{A}$. Recall that the minimization problem for f_a on (X,ρ) is strongly well-posed with respect to $\mathcal{A}$ endowed with the weak topology if $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$ and the following assertion holds:

For each positive number ϵ there exist a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and a positive number δ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(x_a, z) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

Now we consider the metric space $(\mathcal{A}_1 \times \mathcal{A}_2, d)$ where $(\mathcal{A}_i, d_i)$, $i = 1,2$ are complete metric space and

$$d((a_1,a_2),(b_1,b_2)) = d_1(a_1,b_1) + d_2(a_2,b_2), \ (a_1,a_2),(b_1,b_2) \in \mathcal{A}_1 \times \mathcal{A}_2.$$

For the space $\mathcal{A}_2$ we consider the topology induced by the metric d_2 (the strong and weak topologies coincide) and for the space $\mathcal{A}_1$ we consider the strong topology which is induced by the metric d_1 and a weak topology which is weaker than the strong topology. The strong topology of $\mathcal{A}_1 \times \mathcal{A}_2$ is the product of the strong topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$ and the weak topology of $\mathcal{A}_1 \times \mathcal{A}_2$ is the product of the weak topology of $\mathcal{A}_1$ and the topology of $\mathcal{A}_2$.

Assume that with every $a \in \mathcal{A}_1$ a continuous function $\phi_a : X \to R^1$ is associated and with every $a \in \mathcal{A}_2$ a set $S_a \subset X$ is associated. For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \text{ for all } x \in S_{a_2}, \ f_a(x) = \infty \text{ for all } x \in X \setminus S_{a_2}. \tag{8.1}$$

Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 : \ \inf(f_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. We assume that $\mathcal{A}$ is nonempty.

We use the following hypotheses.

(A1) For each $a_1 \in \mathcal{A}_1$, $\inf(\phi_{a_1}) > -\infty$ and for each $a \in \mathcal{A}_1 \times \mathcal{A}_2$ the function f_a is lower semicontinuous.

(A2) For each $a_1 \in \mathcal{A}_1$ and each pair of positive numbers ϵ and D there exists a neighborhood $\mathcal{U}$ of a_1 in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{U}$ and each $x \in X$ which satisfy $\min\{\phi_{a_1}(x), \phi_b(x)\} \le D$ the inequality $|\phi_{a_1}(x) - \phi_b(x)| \le \epsilon$ is valid.

(A3) For each $\gamma \in (0,1)$ there exist $\epsilon(\gamma) > 0$ and $\delta(\gamma) > 0$ such that $\epsilon(\gamma), \delta(\gamma) \to 0$ as $\gamma \to 0$ and the following property holds:

For each $\gamma \in (0,1)$, each $a \in \mathcal{A}_1$, each nonempty set $Y \subset X$ and each $\bar{x} \in Y$ satisfying

$$\phi_a(\bar{x}) \le \inf\{\phi_a(z) : \ z \in Y\} + \delta(\gamma) < \infty$$

there exists $\bar{a} \in \mathcal{A}_1$ such that the following conditions hold:

$$d_1(a, \bar{a}) \le \epsilon(\gamma), \ \phi_{\bar{a}}(z) \ge \phi_a(z), \ z \in X, \ \phi_{\bar{a}}(\bar{x}) \le \phi_a(\bar{x}) + \delta(\gamma);$$

if $y \in Y$ satisfies

$$\phi_{\bar{a}}(y) \le \inf\{\phi_{\bar{a}}(z) : \ z \in Y\} + 2\delta(\gamma),$$

then $\rho(y, \bar{x}) \le \gamma$.

(A4) For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ satisfying $\inf(f_a) < \infty$ and each $\epsilon, \gamma \in (0,1)$ there exist $\bar{a}_2 \in \mathcal{A}_2$, $\bar{r} \in (0,1)$ and $\bar{x} \in S_{\bar{a}_2}$ such that the following properties hold:

$$\{b \in \mathcal{A}_2 : \ d_2(\bar{a}_2, b) \le \bar{r}\} \subset \{b \in \mathcal{A}_2 : \ d_2(a_2, b) \le \epsilon\};$$

if $b \in \mathcal{A}_2$ satisfies $d_2(\bar{a}_2, b) \le \bar{r}$ and if $x \in S_b$, then $\phi_{a_1}(\bar{x}) \le \phi_{a_1}(x) + \gamma$;

for each positive number Δ there exists a positive number $\delta < \bar{r}$ such that if $b \in \mathcal{A}_2$ satisfies $d_2(b, \bar{a}_2) \le \delta$, then there exists $x \in S_b$ for which $\rho(x, \bar{x}) \le \Delta$.

Remark 8.1. We showed in [105] (see also Section 4.14) that if (A3) holds, then the numbers $\epsilon(\gamma)$ and $\delta(\gamma)$ can be chosen such that $0 < \delta(\gamma) \le \epsilon(\gamma) \le \gamma$. In this chapter we always assume that this inequality holds.

The following result obtained in [124] will be proved in the next section.

Proposition 8.2. *Assume that (A1)–(A4) hold. Then (H1) holds.*

8.2 Proof of Proposition 8.2

Let $a = (a_1, a_2) \in \mathcal{A}$ and let $\epsilon, \gamma \in (0,1)$. We may assume that $\inf(f_a) < \infty$. Fix

$$\gamma_0 \in (0, 8^{-1} \min\{\epsilon, \gamma\}).$$

Let $\epsilon(\gamma_0)$, $\delta(\gamma_0) > 0$ be as guaranteed by (A3) (namely, (A3) holds with $\gamma = \gamma_0$, $\epsilon(\gamma) = \epsilon(\gamma_0)$, $\delta(\gamma) = \delta(\gamma_0)$). Fix a positive number

$$\delta_1 < 4^{-1}\delta(\gamma_0). \tag{8.2}$$

In view of (A4) there exist

$$\bar{a}_2 \in \mathcal{A}_2, \ \bar{r} \in (0,1), \ \bar{x} \in S_{\bar{a}_2} \tag{8.3}$$

such that

$$\{b \in \mathcal{A}_2 : \ d_2(\bar{a}_2, b) \le \bar{r}\} \subset \{b \in \mathcal{A}_2 : \ d_2(a_2, b) \le \epsilon(\gamma_0)\}, \tag{8.4}$$

$$\phi_{a_1}(\bar{x}) \le \phi_{a_1}(x) + \delta_1$$
$$\text{for all } x \in \cup\{S_b : \ b \in \mathcal{A}_2 \text{ and } d_2(\bar{a}_2, b) \le \bar{r}\} \tag{8.5}$$

and that the following property holds:

(P1) For each positive number Δ there exists a positive number $r_\Delta < \bar{r}$ such that for each $b \in \mathcal{A}_2$ satisfying $d_2(b, \bar{a}_2) \le r_\Delta$ there exists $x \in S_b$ for which $\rho(x, \bar{x}) \le \Delta$.

By the choice of $\epsilon(\gamma_0)$, $\delta(\gamma_0)$, (A3) (with $a = a_1$, $Y = \cup\{S_b : \ b \in \mathcal{A}_2 \text{ and } d_2(b, \bar{a}_2) \le \bar{r}\}$), (8.2) and (8.5), there exists $\bar{a}_1 \in \mathcal{A}_1$ such that

$$d_1(a_1, \bar{a}_1) \le \epsilon(\gamma_0), \ \phi_{\bar{a}_1}(z) \ge \phi_{a_1}(z) \text{ for all } z \in X, \tag{8.6}$$

$$\phi_{\bar{a}_1}(\bar{x}) \le \phi_{a_1}(\bar{x}) + \delta(\gamma_0)$$

and that the following property holds:

(P2) If $y \in \cup\{S_b : \ b \in \mathcal{A}_2 \text{ and } d_2(b, \bar{a}_2) \le \bar{r}\}$ satisfies

$$\phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) : \ z \in \cup\{S_b : \ b \in \mathcal{A}_2 \text{ and } d_2(b, \bar{a}_2) \le \bar{r}\}\} + 2\delta(\gamma_0),$$

then $\rho(y, \bar{x}) \le \gamma_0$.

Since the function $\phi_{\bar{a}_1}$ is continuous there exists a positive number Δ_0 such that

$$|\phi_{\bar{a}_1}(x) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1 \text{ for all } x \in X \text{ satisfying } \rho(x, \bar{x}) \le \Delta_0. \tag{8.7}$$

By property (P1) there exists

$$r_0 \in (0, \bar{r}) \tag{8.8}$$

such that the following property holds:

(P3) For each $b \in \mathcal{A}_2$ satisfying $d_2(b, \bar{a}_2) \le r_0$ there exists $x \in S_b$ such that $\rho(x, \bar{x}) \le \Delta_0$.

Let $b \in \mathcal{A}_2$ satisfy

$$d_2(b, \bar{a}_2) \le r_0. \tag{8.9}$$

It follows from (8.9) and property (P3) that $S_b \neq \emptyset$. We show that the following property holds:

(P4) For each $y \in S_b$ satisfying

$$\phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) : \ z \in S_b\} + \delta_1$$

the inequalities

$$\rho(y, \bar{x}) \le \gamma_0, \ |\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(\bar{x})| \le 2\delta_1 + \delta(\gamma_0)$$

hold.

Assume that $y \in X$ satisfies

$$y \in S_b,\ \phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_b\} + \delta_1. \tag{8.10}$$

By (8.9) and property (P3) there is $z_0 \in X$ such that

$$z_0 \in S_b,\ \rho(z_0, \bar{x}) \le \Delta_0. \tag{8.11}$$

By (8.11) and (8.7),

$$|\phi_{\bar{a}_1}(z_0) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1. \tag{8.12}$$

Relations (8.10), (8.11), (8.12), (8.6), (8.5) and (8.2) imply that

$$\phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_b\} + \delta_1$$

$$\le \phi_{\bar{a}_1}(z_0) + \delta_1 \le \phi_{\bar{a}_1}(\bar{x}) + 2\delta_1$$

$$\le \phi_{a_1}(\bar{x}) + \delta(\gamma_0) + 2\delta_1$$

$$\le \inf\{\phi_{a_1}(z) :\ z \in \cup\{S_g :\ g \in \mathcal{A}_2 \text{ and } d_2(\bar{a}_2, g) \le \bar{r}\}\} + \delta_1 + \delta(\gamma_0) + 2\delta_1$$

$$\le \inf\{\phi_{\bar{a}_1}(z) :\ z \in \cup\{S_g :\ g \in \mathcal{A}_2 \text{ and } d_2(\bar{a}_2, g) \le \bar{r}\}\} + 2\delta(\gamma_0). \tag{8.13}$$

In view of property (P2), (8.10), (8.9), (8.8) and (8.13),

$$\rho(y, \bar{x}) \le \gamma_0. \tag{8.14}$$

It follows from (8.13), (8.10), (8.9), (8.8) and (8.6) that

$$\phi_{\bar{a}_1}(y) \le \phi_{\bar{a}_1}(\bar{x}) + 2\delta_1$$

$$\le \inf\{\phi_{a_1}(z) :\ z \in \cup\{S_g :\ g \in \mathcal{A}_2 \text{ and } d_2(\bar{a}_2, g) \le \bar{r}\}\} + 3\delta_1 + \delta(\gamma_0)$$

$$\le \phi_{a_1}(y) + 3\delta_1 + \delta(\gamma_0) \le \phi_{\bar{a}_1}(y) + 3\delta_1 + \delta(\gamma_0)$$

and

$$|\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(\bar{x})| \le 2\delta_1 + \delta(\gamma_0).$$

Together with (8.14) this inequality implies that property (P4) holds for each $b \in \mathcal{A}_2$ satisfying (8.9).

Fix

$$D > 4 + |\phi_{\bar{a}_1}(\bar{x})|. \tag{8.15}$$

(A2) implies that there exists an open neighborhood $\mathcal{V}$ of $\bar{a}_1$ in $\mathcal{A}_1$ with the weak topology for which the following property holds:

(P5) If $b \in \mathcal{V}$ and if $x \in X$ satisfies $\min\{\phi_b(x), \phi_{\bar{a}_1}(x)\} \le D + 2$, then $|\phi_{\bar{a}_1}(x) - \phi_b(x)| \le \delta_1/4$ holds.

It follows from property (P5) and (8.15) that for each $b \in \mathcal{V}$

$$|\phi_b(\bar{x}) - \phi_{\bar{a}_1}(\bar{x})| \le \delta_1/4,\ \inf(\phi_b) \le \phi_b(\bar{x}) \le D - 2. \tag{8.16}$$

Now we show that (H1) holds with the open set

$$\mathcal{W} = \mathcal{V} \times \{g \in \mathcal{A}_2 :\ d_2(g, \bar{a}_2) < r_0\},\ x = \bar{x},\ \alpha = \phi_{\bar{a}_1}(\bar{x}),\ \eta = \delta_1/4.$$

Assume that

$$b = (b_1, b_2) \in \mathcal{V} \times \{g \in \mathcal{A}_2 :\ d_2(g, \bar{a}_2) < r_0\}. \tag{8.17}$$

It follows from (8.17) and property (P4) which holds with $b = b_2$ that

$$\begin{aligned}\inf(f_{(\bar{a}_1, b_2)}) &= \inf\{\phi_{\bar{a}_1}(y) :\ y \in S_{b_2} \text{ and } \phi_{\bar{a}_1}(y) \le \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{b_2}\} + \delta_1\} \\ &\in [\phi_{\bar{a}_1}(\bar{x}) - 2\delta_1 - \delta(\gamma_0),\ \phi_{\bar{a}_1}(\bar{x}) + 2\delta_1 + \delta(\gamma_0)]. \end{aligned} \tag{8.18}$$

In view of (8.18), (8.15), (8.2) and the choice of γ_0,

$$\inf(f_{(\bar{a}_1, b_2)}) < D - 2. \tag{8.19}$$

Property (P5) and (8.19) imply that $\inf(f_{(b_1, b_2)}) < \infty$.

Assume now that $z \in X$ satisfies

$$f_b(z) \le \inf(f_b) + \delta_1/4. \tag{8.20}$$

Then

$$z \in S_{b_2},\ \phi_{b_1}(z) \le \inf\{\phi_1(y) :\ y \in S_{b_2}\} + \delta_1/4. \tag{8.21}$$

It follows from (8.19), (8.17) and (P5) that

$$\begin{aligned}\inf(f_{(b_1, b_2)}) &= \inf\{\phi_{b_1}(z) :\ z \in S_{b_2}\} \\ &\le \inf\{\phi_{b_1}(z) :\ z \in S_{b_2} \text{ and } \phi_{\bar{a}_1}(z) \le D - 1\} \\ &\le \inf\{\phi_{\bar{a}_1}(z) + \delta_1/4 :\ z \in S_{b_2} \text{ and } \phi_{\bar{a}_1}(z) \le D - 1\} \\ &= \delta_1/4 + \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{b_2} \text{ and } \phi_{\bar{a}_1}(z) \le D - 1\} \\ &= \delta_1/4 + \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{b_2}\} = \delta_1/4 + \inf(f_{(\bar{a}_1, b_2)}). \end{aligned} \tag{8.22}$$

It follows from (8.22), (8.19), (8.2) and the choice of γ_0 that

$$\inf(f_{(b_1, b_2)}) \le \inf(f_{(\bar{a}_1, b_2)}) + \delta_1/4 < D - 1. \tag{8.23}$$

Relations (8.23), (8.17) and property (P5) imply that

$$\begin{aligned}\inf(f_{(\bar{a}_1, b_2)}) &= \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{b_2}\} \\ &\le \inf\{\phi_{\bar{a}_1}(z) :\ z \in S_{b_2} \text{ and } \phi_{b_1}(z) \le D\} \\ &\le \inf\{\phi_{b_1}(z) + \delta_1/4 :\ z \in S_{b_2} \text{ and } \phi_{b_1}(z) \le D\} \\ &= \delta_1/4 + \inf\{\phi_{b_1}(z) :\ z \in S_{b_2} \text{ and } \phi_{b_1}(z) \le D\} = \delta_1/4 + \inf(f_{(b_1, b_2)}). \end{aligned}$$

Together with (8.23) this inequality implies that

$$|\inf(f_{(\bar{a}_1,b_2)}) - \inf(f_{(b_1,b_2)})| \leq \delta_1/4. \tag{8.24}$$

In view of (8.24) (with $b_2 = \bar{a}_2$)

$$|\inf(f_{(\bar{a}_1,\bar{a}_2)}) - \inf(f_{(b_1,\bar{a}_2)})| \leq \delta_1/4. \tag{8.25}$$

By (8.20), (8.21), (8.23), (8.2) and the choice of γ_0,

$$\phi_{b_1}(z) \leq \delta_1/4 + D - 1 < D.$$

Together with (8.17) and (P5) this inequality implies that

$$|\phi_{b_1}(z) - \phi_{\bar{a}_1}(z)| \leq \delta_1/4. \tag{8.26}$$

Relations (8.26), (8.21), (8.20) and (8.24) imply that

$$\phi_{\bar{a}_1}(z) \leq \phi_{b_1}(z) + \delta_1/4 \leq \inf(f_b) + \delta_1/4 + \delta_1/4$$
$$\leq \inf(f_{(\bar{a}_1,b_2)}) + \delta_1/4 + \delta_1/4 + \delta_1/4. \tag{8.27}$$

It follows from (8.27), (8.21), (8.17) and (P4) (with $b = b_2$, $y = z$) that

$$\rho(z, \bar{x}) \leq \gamma_0, \tag{8.28}$$

$$|\phi_{\bar{a}_1}(z) - \phi_{\bar{a}_1}(\bar{x})| \leq 2\delta_1 + \delta(\gamma_0). \tag{8.29}$$

It follows from (8.2), the choice of γ_0, (8.26) and (8.29) that

$$|\phi_{b_1}(z) - \phi_{\bar{a}_1}(\bar{x})| \leq 3\delta_1 + \delta(\gamma_0) < \gamma.$$

Proposition 8.2 is proved.

8.3 Minimization problems with mixed continuous constraints

We use the convention that $\infty/\infty = 1$ and $\infty - \infty = 0$.

Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces. For each mapping $F : X_1 \to X_2$ set

$$\mathrm{Lip}(F) = \sup\{\rho_2(F(x), F(y))(\rho_1(x, y))^{-1} : \ x, y \in X_1, \ x \neq y\}. \tag{8.30}$$

Let $(X, ||\cdot||)$ be a norm space. For each $x \in X$ and each positive number r put

$$B_X(x, r) = \{y \in X : \ ||y - x|| \leq r\}, \ B^o_X(x, r) = \{y \in X : \ ||y - x|| < r\},$$

$$B_X(r) = B_X(0, r), \ B^o_X(r) = B^o_X(0, r).$$

Denote by $\dim(X)$ the dimension of X. If X is an infinite-dimensional space, then $\dim(X) = \infty$.

Let $(X, ||\cdot||)$, $(Y, ||\cdot||)$ be Banach spaces. Denote by $C(X,Y)$ the set of all continuous mappings $G : X \to Y$. For each $G_1, G_2 \in C(X,Y)$ define

$$\tilde{d}_{XY}(G_1, G_2) = \sup\{||G_1(x) - G_2(x)|| : \ x \in X\}, \tag{8.31}$$

$$d_{XY}(G_1 G_2) = \tilde{d}_{XY}(G_1, G_2)(1 + \tilde{d}_{XY}(G_1, G_2))^{-1}.$$

It is not difficult to see that the metric space $(C(X,Y), d_{XY})$ is complete.

If $(X, ||\cdot||)$ is a norm space, then we always equip X with a metric ρ_X induced by the norm of X:

$$\rho_X(y,z) = ||y - z||, \ y, z \in X.$$

If $(X, ||\cdot||)$, $(Y, ||\cdot||)$ are norm spaces and $L : X \to Y$ is a continuous linear operator, then

$$||L|| = \sup\{||L(x)|| : \ x \in X \text{ and } ||x|| \le 1\}.$$

Let $(X, ||\cdot||)$, $(Y, ||\cdot||)$ and $(Z, ||\cdot||)$ be Banach spaces, Y be finite-dimensional, $\dim(Y) < \dim(X)$ and let Z be ordered by a convex closed cone $Z_+ \subset X$.

Assume that h_* is an interior point of Z_+ and

$$||z|| = \inf\{\lambda \ge 0 : \ -\lambda h_* \le z \le \lambda h_*\}, \ z \in Z. \tag{8.32}$$

Denote by $C_b(X)$ the set of all continuous functions $f : X \to R^1$ which are bounded from below. For each $f_1, f_2 \in C_b(X)$ define

$$\tilde{d}_{cs}(f_1, f_2) = \sup\{|f_1(x) - f_2(x)| : \ x \in X\} + \mathrm{Lip}(f_1 - f_2), \ f_1, f_2 \in C_b(X), \tag{8.33}$$

$$d_{cs}(f_1, f_2) = \tilde{d}_{cs}(f_1, f_2)(1 + \tilde{d}_{cs}(f_1, f_2))^{-1}.$$

It is not difficult to see that the metric space $(C_b(X), d_{cs})$ is complete. We equip the space $C_b(X)$ with the topology induced by the metric d_{cs} which is called the strong topology.

Now we equip the set $C_b(X)$ with a weak topology.

For each positive number ϵ put

$$E_w(\epsilon) = \{(g,h) \in C_b(X) \times C_b(X) :$$

$$|g(x) - h(x)| < \epsilon + \epsilon \max\{|g(x)|, |h(x)|\} \text{ for all } x \in X\}. \tag{8.34}$$

Using the following simple lemma we can easily show that for the set $C_b(X)$ there exists a uniformity which is determined by the base $E_w(\epsilon)$, $\epsilon > 0$. This uniformity induces in $C_b(X)$ the weak topology.

Lemma 8.3. *Let* $a, b \in R^1$, $\epsilon \in (0,1)$, $\Delta \geq 0$ *and*

$$|a - b| < (1 + \Delta)\epsilon + \epsilon \max\{|a|, |b|\}.$$

Then

$$|a - b| < (1 + \Delta)(\epsilon + \epsilon^2(1 - \epsilon)^{-1}) + \epsilon(1 - \epsilon)^{-1} \min\{|a|, |b|\}.$$

Note that Lemma 8.3 is proved in a straightforward manner (see Lemma 1.1 of [105]).

It is not difficult to see that the uniformity determined by the base $E_w(\epsilon)$, $\epsilon > 0$ is metrizable and complete.

We also consider the complete metric spaces

$$(C(X, Y), d_{XY}),\ (C(X, Z), d_{XZ}).$$

Denote by $C_{bu}(X)$ the set of all $f \in C_b(X)$ which are uniformly continuous on X, by $C_{bub}(X)$ the set of all $f \in C_b(X)$ which are uniformly continuous on bounded subsets of X, by $C_{bL}(X)$ the set of all $f \in C_b(X)$ which are Lipschitz on X, by $C_{bLb}(X)$ the set of all $f \in C_b(X)$ which are Lipschitz on all bounded subsets of X and by $C_{bLl}(X)$ the set of all $f \in C_b(X)$ which are locally Lipschitz. It is easy to see that $C_{bu}(X)$, $C_{bub}(X)$, $C_{bL}(X)$, $C_{bLb}(X)$ and $C_{bLl}(X)$ are closed subsets of $(C_b(X), d_{cs})$.

Let $\mathcal{A}_1$ be one of the following spaces

$$C_b(X),\ C_{bu}(X),\ C_{bub}(X),\ C_{bL}(X),\ C_{bLb}(X),\ C_{bLl}(X)$$

and let

$$\mathcal{A}_{21} = C(X, Y),\ \mathcal{A}_{22} = C(X, Z),\ \mathcal{A}_{23} = Y,\ \mathcal{A}_{24} = Z.$$

.

Set $\mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22} \times \mathcal{A}_{23} \times \mathcal{A}_{24}$. For each $a_2 = (G, H, y, z) \in \mathcal{A}_2$ set

$$S_{a_2} = \{x \in X :\ G(x) = y,\ H(x) \leq z\}. \tag{8.35}$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $J_a : X \to R^1 \cup \{\infty\}$ by

$$J_a(x) = a_1(x),\ x \in S_{a_2},\ J_a(x) = \infty,\ x \in X \setminus S_{a_2}. \tag{8.36}$$

The next theorem was obtained in [124].

Theorem 8.4. *Let* $\mathcal{A}$ *be the closure of the set* $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 :\ \inf(J_a) < \infty\}$ *in the space* $\mathcal{A}_1 \times \mathcal{A}_2$ *with the strong topology. There exists a set* $\mathcal{B} \subset \mathcal{A}$ *which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of* $\mathcal{A}$ *such that for any* $a \in \mathcal{B}$ *the minimization problem for* J_a *on* (X, ρ_X) *is strongly well-posed with respect to* $\mathcal{A}$ *endowed with the weak topology.*

Let $y_0 \in Y$, $z_0 \in Z$, $\mathcal{A}_1$ be one of the following spaces

$$C_b(X),\ C_{bu}(X),\ C_{bub}(X),\ C_{bL}(X),\ C_{bLb}(X),\ C_{bLl}(X)$$

and let

$$\mathcal{A}_{21} = C(X,Y),\ \mathcal{A}_{22} = C(X,Z),\ \mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22}.$$

For each $a_2 = (G,H) \in \mathcal{A}_2$ set

$$S_{a_2} = \{x \in X :\ G(x) = y_0,\ H(x) \le z_0\}. \tag{8.37}$$

For each $a = (a_1,a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $\widehat{J}_a : X \to R^1 \cup \{\infty\}$ as follows:

$$\widehat{J}_a(x) = a_1(x),\ x \in S_{a_2},\ \widehat{J}_a(x) = \infty,\ x \in X \setminus S_{a_2}. \tag{8.38}$$

The next theorem was also obtained in [124].

Theorem 8.5. *Let $\mathcal{A}$ be the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 :\ \inf(\widehat{J}_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. There exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem for $\widehat{J}_a$ on (X,ρ_X) is strongly well-posed with respect to $\mathcal{A}$ endowed with the weak topology.*

Remark 8.6. Let $f \in \mathcal{A}_1$, $y_0 \in Y$ and $z_0 \in Z$. Put $G_0(x) = y_0$ and $H_0(x) = z_0$ for all $x \in X$. It is easy to see that $(f, G_0, H_0, y_0, z_0) \in \mathcal{A}$ in the case of Theorem 8.4 and $(f, G_0, H_0) \in \mathcal{A}$ in the case of Theorem 8.5. Thus in Theorems 8.4 and 8.5 the set $\mathcal{A}$ is nonempty.

Since the space Y is finite-dimensional and $\dim(Y) < \dim(X)$ the space Y is linearly isomorphic to a vector subspace of X. We may assume without loss of generality that $Y \subset X$. It is well-known that all norms in a finite-dimensional vector space are equivalent. Therefore we may assume without loss of generality that the norm of Y is induced by the norm of X.

Since Y is a finite-dimensional vector subspace of X there exists a linear continuous operator $L_0 : X \to Y$ such that

$$L_0(x) = -x \text{ for all } x \in Y. \tag{8.39}$$

Theorems 8.4 and 8.5 are obtained as realizations of the variational principles established in Sections 8.1 and 8.2

8.4 An auxiliary result for (A2)

Lemma 8.7. *Assume that $f \in C_b(X)$ and that D, ϵ is a pair of positive numbers. Then there exists a neighborhood $\mathcal{U}$ of $f \in C_b(X)$ with the weak topology such that for each $g \in \mathcal{U}$ and each $x \in X$ satisfying $\min\{f(x), g(x)\} \le D$ the inequality $|f(x) - g(x)| \le \epsilon$ holds.*

Proof: Since the function f is bounded from below there exists $c_0 > 0$ for which

$$f(x) \geq -c_0 \text{ for all } x \in X.$$

Fix $\epsilon_0 > 0$ such that

$$4\epsilon_0(D + 2c_0 + 4) < \min\{1, \epsilon\} \tag{8.40}$$

and $\epsilon_1 > 0$ which satisfies

$$4(\epsilon_1 + \epsilon_1(1 - \epsilon_1)^{-1}) < \epsilon_0. \tag{8.41}$$

Put

$$\mathcal{U} = \{g \in C_b(X) :\ (f, g) \in E_w(\epsilon_1)\} \tag{8.42}$$

(see (8.34)).

Assume that

$$g \in \mathcal{U},\ x \in X \text{ and } \min\{f(x), g(x)\} \leq D. \tag{8.43}$$

It follows from the choice of $\mathcal{U}$ and (8.34) that

$$|f(z) - g(z)| < \epsilon_1 + \epsilon_1 \max\{|f(z)|, |g(z)|\},\ z \in X.$$

By this relation, (8.41) and Lemma 8.3, for all $z \in X$,

$$|f(z) - g(z)| < \epsilon_0 + \epsilon_0 \min\{|f(z)|, |g(z)|\} \tag{8.44}$$

and it follows from the choice of c_0 and (8.40) that

$$g(z) \geq f(z) - \epsilon_0 - \epsilon_0|f(z)| \geq -1 - 2c_0.$$

The relation above, (8.44), the choice of c_0, (8.43) and (8.40) imply that

$$|f(x) - g(x)| < \epsilon_0 + \epsilon_0[\min\{f(x),\ g(x)\} + 4c_0 + 4]$$

$$< \epsilon_0 + \epsilon_0(D + 4c_0 + 4) < \epsilon.$$

This completes the proof of Lemma 8.7.

8.5 An auxiliary result for (A3)

Let $\mathcal{A}_1$ be one of the following spaces:

$$C_b(X),\ C_{bu}(X),\ C_{bub}(X),\ C_{bL}(X),\ C_{bLb}(X),\ C_{bLl}(X).$$

For each $\gamma \in (0, 1)$ choose $\epsilon(\gamma) > 0$, $\delta(\gamma) > 0$ and $\epsilon_0(\gamma) > 0$ which satisfy

$$4\epsilon_0(\gamma) < \epsilon(\gamma) < \gamma, \tag{8.45}$$

$$\delta(\gamma) < 8^{-1}\epsilon_0(\gamma)^2. \tag{8.46}$$

Lemma 8.8. *Let* $\gamma \in (0,1)$, $f \in \mathcal{A}_1$, M *be a nonempty subset of* X *and let* $\bar{x} \in M$ *satisfy*

$$f(\bar{x}) \le \inf\{f(z) : \ z \in M\} + \delta(\gamma) < \infty. \tag{8.47}$$

Define a function $\bar{f} : X \to R^1 \cup \{\infty\}$ *by*

$$\bar{f}(x) = f(x) + \epsilon_0(\gamma) \min\{1, ||x - \bar{x}||\}, \ x \in X. \tag{8.48}$$

Then

$$\bar{f} \in \mathcal{A}_1, \ d_{cs}(f, \bar{f}) \le \epsilon(\gamma), \tag{8.49}$$

$$\bar{f}(z) \ge f(z), \ z \in X, \ \bar{f}(\bar{x}) = f(\bar{x})$$

and for each $y \in M$ *satisfying*

$$\bar{f}(y) \le \inf\{\bar{f}(z) : \ z \in M\} + 2\delta(\gamma) \tag{8.50}$$

the inequality $||y - \bar{x}|| \le \gamma$ *is valid.*

Proof: It is easy to see that $\bar{f} \in \mathcal{A}_1$. Clearly, (8.49) holds.

Assume that $y \in M$ satisfies (8.50). By (8.48), (8.49) , (8.50) and (8.47),

$$f(y) + \epsilon_0(\gamma) \min\{1, ||y - \bar{x}||\} = \bar{f}(y) \le \bar{f}(\bar{x}) + 2\delta(\gamma)$$

$$\le f(\bar{x}) + 2\delta(\gamma) \le f(y) + 3\delta(\gamma)$$

and (8.45) and (8.46) imply that

$$\min\{1, ||y - \bar{x}||\} \le 3\delta(\gamma)\epsilon_0(\gamma)^{-1} \le \epsilon_0(\gamma) < \gamma.$$

Lemma 8.8 is proved.

8.6 An auxiliary result for (A4)

Lemma 8.9. *Let* $\epsilon, \gamma \in (0,1)$, $f_0 \in C_b(X)$, $A_0 \in C(X,Y)$, $B_0 \in C(X,Z)$, $y_0 \in Y$, $z_0 \in Z$ *and let*

$$\inf\{f_0(x) : \ x \in X \ \textit{and} \ A_0(x) = y_0, \ B_0(x) \le z_0\} < \infty. \tag{8.51}$$

Then there exist a positive number $\bar{r} < \epsilon/8$,

$$\bar{A} \in C(X,Y), \ \bar{B} \in C(X,Z) \ \textit{and} \ \bar{x} \in X$$

such that the following properties hold:

$$\bar{A}(\bar{x}) = y_0, \ \bar{B}(\bar{x}) \le z_0; \tag{8.52}$$

$$\{A \in C(X,Y) : \ d_{XY}(A, \bar{A}) \le \bar{r}\} \subset \{A \in C(X,Y) : \ d_{XY}(A, A_0) \le \epsilon/4\}; \tag{8.53}$$

$$\{B \in C(X,Z) : d_{XZ}(B,\bar{B}) \le \bar{r}\} \subset \{B \in C(X,Z) : d_{XZ}(B,B_0) \le \epsilon/4\}. \tag{8.54}$$

(P6) For each $A \in C(X,Y)$, $B \in C(X,Z)$, $y \in Y$, $z \in Z$ and each $x \in X$ which satisfy

$$d_{XY}(A,\bar{A}) \le \bar{r},\ d_{XZ}(B,\bar{B}) \le \bar{r},\ ||y-y_0|| \le \bar{r},\ ||z-z_0|| \le \bar{r} \tag{8.55}$$

and

$$A(x) = y,\ B(x) \le z \tag{8.56}$$

the inequality

$$f_0(\bar{x}) \le f_0(x) + \gamma \tag{8.57}$$

is valid.

(P7) For each positive number Δ there exists a positive number $\delta < \bar{r}$ such that for each $A \in C(X,Y)$, $B \in C(X,Z)$, $y \in Y$ and $z \in Z$ which satisfy

$$d_{XY}(A,\bar{A}) \le \delta,\ d_{XZ}(B,\bar{B}) \le \delta,\ ||y-y_0|| \le \delta,\ ||z-z_0|| \le \delta \tag{8.58}$$

there exists $x \in X$ satisfying

$$A(x) = y,\ B(x) \le z,\ ||x-\bar{x}|| \le \Delta. \tag{8.59}$$

Proof: Fix $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$ which satisfy

$$\epsilon_2 < \epsilon_1 < \epsilon_0 < \epsilon,\ \epsilon_1 < \gamma/2 \tag{8.60}$$

and $\lambda > 0$ for which

$$\lambda(||L_0||+1) < 1/8. \tag{8.61}$$

For each $r \in [0,1]$ define

$$Q_r = \{x \in X : ||A_0(x) - y_0|| \le r \text{ and } B_0(x) \in z_0 - Z_+ + B_Z(r)\} \tag{8.62}$$

and put

$$\mu(r) = \inf\{f_0(x) : x \in Q_r\}. \tag{8.63}$$

It is easy to see that $\mu(r)$ is finite for all $r \in [0,1]$ and that the function μ is monotone decreasing. There exists a positive number

$$r_0 < \epsilon/64 \tag{8.64}$$

such that μ is continuous at r_0. Fix a positive number $r_1 < r_0$ for which

$$|\mu(r_1) - \mu(r_0)| < \epsilon_1. \tag{8.65}$$

There exists $\bar{x} \in X$ which satisfies

$$\bar{x} \in Q_{r_1},\ f_0(\bar{x}) \le \mu(r_1) + \epsilon_1. \tag{8.66}$$

It follows from (8.66) and (8.62) that

$$||A_0(\bar{x}) - y_0|| \le r_1,\ B_0(\bar{x}) \in z_0 - Z_+ + B_Z(r_1). \tag{8.67}$$

Thus there exists $h_0 \in Z$ such that

$$||h_0|| \le r_1,\ B_0(\bar{x}) \le z_0 + h_0. \tag{8.68}$$

Fix $r_2 > 0$, $r_3 > 0$ such that

$$r_2 < (r_0 - r_1)/16,\ 16r_3 < 4r_2 < r_1;$$

$$||A_0(\bar{x}) - A_0(x)|| \le (r_0 - r_1)/4 \text{ for each } x \in B_X(\bar{x}, r_2) \tag{8.69}$$

and put

$$\bar{r} = r_3.$$

Urysohn's theorem implies that there exists a continuous function $\phi : X \to [0,1]$ which satisfies

$$\phi(x) = 1,\ x \in B_X(\bar{x}, r_2/2),$$
$$\phi(x) = 0,\ x \in X \setminus B_X^o(\bar{x}, r_2). \tag{8.70}$$

For each $x \in X$ define

$$\bar{A}(x) = \phi(x)[y_0 + \lambda L_0(x - \bar{x})] + (1 - \phi(x))A_0(x), \tag{8.71}$$

$$\bar{B}(x) = B_0(x) - r_2 h_* - h_0 \tag{8.72}$$

(see (8.32)).

It is easy to see that $\bar{A} \in C(X, Y)$, $\bar{B} \in C(X, Z)$. It follows from (8.71), (8.72), (8.70) and (8.68) that

$$\bar{A}(\bar{x}) = y_0,\ \bar{B}(\bar{x}) = B_0(\bar{x}) - r_2 h_* - h_0 \le z_0 - r_2 h_*. \tag{8.73}$$

By (8.71) and (8.70),

$$d_{XY}(\bar{A}, A_0) \le \tilde{d}_{XY}(\bar{A}, A_0) = \sup\{||A_0(u) - \bar{A}(u)|| :\ u \in X\}$$
$$= \sup\{||A_0(u) - \bar{A}(u)|| :\ u \in B_X(\bar{x}, r_2)\}$$
$$= \sup\{\phi(u)||y_0 + \lambda L_0(u - \bar{x}) - A_0(u)|| :\ u \in B_X(\bar{x}, r_2)\}$$
$$\le \sup\{||y_0 - A_0(u)|| + \lambda||L_0||||u - \bar{x}|| :\ u \in B_X(\bar{x}, r_2)\}$$
$$\le \sup\{||y_0 - A_0(u)|| :\ u \in B_X(\bar{x}, r_2)\} + \lambda||L_0||r_2$$
$$\le \lambda||L_0||r_2 + ||y_0 - A_0(\bar{x})|| + ||A_0(\bar{x}) - A_0(u)|| :\ u \in B_X(\bar{x}, r_2)\}. \tag{8.74}$$

Relations (8.74), (8.67), (8.61) and (8.69) imply that

$$d_{XY}(\bar{A}, A_0) = \tilde{d}_{XY}(\bar{A}, A_0)$$

$$\le \lambda||L_0||r_2 + ||y_0 - A_0(\bar{x})|| + \sup\{||A_0(\bar{x}) - A_0(u)|| :\ u \in B_X(\bar{x}, r_2)\}$$
$$\le r_2/8 + r_1 + \sup\{||A_0(\bar{x}) - A_0(u)|| :\ u \in B_X(\bar{x}, r_2)\}$$

$$\leq r_2/8 + r_1 + (r_0 - r_1)/4 < (r_0 - r_1)/16 + (r_0 - r_1)/4 + r_1 = (r_0 + r_1)/2.$$

Therefore

$$d_{XY}(\bar{A}, A_0) \leq \tilde{d}_{XY}(\bar{A}, A_0) \leq (r_0 + r_1)/2. \tag{8.75}$$

It follows from (8.72), (8.68), (8.32) and (8.69) that

$$d_{XZ}(\bar{B}, B_0) \leq \tilde{d}_{XZ}(\bar{B}, B_0) \leq ||r_2 h_*|| + ||h_0||$$

$$\leq r_1 + r_2 < (r_0 - r_1)/16 + r_1 < (r_0 + r_2)/2. \tag{8.76}$$

By (8.75), (8.76), equality $\bar{r} = r_3$, (8.69) and (8.64), relations (8.53) and (8.54) are true.

Assume that

$$A \in C(X, Y),\ B \in C(X, Z),\ y \in Y,\ z \in Z$$

and that $x \in X$ satisfies (8.55) and (8.56). We show that $x \in Q_{r_0}$. It follows from (8.55), (8.31), (8.64) and (8.69) that

$$\tilde{d}_{XY}(A, \bar{A}) \leq d_{XY}(A, \bar{A})(1 - d_{XY}(A, \bar{A}))^{-1} \leq 2\bar{r}, \tag{8.77}$$

$$\tilde{d}_{XZ}(B, \bar{B}) \leq d_{XZ}(B, \bar{B})(1 - d_{XZ}(B, \bar{B}))^{-1} \leq 2\bar{r}.$$

Relations (8.77), (8.75) and (8.76) imply that

$$\tilde{d}_{XY}(A, A_0) \leq \tilde{d}_{XY}(A, \bar{A}) + \tilde{d}_{XY}(\bar{A}, A_0)$$

$$\leq 2\bar{r} + (r_0 + r_1)/2, \tag{8.78}$$

$$\tilde{d}_{XZ}(B, B_0) \leq \tilde{d}_{XZ}(B, \bar{B}) + \tilde{d}_{XZ}(\bar{B}, B_0) \leq 2\bar{r} + (r_0 + r_1)/2.$$

By (8.78), (8.31), (8.55), (8.69) and (8.56),

$$||A_0(x) - y_0|| \leq ||A_0(x) - A(x)|| + ||A(x) - y|| + ||y - y_0||$$

$$\leq 2\bar{r} + (r_0 + r_1)/2 + \bar{r} < r_0.$$

It follows from (8.31), (8.32), (8.55), (8.78), (8.56) and (8.69) that

$$B_0(x) - z_0 = B_0(x) - B(x) + B(x) - z + z - z_0$$

$$\leq \tilde{d}_{XZ}(B, B_0) h_* + ||z - z_0|| h_*$$

$$\leq (2\bar{r} + (r_0 + r_1)/2) h_* + \bar{r} h_* = (3\bar{r} + (r_0 + r_1)/2) h_* \leq r_0 h_*.$$

Therefore

$$||A_0(x) - y_0|| < r_0,\ B_0(x) - z_0 \leq r_0 h_*.$$

Combined with (8.62) and (8.32) this implies that

$$x \in Q_{r_0}. \tag{8.79}$$

In view of (8.79), (8.63), (8.65), (8.60) and (8.66),

$$f_0(x) \geq \mu(r_0) \geq \mu(r_1) - \epsilon_1 \geq f_0(\bar{x}) - 2\epsilon_1 \geq f_0(\bar{x}) - \gamma.$$

Therefore we have shown that property (P6) holds.

Now we show that property (P7) holds.

Let Δ be a positive number. Fix $\Delta_1 > 0$ such that

$$\Delta_1 < \Delta,\ \Delta_1 < r_2/2, \tag{8.80}$$

$$||\bar{B}(u) - \bar{B}(\bar{x})|| \leq r_2/8 \text{ for all } u \in B_X(\bar{x}, \Delta_1). \tag{8.81}$$

Fix $\delta > 0$ for which

$$4\delta < \min\{1, \Delta_1, \bar{r}/2\}\lambda. \tag{8.82}$$

Assume that $A \in C(X, Y)$, $B \in C(X, Z)$, $y \in Y$, $z \in Z$ satisfy

$$d_{XY}(A, \bar{A}) \leq \delta,\ d_{XZ}(B, \bar{B}) \leq \delta,\ ||y - y_0|| \leq \delta,\ ||z - z_0|| \leq \delta. \tag{8.83}$$

By (8.31), (8.83) and (8.82),

$$\tilde{d}_{XY}(A, \bar{A}) = d_{XY}(A, \bar{A})(1 - d_{XY}(A, \bar{A}))^{-1} \leq \delta(1-\delta)^{-1} \leq 2\delta, \tag{8.84}$$

$$\tilde{d}_{XZ}(B, \bar{B}) = d_{XZ}(B, \bar{B})(1 - d_{XZ}(B, \bar{B}))^{-1} \leq \delta(1-\delta)^{-1} \leq 2\delta. \tag{8.85}$$

We show that there exists $x \in X$ which satisfies

$$A(x) = y,\ B(x) \leq z,\ ||x - \bar{x}|| \leq \Delta_1. \tag{8.86}$$

For each $x \in X$ put

$$D(x) = A(x) - y + x. \tag{8.87}$$

We show that

$$D(B_X(\bar{x}, \Delta_1) \cap (\bar{x} + Y)) \subset B_X(\bar{x}, \Delta_1) \cap (\bar{x} + Y). \tag{8.88}$$

Let

$$x \in B_X(\bar{x}, \Delta_1) \cap (\bar{x} + Y). \tag{8.89}$$

By (8.71), (8.70), (8.89), (8.80) and (8.39),

$$\bar{A}(x) = y_0 + \lambda L_0(x - \bar{x}) \in Y. \tag{8.90}$$

Relations (8.87) and (8.89) imply that

$$D(x) = A(x) - y + x \in Y + (\bar{x} + Y) = \bar{x} + Y. \tag{8.91}$$

It follows from (8.87), (8.90), (8.39), (8.89), (8.83), (8.84), (8.31) and (8.61) that

$$||D(x) - \bar{x}|| = ||A(x) - y + x - \bar{x}|| = ||\bar{A}(x) - y_0 + x - \bar{x} + (y_0 - y) + (A(x) - \bar{A}(x))||$$

$$\leq ||\lambda L_0(x - \bar{x}) + x - \bar{x}|| + ||y_0 - y|| + ||A(x) - \bar{A}(x)|| \tag{8.92}$$
$$\leq ||(1 - \lambda)(x - \bar{x})|| + 3\delta \leq (1 - \lambda)||x - \bar{x}|| + 3\delta.$$

By (8.89), (8.82) and (8.92),

$$||D(x) - \bar{x}|| \leq (1 - \lambda)||x - \bar{x}|| + 3\delta = (1 - \lambda)\Delta_1 + 3\delta < \Delta_1. \tag{8.93}$$

In view of this inequality and (8.91),

$$D(x) \in B_X(\bar{x}, \Delta_1) \cap (\bar{x} + Y). \tag{8.94}$$

Thus we have shown that (8.88) holds. Since the space Y is finite dimensional the Brouwer fixed point theorem implies that there exists $x_* \in X$ which satisfies

$$x_* \in B_X(\bar{x}, \Delta_1) \cap (\bar{x} + Y), \tag{8.95}$$
$$x_* = D(x_*) = A(x_*) - y + x_*.$$

Hence

$$A(x_*) = y. \tag{8.96}$$

By (8.32), (8.85), (8.31), (8.73) and (8.83),

$$B(x_*) = \bar{B}(x_*) + (B(x_*) - \bar{B}(x_*)) \leq \bar{B}(x_*) + ||B(x_*) - \bar{B}(x_*)||h_*$$
$$\leq \bar{B}(x_*) + 2\delta h_* = (\bar{B}(x_*) - \bar{B}(\bar{x})) + \bar{B}(\bar{x}) + 2\delta h_*$$
$$\leq (\bar{B}(x_*) - \bar{B}(\bar{x})) + z_0 - r_2 h_* + 2\delta h_*$$
$$= (\bar{B}(x_*) - \bar{B}(\bar{x})) + z + (z_0 - z) - r_2 h_* + 2\delta h_*$$
$$\leq \bar{B}(x_*) - \bar{B}(\bar{x}) + z + 3\delta h_* - r_2 h_*. \tag{8.97}$$

It follows from (8.81), (8.32), (8.95), (8.82), (8.69) and (8.97) that

$$B(x_*) \leq z - r_2 h_* + 3\delta h_* + \bar{B}(x_*) - \bar{B}(\bar{x})$$
$$\leq z - r_2 h_* + 3\delta h_* + (r_2/8) h_* \leq$$
$$z - r_2 h_* + 2^{-1} r_2 h_* \leq z. \tag{8.98}$$

By (8.96), (8.95) and (8.98),

$$A(x_*) = y, \ B(x_*) \leq z, \ x_* \in B_X(\bar{x}, \Delta_1).$$

Thus property (P7) holds. Lemma 8.9 is porved.

8.7 Proof of Theorems 8.4 and 8.5

It is clear that (A1) and (H2) hold. In view of Theorem 4.1 we need to show that (H1) holds. Proposition 8.2 implies that it is sufficient to show that (A2), (A3) and (A4) hold. Lemma 8.7 implies (A2). (A3) follows from Lemma 8.8. It follows from Lemma 8.9 that (A4) holds. This completes the proof of the theorems.

8.8 An abstract implicit function theorem

Let $(X, ||\cdot||)$ be a norm space. For each $x \in X$ and each positive number r put

$$B_X(x,r) = \{y \in X : \ ||y-x|| \le r\}, \ B_X^o(x,r) = \{y \in X : \ ||y-x|| < r\},$$

$$B_X(r) = B_X(0,r), \ B_X^o(r) = B_X^o(0,r).$$

We always equip the space X with the norm topology and with a metric ρ_X induced by the norm of X:

$$\rho_X(y,z) = ||y-z||, \ y,z \in X.$$

Denote by $\dim(X)$ the dimension of X. If X is an infinite-dimensional space, then $\dim(X) = \infty$. We denote by I_X the identity operator $I_X : \ X \to X$ such that $I_X x = x$ for all $x \in X$.

If $(X, ||\cdot||)$, $(Y, ||\cdot||)$ are norm spaces and $L : X \to Y$ is a continuous linear operator, then

$$||L|| = \sup\{||L(x)|| : \ x \in X \text{ and } ||x|| \le 1\}.$$

Let $(X, ||\cdot||)$ and $(Y, ||\cdot||)$ be norm spaces and let W be a nonempty open subset of X. Denote by $C^1(W,Y)$ the set of all mappings $f : W \to Y$ such that for each $x \in W$ there exists a Frechet derivative $f'(x) : X \to Y$ of f at x and that the mapping $x \to f'(x)$, $x \in W$ is continuous.

If $X = X_1 \times X_2$ where $(X_i, ||\cdot||)$, $i = 1,2$ are normed spaces and $f \in C^1(W,Y)$, then the partial derivative of f at a point $w \in W$ with respect to x_i $(i = 1,2)$ is denoted by $(\partial f/\partial x_i)(w)$ or $f_{x_i}(w)$.

We will prove the following result established in [125].

Theorem 8.10. *Let X be a topological space, $(Y, ||\cdot||)$ and $(Z, ||\cdot||)$ be Banach spaces, W be a neighborhood of (x_0, y_0) in $X \times Y$ and $\bar{\Psi} : W \to Z$ be a mapping which satisfy the following conditions:*

$$\bar{\Psi}(x_0, y_0) = z_0; \tag{8.99}$$

the mapping $x \to \bar{\Psi}(x, y_0)$, is continuous at x_0;

there exists a linear continuous operator $A : Y \to Z$ such that for each positive number ϵ there exist a positive number δ and a neighborhood $\mathcal{V}$ of x_0 in X such that if $x \in \mathcal{V}$ and $y', y'' \in B_Y(y_0, \delta)$, then

$$||\bar{\Psi}(x,y') - \bar{\Psi}(x,y'') - A(y'-y'')|| < \epsilon||y'-y''|| \tag{8.100}$$

and that

$$AY = Z. \tag{8.101}$$

Then there exist a pair of positive constants K, σ and a neighborhood $\mathcal{U}$ of (x_0, z_0) in $X \times Z$ such that for each mapping $\Psi : W \to Z$ which satisfies

$$||(\Psi - \bar{\Psi})(x, y)|| \leq \sigma \textit{ for all } (x, y) \in W \tag{8.102}$$

and

$$||(\Psi - \bar{\Psi})(x, y') - (\Psi - \bar{\Psi})(x, y'')|| \leq \sigma ||y' - y''|| \tag{8.103}$$

for each $(x, y'), (x, y'') \in W$*, there exists a mapping* $\tilde{\Phi} : \mathcal{U} \to Y$ *such that for each* $(x, z) \in \mathcal{U}$

$$(x, y_0),\ (x, \tilde{\Phi}(x, z)) \in W,\ \Psi(x, \tilde{\Phi}(x, z)) = z, \tag{8.104}$$

$$||\tilde{\Phi}(x, z) - y_0|| \leq K||\Psi(x, y_0) - z||. \tag{8.105}$$

We need the following lemma proved in [1].

Lemma 8.11. *Let* T *be a topological space,* $(Y, ||\cdot||)$ *be a Banach space,* V *be a neighborhood of* (t_0, y_0) *in* $T \times Y$ *and let* $\phi : V \to Y$*. Assume that there exist a neighborhood* U *of* t_0 *in* T*,* $\beta > 0$ *and* $\theta \in (0, 1)$ *such that for each* $t \in U$ *and each* $y \in B_Y^o(y_0, \beta)$

$$\phi(t, y) \textit{ is defined and } (t, \Phi(t, y)) \in V, \tag{8.106}$$

$$||\Phi(t, \Phi(t, y)) - \Phi(t, y)|| \leq \theta ||\Phi(t, y) - y||, \tag{8.107}$$

$$||\Phi(t, y_0) - y_0|| < \beta(1 - \theta). \tag{8.108}$$

Then for each $t \in U$ *the sequence* $\{y_n(t)\}_{n=0}^{\infty}$ *defined by*

$$y_0(t) = y_0,\ y_n(t) = \phi(t, y_{n-1}(t)) \textit{ for all integers } n \geq 1 \tag{8.109}$$

is well-defined and is contained in $B_Y^o(y_0, \beta)$ *and* $y_n(t)$ *converges to* $\Phi(t) \in Y$ *as* $n \to \infty$ *uniformly on* U *and*

$$||\Phi(t) - y_0|| \leq ||\Phi(t, y_0) - y_0||(1 - \theta)^{-1} \textit{ for all } t \in U. \tag{8.110}$$

We also use the following auxiliary result. For its proof see [1].

Lemma 8.12. *Let* $(X, ||\cdot||)$*,* $(Y, ||\cdot||)$ *be Banach spaces and let* $\Lambda : X \to Y$ *be a continuous linear operator such that* $\Lambda X = Y$*. Then there exist a mapping* $M : Y \to X$ *and a constant* $c > 0$ *such that*

$$\Lambda \circ M = I_Y \textit{ and } ||My|| \leq c||y|| \textit{ for all } y \in Y.$$

8.9 Proof of Theorem 8.10

Since $AY = Z$ (see (8.101)) Lemma 8.12 implies that there exist a mapping $C : Z \to Y$ and a number $c_0 > 0$ for which

$$ACz = z \text{ for all } z \in Z, \tag{8.111}$$

$$||Cz|| \le c_0||z|| \text{ for all } z \in Z. \tag{8.112}$$

Set $T = X \times Z$, choose $\theta \in (0,1)$ and set

$$\epsilon = \theta/c_0. \tag{8.113}$$

The assumptions of the theorem (see (8.100)) imply that there exist a neighborhood V_1 of x_0 and a positive constant β_0 such that

$$V_1 \times B_Y(y_0, \beta_0) \subset W, \tag{8.114}$$

$$||\bar{\Psi}(x,y') - \bar{\Psi}(x,y'') - A(y'-y'')|| < 4^{-1}||y'-y''||\epsilon \tag{8.115}$$

for each $x \in V_1$ and each $y', y'' \in B_Y(y_0, \beta_0)$.

Put

$$\mathcal{V} = V_1 \times Z \times B_Y^o(y_0, \beta_0) \tag{8.116}$$

and

$$\beta = \beta_0(c_0||A|| + 2)^{-1}. \tag{8.117}$$

By the assumptions of Theorem 8.10, the mapping

$$(x,z) \to \bar{\Psi}(x,y_0) - z$$

is continuous at the point (x_0, z_0). Together with (8.99) this fact implies that there exist a neighborhood V_2 of x_0 and a positive constant γ such that

$$V_2 \subset V_1 \tag{8.118}$$

and that for each $(x,z) \in V_2 \times B_Z^o(z_0, \gamma)$,

$$||\bar{\Psi}(x,y_0) - z|| < 4^{-1}\beta(1-\theta)c_0^{-1}. \tag{8.119}$$

Define

$$\mathcal{U} = \mathcal{V}_2 \times B_Z^o(z_0, \gamma). \tag{8.120}$$

Fix a positive number

$$\sigma < \min\{4^{-1}\beta(1-\theta)c_0^{-1},\ \epsilon/4,\ \theta(c_0+1)^{-1}/4\} \tag{8.121}$$

and set

$$K = c_0(1-\theta)^{-1}. \tag{8.122}$$

Assume that a map $\Psi : W \to Z$ satisfies (8.102) and (8.103) for each $(x,y'),\ (x,y'') \in W$. Define a mapping $\Phi : \mathcal{V} \to Y$ by

$$\Phi(x,z,y) = y + C(z - \Psi(x,y)),\ (x,z,y) \in \mathcal{V}. \tag{8.123}$$

Now we show that the conditions of Lemma 8.11 hold with $T = X \times Z$, $t_0 = (x_0, z_0)$, $V = \mathcal{V}$ and $U = \mathcal{U}$.

Assume that

$$(x,z) \in \mathcal{U},\ y \in B_Y^o(y_0,\beta). \tag{8.124}$$

By (8.123), (8.112), (8.124), (8.116), (8.120), (8.118), (8.117), (8.103), (8.115) and (8.114),

$$||\Phi(x,z,y)-y_0|| = ||y-y_0+C(z-\Psi(x,y))||$$

$$\leq ||y-y_0|| + c_0||z-\Psi(x,y)||$$

$$\leq ||y-y_0|| + c_0||z-\Psi(x,y_0)-\Psi(x,y)+\Psi(x,y_0)+A(y-y_0)-A(y-y_0)||$$

$$\leq ||y-y_0||+c_0||z-\Psi(x,y_0)||+c_0||\Psi(x,y)-\Psi(x,y_0)+A(y-y_0)||+c_0||A||||y-y_0||$$

$$\leq ||y-y_0|| + c_0||z-\Psi(x,y_0)|| + c_0||A||||y-y_0||$$

$$+c_0||\Psi(x,y)-\bar{\Psi}(x,y)-[\Psi(x,y_0)-\bar{\Psi}(x,y_0)]||$$

$$+c_0||\bar{\Psi}(x,y)-\bar{\Psi}(x,y_0)-A(y-y_0)||$$

$$\leq ||y-y_0||+c_0||A||||y-y_0||+c_0||z-\Psi(x,y_0)||+c_0\sigma||y-y_0||+4^{-1}c_0\epsilon||y-y_0||$$

$$= (1+c_0\sigma+c_0\epsilon/4)||y-y_0|| + c_0||A||||y-y_0|| + c_0||z-\Psi(x,y_0)||$$

$$= (1+c_0\sigma+c_0\epsilon/4+c_0||A||)||y-y_0|| + c_0||z-\Psi(x,y_0)||.$$

By the relation above, (8.113) and (8.121),

$$||\Phi(x,z,y)-y_0|| \leq c_0||z-\Psi(x,y_0)|| + ||y-y_0||(1+c_0||A||+\theta/4+\theta/4)$$

$$\leq c_0||z-\Psi(x,y_0)|| + ||y-y_0||(1+c_0||A||+\theta/2). \tag{8.125}$$

It follows from (8.125), (8.119), (8.124), (8.120), (8.102), (8.121) and (8.117) that

$$||\Phi(x,z,y)-y_0|| \leq ||y-y_0||(1+c_0||A||+\theta/2)$$

$$+c_0||z-\bar{\Psi}(x,y_0)|| + c_0||\bar{\Psi}(x,y_0)-\Psi(x,y_0)||$$

$$\leq \beta(1+c_0||A||+\theta/2) + 4^{-1}c_0\beta(1-\theta)c_0^{-1} + c_0\sigma$$

$$\leq \beta(1+c_0||A||+\theta/2) + 2^{-1}\beta(1-\theta) < \beta(2+c_0||A||) = \beta_0. \tag{8.126}$$

Therefore the first condition of Lemma 8.11 holds (see (8.106)).

By (8.119), (8.120), (8.124), (8.102), (8.121) and (8.125) with $y = y_0$,

$$||\Phi(x,z,y_0)-y_0|| \leq c_0||z-\Psi(x,y_0)|| \leq c_0||z-\bar{\Psi}(x,y_0)||$$

$$+c_0||\bar{\Psi}(x,y_0)-\Psi(x,y_0)|| \leq 4^{-1}\beta(1-\theta) + 4^{-1}\beta(1-\theta) < \beta(1-\theta). \tag{8.127}$$

Thus the third condition of Lemma 8.11 holds (see (8.108)).

It follows from (8.126), (8.124), (8.123), (8.116), (8.120) and (8.118) that

$$\Phi(x,z,\Phi(x,z,y)) - \Phi(x,z,y) = \Phi(x,z,y)$$

$$+C(z-\Psi(x,\Phi(x,z,y))) - \Phi(x,z,y) = C(z-\Psi(x,\Phi(x,z,y))). \tag{8.128}$$

Relations (8.123) and (8.111) imply that

$$A\Phi(x,z,y) = Ay + AC(z - \Psi(x,y)) = Ay + z - \Psi(x,y)$$

and

$$z = \Psi(x,y) + A(\Phi(x,z,y) - y). \tag{8.129}$$

By (8.128), (8.112) and (8.129),

$$||\Phi(x,z,\Phi(x,z,y)) - \Phi(x,z,y)|| \le c_0||z - \Psi(x,\Phi(x,z,y))||$$

$$= c_0||\Psi(x,\Phi(x,z,y)) - \Psi(x,y) - A(\Phi(x,z,y) - y)||$$
$$\le c_0||\bar{\Psi}(x,\Phi(x,z,y)) - \bar{\Psi}(x,y) - A(\Phi(x,z,y) - y)||$$
$$+c_0||\Psi(x,\Phi(x,z,y)) - \bar{\Psi}(x,\Phi(x,z,y)) - [\Psi(x,y) - \bar{\Psi}(x,y)]||.$$

Together with (8.115), (8.124), (8.120), (8.118), (8.126), (8.117), (8.103), (8.114), (8.121) and (8.113) this relation implies that

$$||\Phi(x,z,\Phi(x,z,y)) - \Phi(x,z,y)||$$

$$\le 4^{-1}c_0\epsilon||y - \Phi(x,z,y)|| + c_0\sigma||y - \Phi(x,z,y)||$$
$$\le 2^{-1}c_0\epsilon||y - \Phi(x,z,y)|| \le \theta||y - \Phi(x,z,y)||.$$

Therefore the second condition of Lemma 8.11 holds (see (8.107)).

Put

$$t = (x,z). \tag{8.130}$$

Since all the conditions of Lemma 8.11 hold the sequence $\{y_n(t)\}_{n=0}^{\infty}$ defined by

$$y_0(t) = y_0,\ y_n(t) = \Phi(t, y_{n-1}(t)) \text{ for all natural numbers } n \tag{8.131}$$

is well-defined and is contained in $B_Y^o(y_0,\beta)$ and $y_n(t)$ converges to $\tilde{\Phi}(t) = \tilde{\Phi}(x,z) \in Y$ as $n \to \infty$ in the norm topology uniformly on $\mathcal{U}$ and

$$||\tilde{\Phi}(x,z) - y_0|| \le ||\Phi(x,z,y_0) - y_0||(1-\theta)^{-1}. \tag{8.132}$$

It follows from (8.124), (8.120), (8.118) and (8.114) that

$$(x, y_0) \in W. \tag{8.133}$$

Clearly,

$$||\tilde{\Phi}(x,z) - y_0|| \le \beta. \tag{8.134}$$

By (8.124), (8.120), (8.118), (8.114), (8.117) and (8.134),

$$(x, \tilde{\Phi}(x,z)) \in W. \tag{8.135}$$

By (8.132), (8.123), (8.112) and (8.122),

$$||\tilde{\Phi}(x,z) - y_0|| \le ||\Phi(x,z,y_0) - y_0||(1-\theta)^{-1} = ||C(z - \Psi(x,y_0))||(1-\theta)^{-1}$$

$$\leq c_0||z - \Psi(x, y_0)||(1 - \theta)^{-1} = K||z - \Psi(x, y_0)||. \tag{8.136}$$

In view of (8.135), (8.131), (8.130), (8.114), (8.124), (8.120), (8.118), the inclusion $\{y_i(t)\}_{i=0}^{\infty} \subset B_Y^o(y_0, \beta)$, (8.117), (8.103), (8.123) and (8.115), for each integer $n \geq 1$,

$$||\Psi(x, \tilde{\Phi}(x, z)) - z|| \leq ||\Psi(x, \tilde{\Phi}(x, z)) - \Psi(x, y_n(x, z))|| + ||z - \Psi(x, y_n(x, z))||$$

$$\leq ||\bar{\Psi}(x, \tilde{\Phi}(x, z)) - \bar{\Psi}(x, y_n(x, z))|| + ||(\Psi - \bar{\Psi})(x, \tilde{\Phi}(x, z)) - (\Psi - \bar{\Psi})(x, y_n(x, z))||$$

$$+||AC(z - \Psi(x, y_n(x, z)))||$$

$$\leq ||\bar{\Psi}(x, \tilde{\Phi}(x, z)) - \bar{\Psi}(x, y_n(x, z)) - A(\tilde{\Phi}(x, z) - y_n(x, z))||$$

$$+||A(\tilde{\Phi}(x, z) - y_n(x, z))|| + \sigma||\tilde{\Phi}(x, z) - y_n(x, z)|| + ||A||||y_{n+1}(x, z) - y_n(x, z)||$$

$$\leq 4^{-1}\epsilon||\tilde{\Phi}(x, z) - y_n(x, z)|| + ||A||||\tilde{\Phi}(x, z) - y_n(x, z)||$$

$$+\sigma||\tilde{\Phi}(x, z) - y_n(x, z)|| + ||A||||y_{n+1}(x, z) - y_n(x, z)|| \to 0 \text{ as } n \to \infty.$$

Hence

$$\Psi(x, \tilde{\Phi}(x, z)) = z. \tag{8.137}$$

Theorem 8.10 follows from (8.133), (8.135), (8.137) and (8.136).

8.10 An extension of the classical implicit function

Theorem 8.13. *Let $(X, ||\cdot||)$, $(Y, ||\cdot||)$ and $(Z, ||\cdot||)$ be Banach spaces, W be an open neighborhood of (x_0, y_0) in $X \times Y$ and let a mapping $\bar{\Psi} : W \to Z$ belong to $C^1(W, Z)$. Assume that*

$$\bar{\Psi}(x_0, y_0) = 0 \tag{8.138}$$

and that there exists an inverse operator $[\bar{\Psi}_y(x_0, y_0)]^{-1} : Z \to Y$ which is linear and continuous.

Let ϵ_0 be a positive number. Then there exist

$$\sigma > 0, \ \delta > 0, \ \epsilon \in (0, \epsilon_0) \tag{8.139}$$

such that for each mapping $\Psi : W \to Z$ which satisfies

$$||(\Psi - \bar{\Psi})(x, y)|| \leq \sigma \text{ for all } (x, y) \in \mathcal{W}, \tag{8.140}$$

$$||(\Psi - \bar{\Psi})(x, y') - (\Psi - \bar{\Psi})(x, y'')|| \leq \sigma||y' - y''|| \tag{8.141}$$

for each (x, y'), $(x, y'') \in W$, there exists a mapping $\Phi : B_X^o(x_0, \delta) \to Y$ such that

$$\text{if } \Psi = \bar{\Psi}, \text{ then } \Phi(x_0) = y_0, \tag{8.142}$$

$$(x, \Phi(x)) \in W \text{ and } \Psi(x, \Phi(x)) = 0 \text{ for all } x \in B_X^o(x_0, \delta), \tag{8.143}$$

$$||\Phi(x) - y_0|| < \epsilon \text{ for each } x \in B_X^o(x_0, \delta), \tag{8.144}$$

and that for each $x \in B_X^o(x_0, \delta)$ and each $y \in B_Y^o(y_0, \delta)$ the equality $\Psi(x, y) = 0$ holds if and only if $y = \Phi(x)$.

Proof: We show that $\bar{\Psi}$ satisfies the conditions of Theorem 8.10 with $z_0 = 0$. It is easy to see that the mapping $x \to \bar{\Psi}(x, y_0)$ is continuous at x_0. Put

$$A = \bar{\Psi}_y(x_0, y_0). \tag{8.145}$$

Since $\bar{\Psi} \in C^1(W, Z)$ by the mean value theorem for each positive number ϵ there exist a positive number δ and a neighborhood $\mathcal{V}$ of x_0 such that (8.100) holds for each $x \in \mathcal{V}$ and each $y', y'' \in B_Y(y_0, \delta)$. It is easy to see that (8.101) holds. In view of Theorem 8.10 there exist a pair of positive constants K_0, σ_0 and a neighborhood $\mathcal{U}$ of x_0 in X such that the following property holds:

(P1) For each mapping $\Psi : W \to Z$ which satisfies

$$||(\Psi - \bar{\Psi})(x, y)|| \leq \sigma_0 \text{ for all } (x, y) \in W \tag{8.146}$$

and

$$||(\Psi - \bar{\Psi})(x, y') - (\Psi - \bar{\Psi})(x, y'')|| \leq \sigma_0 ||y' - y''|| \tag{8.147}$$

for each $(x, y'), (x, y'') \in W$ there exists a mapping $\Phi : \mathcal{U} \to Y$ such that

$$(x, \Phi(x)) \in W \text{ and } \Psi(x, \Phi(x)) = 0 \text{ for all } x \in \mathcal{U}, \tag{8.148}$$

$$||\Phi(x) - y_0|| \leq K_0 ||\Psi(x, y_0)|| \text{ for all } x \in \mathcal{U}. \tag{8.149}$$

Since $\bar{\Psi} \in C^1(W, Z)$ for each positive number κ there exists a positive number

$$\epsilon(\kappa) < \epsilon_0 \tag{8.150}$$

such that

$$B_X^o(x_0, \epsilon(\kappa)) \times B_Y^o(y_0, \epsilon(\kappa)) \subset W \tag{8.151}$$

and that for each

$$x_1, x_2 \in B_X^o(x_0, \epsilon(\kappa)),\ y_1, y_2 \in B_Y^o(y_0, \epsilon(\kappa)) \tag{8.152}$$

the following inequality holds:

$$||\bar{\Psi}(x_1, y_1) - \bar{\Psi}(x_2, y_2) - \bar{\Psi}_x(x_0, y_0)(x_1 - x_2) - \bar{\Psi}_y(x_0, y_0)(y_1 - y_2)||$$
$$\leq \kappa \max\{||x_1 - x_2||,\ ||y_1 - y_2||\}. \tag{8.153}$$

Put

$$\bar{\kappa} = 4^{-1} ||(\bar{\Psi}_y(x_0, y_0))^{-1}||^{-1},\ \epsilon = \epsilon(\bar{\kappa}). \tag{8.154}$$

Since $\bar{\Psi}$ is continuous and $\bar{\Psi}(x_0, y_0) = 0$ (see (8.138)) there exists a positive number

$$\delta < \epsilon \tag{8.155}$$

such that

$$B_X(x_0, \delta) \subset \mathcal{U}, \tag{8.156}$$

$$||\bar{\Psi}(x, y_0)|| < \epsilon K_0^{-1} 4^{-1} \text{ for each } x \in B_X(x_0, \delta). \tag{8.157}$$

Choose a positive number

$$\sigma < \min\{\sigma_0, (4K_0)^{-1}\bar{\kappa}\epsilon\}. \tag{8.158}$$

Assume that a map $\Psi : W \to Z$ satisfies (8.140) and (8.141) for each $(x, y'),\ (x, y'') \in W$. Let a mapping $\Phi : \mathcal{U} \to Y$ be as guaranteed by the property (P1). Then (8.148) and (8.149) hold. Relations (8.149) and (8.138) imply that (8.142) holds. It follows from (8.148) and (8.156) that

$$(x, \Phi(x)) \in W \text{ and } \Psi(x, \Phi(x)) = 0 \text{ for all } x \in B^o_X(x_0, \delta)$$

and (8.143) is true. Assume that

$$x \in B^o_X(x_0, \delta). \tag{8.159}$$

It follows from (8.149), (8.156), (8.159), (8.158) and (8.140) that

$$||\Phi(x) - y_0|| \le K_0||\Psi(x, y_0)|| \le K_0(||\bar{\Psi}(x, y_0)|| + ||\Psi(x, y_0) - \bar{\Psi}(x, y_0)||)$$
$$\le K_0\epsilon K_0^{-1}4^{-1} + K_0\sigma < \epsilon/4 + \epsilon/4.$$

Therefore (8.144) holds. Assume that

$$x \in B^o_X(x_0, \delta),\ y \in B^o_Y(y_0, \epsilon),\ \Psi(x, y) = 0. \tag{8.160}$$

We show that $y = \Phi(x)$. By (8.160), (8.143), (8.154), (8.144), (8.141), (8.151), (8.158), (8.155) and the choice of $\epsilon(\bar{\kappa})$ (see (8.152), (8.153)),

$$||y - \Phi(x)|| = ||(\bar{\Psi}_y(x_0, y_0))^{-1}\bar{\Psi}_y(x_0, y_0)(y - \Phi(x))||$$
$$\le ||(\bar{\Psi}_y(x_0, y_0))^{-1}||||\bar{\Psi}(x, y) - \Psi(x, \Phi(x)) - \bar{\Psi}_y(x_0, y_0)(y - \Phi(x))||$$
$$\le ||(\bar{\Psi}_y(x_0, y_0))^{-1}||||\bar{\Psi}(x, y) - \bar{\Psi}(x, \Phi(x)) - \bar{\Psi}_y(x_0, y_0)(y - \Phi(x))||$$
$$+||(\bar{\Psi}_y(x_0, y_0))^{-1}||||(\Psi - \bar{\Psi})(x, y) - [\Psi(x, \Phi(x)) - \bar{\Psi}(x, \Phi(x))]||$$
$$\le ||(\bar{\Psi}_y(x_0, y_0))^{-1}||\bar{\kappa}||y - \Phi(x)||$$
$$+||(\bar{\Psi}_y(x_0, y_0))^{-1}||||(\Psi - \bar{\Psi})(x, y) - [\Psi(x, \Phi(x)) - \bar{\Psi}(x, \Phi(x))]||$$
$$\le ||(\bar{\Psi}_y(x_0, y_0))^{-1}||\bar{\kappa}||y - \Phi(x)|| + \sigma||y - \Phi(x)||||(\bar{\Psi}_y(x_0, y_0))^{-1}||$$
$$\le ||(\bar{\Psi}_y(x_0, y_0))^{-1}||2\bar{\kappa}||y - \Phi(x)|| \le 2^{-1}||y - \Phi(x)||.$$

This relation implies that $y = \Phi(x)$. This completes the proof of Theorem 8.13.

Note that Theorem 8.13 was obtained in [125].

8.11 Minimization problems with mixed smooth constraints

We use the convention that $\infty/\infty = 1$ and $\infty - \infty = 0$.

Let $(X, ||\cdot||)$ and $(Y, ||\cdot||)$ be Banach spaces. Denote by $C(X,Y)$ the set of all continuous mappings $f : X \to Y$.

Let $k \geq 1$ be an integer. For each mapping $f : X \to Y$ we denote by $f^{(k)}(x)$ its Frechet derivative of the order k at a point $x \in X$ if it exists. Denote by $C^k(X,Y)$ the set of all mappings $f : X \to Y$ such that $f^{(k)}(x)$ exists at any point $x \in X$ and the mapping $x \to f^{(k)}(x)$, $x \in X$ is continuous. We set $C^0(X,Y) = C(X,Y)$, $f^{(0)} = f$, $f \in C(X,Y)$.

Let $k \geq 0$ be an integer. For each $f, g \in C^{(k)}(X,Y)$ define

$$\tilde{d}^{(k)}_{X,Y}(f,g) = \sup\{||f^{(j)}(x) - g^{(j)}(x)|| : \ j = 0, \dots, k, \ x \in X\},$$

$$d^{(k)}_{X,Y}(f,g) = \tilde{d}^{(k)}_{X,Y}(f,g)(1 + \tilde{d}^{(k)}_{X,Y}(f,g))^{-1}.$$

It is known that $(C^{(k)}(X,Y), d^{(k)}_{X,Y})$ is a complete metric space [74, 35].

Let $k \geq 1$ be an integer, $(X, ||\cdot||)$, $(Y, ||\cdot||)$ and $(Z, ||\cdot||)$ be Banach spaces, the space Y be finite-dimensional and let $\dim(Y) < \dim(X)$.

We assume that there exists a function $\lambda_* \in C^k(X, R^1)$ such that

$$0 \leq \lambda_*(x) \leq 1, \ x \in X, \tag{8.161}$$

$$\mathrm{supp}(\lambda_*) = \{x \in X : \ \lambda_*(x) > 0\} \subset B_X(0,1)\}, \tag{8.162}$$

$$\lambda_*(z) > 0 \text{ for some } z \in X, \tag{8.163}$$

$$\sup\{||\lambda_*^{(j)}(z)|| : \ z \in X, \ j = 0, \dots, k\} < \infty. \tag{8.164}$$

We assume that the space Z is ordered by a convex closed cone $Z_+ \subset X$, h_* is an interior point of Z_* and that

$$||z|| = \inf\{\lambda \geq 0 : \ -\lambda h_* \leq z \leq \lambda h_*\}, \ z \in Z. \tag{8.165}$$

Remark 8.14. Note that the function λ_* exists if the space X is Hilbert. It also exists if $k = 1$ and X has an equivalent Frechet differentiable norm. For more details concerning the existence of the function λ_* see [31].

Let $p, q, l \geq 0$ be integers such that $p \geq 1$ and $p, l \leq k$. We consider the complete metric spaces

$$(C^{(p)}(X,Y), d^{(p)}_{X,Y}), \ (C^{(q)}(X,Z), d^{(q)}_{X,Z})$$

and the complete metric space $(C^{(l)}(X,R^1), d^{(l)}_{X,R^1})$. Denote by $C_b^{(l)}(X)$ the set of all functions $f \in C^{(l)}(X,R^1)$ which are bounded from below. It is easy to see that $C_b^{(l)}(X)$ is a closed subset of $(C^{(l)}(X,R^1), d_{X,R^1}^{(l)})$. We consider the complete metric space $(C_b^{(l)}(X), d_{X,R^1}^{(l)})$.

We equip the space $C^{(p)}(X,Y)$ with the topology induced by the metric $d_{X,Y}^{(p)}$ (the strong and the weak topologies coincide), the space $C^{(q)}(X,Z)$ with the topology induced by the metric $d_{X,Z}^{(q)}$ (the strong and the weak topologies coincide) and we equip the space $C_b^{(l)}(X)$ with the strong topology induced by the metric $d_{X,R^1}^{(l)}$.

We also equip the space $C_b^{(l)}(X)$ with the weak topology induced by the uniformity determined by the base

$$E_w(\epsilon) = \{(g,h) \in C_b^{(l)}(X) \times C_b^{(l)}(X) :$$

$$|g(x) - h(x)| < \epsilon + \epsilon \max\{|g(x)|, |h(x)|\} \text{ for all } x \in X\},$$

where $\epsilon > 0$. It was shown in Section 8.3 (see Lemma 8.3) that for the space $C_b^{(l)}(X)$ there exists a uniformity which is determined by the base $E_w(\epsilon)$, $\epsilon > 0$. Evidently, the space $C_b^{(l)}(X)$ with this uniformity is metrizable.

Set

$$\mathcal{A}_1 = C_b^{(l)}(X),\ \mathcal{A}_{21} = C^{(p)}(X,Y),\ \mathcal{A}_{22} = C^{(q)}(X,Z),$$

$$\mathcal{A}_{23} = Y,\ \mathcal{A}_{24} = Z,\ \mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22} \times \mathcal{A}_{23} \times \mathcal{A}_{24}.$$

For each $a_2 = (G,H,y,z) \in \mathcal{A}_2$ put

$$S_{a_2} = \{x \in X :\ G(x) = y,\ H(x) \le z\}. \tag{8.166}$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $J_a : X \to R^1 \cup \{\infty\}$ by

$$J_a(x) = a_1(x),\ x \in S_{a_2},\ J_a(x) = \infty,\ x \in X \setminus S_{a_2}. \tag{8.167}$$

We prove the following theorem which was obtained in [125].

Theorem 8.15. *Let $\mathcal{A}$ be the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 :\ \inf(J_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. Then there exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem for J_a on $(X, ||\cdot||)$ is strongly well-posed with respect to the space $\mathcal{A}$ with the weak topology.*

Let $y_0 \in Y$, $z_0 \in Z$ and

$$\mathcal{A}_1 = C_b^{(l)}(X),\ \mathcal{A}_{21} = C^{(p)}(X,Y),\ \mathcal{A}_{22} = C^{(q)}(X,Z),\ \mathcal{A}_2 = \mathcal{A}_{21} \times \mathcal{A}_{22}.$$

For each $a_2 = (G,H) \in \mathcal{A}_2$ set

$$S_{a_2} = \{x \in X :\ Gx = y_0,\ Hx \le z_0\}. \tag{8.168}$$

For each $a = (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ define $\widehat{J}_a : X \to R^1 \cup \{\infty\}$ by

$$\widehat{J}_a(x) = a_1(x),\ x \in S_{a_2},\ \widehat{J}_a(x) = \infty,\ x \in X \setminus S_{a_2}. \tag{8.169}$$

We also prove the following theorem which was obtained in [125].

Theorem 8.16. *Let $\mathcal{A}$ be the closure of the set $\{a \in \mathcal{A}_1 \times \mathcal{A}_2 :\ \inf(\widehat{J}_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2$ with the strong topology. Then there exists a set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem for $\widehat{J}_a$ on $(X, ||\cdot||)$ is strongly well-posed with respect to the space $\mathcal{A}$ with the weak topology.*

Note that Theorems 8.15 and 8.16 are obtained as realizations of the variational principles established in Sections 8.1 and 8.2

Clearly, without loss of generality we may assume that $Y \subset X$ and that the norm of Y is induced by the norm of X.

It is easy to see that there exists a linear continuous operator $L_0 : X \to Y$ such that

$$L_0 x = x,\ x \in Y. \tag{8.170}$$

Put

$$S = L_0^{-1}(0) = \{x \in X :\ L_0 x = 0\}.$$

For each $x \in X$ we have

$$x = L_0 x + (x - L_0 x) \in Y + S.$$

It is easy to see that for each $x \in X$ there is a unique pair $u, v \in X$ such that $u \in Y$, $v \in S$ and $x = u + v$. The linear continuous operator

$$x \to (L_0 x, x - L_0 x),\ x \in X$$

acting from X to $Y \times S$ has a continuous inverse operator. Note that S is equipped with the norm induced by the norm of X. Evidently, we may assume without loss of generality that

$$X = S \times Y \tag{8.171}$$

where $(S, ||\cdot||)$ is a Banach space and

$$||(s, y)|| = ||s|| + ||y||,\ (s, y) \in S \times Y. \tag{8.172}$$

8.12 Auxiliary results

Lemma 8.17. *There exists a function $\tilde{\lambda} \in C^k(X, R^1)$ such that*

$$0 \le \tilde{\lambda}(x) \le 1, \; x \in X,$$

$$supp(\tilde{\lambda}) = \{x \in X : \; \tilde{\lambda}(x) > 0\} \subset B_X(0,1), \tag{8.173}$$

$$\tilde{\lambda}(x) = 1 \textit{ for all } x \textit{ belonging to a neighborhood of zero,} \tag{8.174}$$

$$\sup\{||\tilde{\lambda}^{(j)}(x)|| : \; j = 0, \dots, k, \; x \in X\} < \infty. \tag{8.175}$$

Proof: We may assume without loss of generality that

$$\sup\{\lambda_*(z) : \; z \in X\} = 1. \tag{8.176}$$

There exists a monotone increasing C^∞-function $\eta : R^1 \to R^1$ such that

$$\eta(0) = 0, \; \eta(x) = \eta(c_0) \text{ for all } x \ge c_0,$$

where $c_0 > 0$ is a constant. We may assume without loss of generality that there is a $c_1 > 0$ such that

$$\eta(c_0) = 1, \; c_0 < c_1 \in \lambda_*(B_X(0,1)). \tag{8.177}$$

Consider the function $\eta \circ \lambda_*$. Clearly, $\eta \circ \lambda_* \in C^k(X, R^1)$,

$$\{x \in X : \; (\eta \circ \lambda_*)(x) \ne 0\} \subset B_X(0,1), \tag{8.178}$$

$$0 \le (\eta \circ \lambda_*)x \le 1, \; x \in X$$

and there exist $x_0 \in X$ and a positive number r_0 for which

$$(\eta \circ \lambda_*)(x) = 1 \text{ for all } x \in B_X(x_0, r_0). \tag{8.179}$$

Fix $c_2 > 2$ and define

$$\tilde{\lambda}(x) = (\eta \circ \lambda_*)(c_2 x + x_0), \; x \in X. \tag{8.180}$$

It is easy to see that $\tilde{\lambda} \in C^k(X, R^1)$,

$$0 \le \tilde{\lambda}(x) \le 1 \text{ for all } x \in X \tag{8.181}$$

and (8.175) is true.

Assume that $x \in X$ satisfies $\tilde{\lambda}(x) > 0$. Then (8.180) implies that

$$0 < (\eta \circ \lambda_*)(c_2 x + x_0).$$

It follows from the inequality above and (8.178) that $||c_2 x + x_0|| \le 1$. By this inequality, (8.178), (8.179) and the equality $x = c_2^{-1}((c_2 x + x_0) - x_0)$,

$$||x|| \le c_2^{-1}(||c_2 x + x_0|| + ||x_0||) \le c_2^{-1} + c_2^{-1}||x_0|| \le 2c_2^{-1} < 1.$$

Therefore

$$\{x \in X : \; \tilde{\lambda}(x) > 0\} \subset B_X(0,1).$$

Combined with (8.181) this inclusion implies that (8.173) is true.

Let $x \in B_X(0, c_2^{-1} r_0)$. Then $c_2 x + x_0 \in B_X(x_0, r_0)$ and (8.179) and (8.180) imply that

$$\tilde{\lambda}(x) = (\eta \circ \lambda_*)(c_2 x + x_0) = 1.$$

Lemma 8.17 is proved.

For each $\gamma \in (0, 1)$ set

$$\lambda_\gamma(z) = \tilde{\lambda}(\gamma^{-1} z),\ z \in X \tag{8.182}$$

and choose $\epsilon(\gamma) > 0,\ \epsilon_0(\gamma) > 0,\ \delta(\gamma) > 0$ such that

$$\begin{aligned} \epsilon_0(\gamma) < \epsilon(\gamma) < \gamma, \\ \epsilon_0(\gamma)[2 + \sup\{||\lambda_\gamma^{(j)}(z)|| :\ z \in X, j = 0, \dots, k\}] < \epsilon(\gamma), \\ \delta(\gamma) < \epsilon_0(\gamma)\gamma/12. \end{aligned} \tag{8.183}$$

The following lemma is an auxiliary result for (A3) (see Section 8.1).

Lemma 8.18. *Let $\gamma \in (0, 1)$, $f \in C_b^{(l)}(X)$, M be a nonempty subset of X, $\bar{x} \in M$ and let*

$$f(\bar{x}) \le \inf\{f(z) :\ z \in M\} + \delta(\gamma) < \infty. \tag{8.184}$$

Define a function $\bar{f} : X \to R^1$ by

$$\bar{f}(x) = f(x) + \epsilon_0(\gamma)(1 - \lambda_\gamma(x - \bar{x})),\ x \in X. \tag{8.185}$$

Then $\bar{f} \in C_b^{(l)}(X)$,

$$d_{X,R^1}^{(l)}(f, \bar{f}) \le \epsilon(\gamma),\ \bar{f}(z) \ge f(z) \text{ for all } x \in X, \tag{8.186}$$

$$\bar{f}(\bar{x}) = f(\bar{x})$$

and for each $y \in M$ satisfying

$$\bar{f}(y) \le \inf\{\bar{f}(z) :\ z \in M\} + 2\delta(\gamma) \tag{8.187}$$

the inequality $||y - \bar{x}|| \le \gamma$ is valid.

Proof: It is clear that $\bar{f} \in C_b^{(l)}(X)$. Evidently, (8.186) is true. Assume that $y \in M$ satisfies (8.187). By (8.184)–(8.187),

$$\begin{aligned} f(y) + \epsilon_0(\gamma)(1 - \lambda_\gamma(y - \bar{x})) = \bar{f}(y) \le \bar{f}(\bar{x}) + 2\delta(\gamma) \\ = f(\bar{x}) + 2\delta(\gamma) \le f(y) + 3\delta(\gamma). \end{aligned} \tag{8.188}$$

In view of (8.188),

$$1 - \lambda_\gamma(y - \bar{x}) \le 3\delta(\gamma)\epsilon_0(\gamma)^{-1}.$$

It follows from the inequality above, (8.182) and (8.183) that

$$\tilde{\lambda}(\gamma^{-1}(y - \bar{x})) = \lambda_\gamma(y - \bar{x}) \ge 1 - 3\delta(\gamma)\epsilon_0(\gamma)^{-1} \ge 1 - \gamma/4 \ge 3/4. \tag{8.189}$$

By (8.189) and (8.173),

$$||\gamma^{-1}(y - \bar{x})|| \le 1 \text{ and } ||y - \bar{x}|| \le \gamma. \tag{8.190}$$

This completes the proof of Lemma 8.18.

8.13 An auxiliary result for hypothesis (A4)

We assume that the infimum over empty set is ∞.

Lemma 8.19. *Let* $\epsilon, \gamma \in (0,1)$*,* $f_0 \in C_b^{(l)}(X)$*,* $A_0 \in C^{(p)}(X,Y)$*,* $B_0 \in C^{(q)}(X,Z)$*,* $y_0 \in Y$*,* $z_0 \in Z$ *and let*

$$\inf\{f_0(x) : \; x \in X \text{ and } A_0x = y_0, \; B_0x \le z_0\} < \infty. \tag{8.191}$$

Then there exist

$$\bar{r} \in (0, 8^{-1}\epsilon),$$
$$\bar{A} \in C^{(p)}(X,Y), \; \bar{B} \in C^{(q)}(X,Z), \; \bar{x} \in X \tag{8.192}$$

such that the following properties hold:

$$\bar{A}\bar{x} = y_0, \; \bar{B}\bar{x} \le z_0; \tag{8.193}$$

$$\{A \in C^{(p)}(X,Y) : \; d_{X,Y}^{(p)}(A, \bar{A}) \le \bar{r}\}$$
$$\subset \{A \in C^{(p)}(X,Y) : \; d_{X,Y}^{(p)}(A, A_0) \le 4^{-1}\epsilon\}; \tag{8.194}$$
$$\{B \in C^{(q)}(X,Z) : \; d_{X,Z}^{(q)}(B, \bar{B}) \le \bar{r}\}$$
$$\subset \{B \in C^{(q)}(X,Z) : \; d_{X,Z}^{(q)}(B, B_0) \le 4^{-1}\epsilon\}. \tag{8.195}$$

(P2) For each $A \in C^{(p)}(X,Y)$*, each* $B \in C^{(q)}(X,Z)$*, each* $y \in Y$*, each* $z \in Z$ *and each* $x \in X$ *which satisfy*

$$d_{X,Y}^{(p)}(A, \bar{A}) \le \bar{r}, \; d_{X,Z}^{(q)}(B, \bar{B}) \le \bar{r}, \; ||y - y_0|| \le \bar{r}, \; ||z - z_0|| \le \bar{r},$$
$$Ax = y, \; Bx \le z \tag{8.196}$$

the inequality

$$f_0(\bar{x}) \le f_0(x) + \gamma \tag{8.197}$$

holds.

(P3) For each positive number Δ *there exists a positive number* $\delta < \bar{r}$ *such that for each* $A \in C^{(p)}(X,Y)$*, each* $B \in C^q(X,Z)$*, each* $y \in Y$ *and each* $z \in Z$ *which satisfy*

$$d_{X,Y}^{(p)}(A, \bar{A}) \le \delta, \; d_{X,Z}^{(q)}(B, \bar{B}) \le \delta, \; ||y - y_0|| \le \delta, \; ||z - z_0|| \le \delta \tag{8.198}$$

there exists $x \in X$ *such that*

$$Ax = y, \; Bx \le z, \; ||x - \bar{x}|| \le \Delta. \tag{8.199}$$

Proof: Let a function $\tilde{\lambda} \in C^{(k)}(X, R^1)$ be as guaranteed by Lemma 8.17. Namely, $\tilde{\lambda}$ satisfies

$$0 \leq \tilde{\lambda}(x) \leq 1, \ x \in X,$$

$$\{x \in X : \ \tilde{\lambda}(x) > 0\} \subset B_X(0, 1), \tag{8.200}$$

$$\tilde{\lambda}(x) = 1 \text{ for all } x \text{ in a neighborhood of } 0, \tag{8.201}$$

$$\sup\{||\tilde{\lambda}^{(j)}(x)|| : \ x \in X, \ j = 0, \dots, k\} < \infty. \tag{8.202}$$

Fix $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\epsilon_2 > 0$ for which

$$\epsilon_2 < \epsilon_1 < \epsilon_0 < \epsilon \text{ and } \epsilon_0 < \gamma/4. \tag{8.203}$$

For each $r \in [0, 1]$ set

$$Q_r = \{x \in X : \ ||A_0 x - y_0|| \leq r, \ B_0 x \in z_0 - Z_+ + B_Z(r)\} \tag{8.204}$$

and put

$$\mu(r) = \inf\{f_0(x) : \ x \in Q_r\}. \tag{8.205}$$

It is clear that $\mu(r)$ is finite for all $r \in [0, 1]$ and the function μ is monotone decreasing. There exists a positive number

$$r_0 < \epsilon/64 \tag{8.206}$$

such that μ is continuous at r_0. Fix a positive number $r_1 < r_0$ such that

$$|\mu(r_1) - \mu(r_0)| < \epsilon_1. \tag{8.207}$$

There exists $\bar{x} \in X$ such that

$$\bar{x} \in Q_{r_1}, \ f_0(\bar{x}) \leq \mu(r_1) + \epsilon_1. \tag{8.208}$$

Let

$$\bar{x} = (\bar{s}, \bar{y}) \in S \times Y, \text{ where } \bar{s} \in S, \ \bar{y} \in Y \tag{8.209}$$

(see (8.171)). It follows from (8.208) and (8.204) that

$$||A_0 \bar{x} - y_0|| \leq r_1, \ B_0 \bar{x} \in z_0 - Z_+ + B_Z(r_1). \tag{8.210}$$

Therefore there exists $h_0 \in Z$ such that

$$||h_0|| \leq r_1, \ B_0 \bar{x} \leq z_0 + h_0. \tag{8.211}$$

Fix $r_2 > 0, r_3 > 0$ such that

$$r_2 < 16^{-1}(r_0 - r_1), \ 16 r_3 < 4 r_2 < r_1. \tag{8.212}$$

Put

$$\phi(x) = \tilde{\lambda}(x - \tilde{x}), \ x \in X. \tag{8.213}$$

By (8.213), (8.200), and (8.201),

$$0 \le \phi(x) \le 1,\ x \in X,$$

$$\{x \in X:\ \phi(x) > 0\} \subset B_X(\bar{x}, 1), \tag{8.214}$$

$$\phi(x) = 1 \text{ for all } x \text{ in a neighborhood of } \bar{x} \text{ in } X, \tag{8.215}$$

$$\sup\{||\phi^{(j)}(z)||:\ z \in X,\ j = 0, \dots, k\} < \infty. \tag{8.216}$$

Put

$$A_1(s, y) = A_0(s, y) - A_0\bar{x} + y_0,\ (s, y) \in S \times Y. \tag{8.217}$$

Relations (8.217) and (8.210) imply that

$$d^{(p)}_{X,Y}(A_1, A_0) \le \tilde{d}^{(p)}_{X,Y}(A_1, A_0) = \tilde{d}^{(0)}_{X,Y}(A_1, A_0) = ||A_0\bar{x} - y_0|| \le r_1. \tag{8.218}$$

Denote by $\mathcal{L}(Y, Y)$ the set of all linear mappings $L: Y \to Y$. For each $L \in \mathcal{L}(Y, Y)$ denote by L_ϕ a mapping $L_\phi : X \to Y$ defined by

$$L_\phi(s, y) = \phi(s, y)L(y - \bar{y}),\ (s, y) \in S \times Y = X. \tag{8.219}$$

It is clear to see that $L_\phi \in C^{(k)}(X, Y)$, $L \in \mathcal{L}(Y, Y)$ and the mapping $L \to L_\phi$, $L \in \mathcal{L}(Y, Y)$ from $\mathcal{L}(Y, Y)$ to $C^k(X, Y)$ is continuous. Therefore there exists a positive number ϵ_3 such that

$$\sup\{||L^{(j)}_\phi(z)||:\ j = 0, \dots, k,\ z \in X\} \le 16^{-1}(r_0 - r_1) \tag{8.220}$$

$$\text{for each } L \in \mathcal{L}(Y, Y) \text{ satisfying } ||L|| \le \epsilon_3.$$

There exists $L \in \mathcal{L}(Y, Y)$ such that

$$||L|| \le \epsilon_3 \text{ and } L + (\partial A_1/\partial y)(\bar{s}, \bar{y}) \text{ is invertible.} \tag{8.221}$$

It is easy to see that (8.220) holds. For each $x = (s, y) \in S \times Y = X$ define

$$\bar{A}x = \bar{A}(s, y) = \phi(s, y)[A_0(s, y) - A_0\bar{x} + y_0 + L(y - \bar{y})] \tag{8.222}$$

$$+(1 - \phi(s, y))[A_0(s, y) - A_0\bar{x} + y_0],$$

$$\bar{B}x = B_0x - r_2h_* - h_0. \tag{8.223}$$

Evidently,

$$\bar{A} \in C^{(p)}(X, Y),\ \bar{B} \in C^{(q)}(X, Z). \tag{8.224}$$

It follows from (8.222), (8.215), (8.209), (8.223) and (8.211) that

$$\bar{A}\bar{x} = y_0,\ \bar{B}\bar{x} = B_0\bar{x} - r_2h_* - h_0 \le z_0 - r_2h_*. \tag{8.225}$$

By (8.222), (8.215), (8.209) and (8.217)

$$\bar{A}(s, y) = A_0(s, y) - A_0\bar{x} + y_0 + L(y - \bar{y}) = A_1(s, y) + L(y - \bar{y}) \tag{8.226}$$

in a neighborhood of $\bar{x}$ in $S \times Y$.

Put

$$\bar{r} = r_3. \tag{8.227}$$

By (8.223), (8.165), (8.211) and (8.212),

$$d^{(q)}_{X,Z}(\bar{B}, B_0) \le \tilde{d}^{(q)}_{X,Z}(\bar{B}, B_0) = \tilde{d}^{(0)}_{X,Z}(\bar{B}, B_0)$$

$$\le ||r_2 h_*|| + ||h_0|| \le r_2 + r_1 < r_1 + 16^{-1}(r_0 - r_1) < (r_0 + r_1)/2. \tag{8.228}$$

Relations (8.222), (8.217) and (8.220) imply that

$$\tilde{d}^{(p)}_{X,Y}(\bar{A}, A_1) = \sup\{||L^{(j)}_{\phi}(z)|| : \ z \in X, \ j = 0, \dots, k\} \le 16^{-1}(r_0 - r_1). \tag{8.229}$$

It follows from (8.229) and (8.218) that

$$\tilde{d}^{p}_{X,Y}(\bar{A}, A_0) \le \tilde{d}^{p}_{X,Y}(\bar{A}, A_1) + \tilde{d}^{p}_{X,Y}(A_1, A_0) \le 16^{-1}(r_0 - r_1) + r_1 \le 2^{-1}(r_0 + r_1). \tag{8.230}$$

Relations (8.227), (8.212), (8.205), (8.224), (8.225), (8.228), (8.230) and (8.205) imply that (8.192)–(8.195) are true. We will show that the property (P2) holds.

Assume that $A \in C^{(p)}(X, Y)$, $B \in C^{(q)}(X, Z)$, $y \in Y$, $z \in Z$ and that $x \in X$ satisfies (8.196). We show that $x \in Q_{r_0}$.

It follows from (8.196), (8.227), (8.212) and (8.206) that

$$\tilde{d}^{(p)}_{X,Y}(A, \bar{A}) \le d^{(p)}_{X,Y}(A, \bar{A})(1 - d^{(p)}_{X,Y}(A, \bar{A}))^{-1} \le 2\bar{r},$$

$$\tilde{d}^{(q)}_{X,Z}(B, \bar{B}) \le d^{(q)}_{X,Z}(B, \bar{B})(1 - d^{(q)}_{X,Z}(B, \bar{B}))^{-1} \le 2\bar{r}.$$

In view of these relations, (8.230) and (8.228),

$$\tilde{d}^{(p)}_{X,Y}(A, A_0) \le \tilde{d}^{(p)}_{X,Y}(A, \bar{A}) + \tilde{d}^{(p)}_{X,Y}(\bar{A}, A_0) \le 2\bar{r} + (r_0 + r_1)/2, \tag{8.231}$$

$$\tilde{d}^{(q)}_{X,Z}(B, B_0) \le \tilde{d}^{(q)}_{X,Z}(B, \bar{B}) + \tilde{d}^{(q)}_{X,Z}(\bar{B}, B_0) \le 2\bar{r} + (r_0 + r_1)/2. \tag{8.232}$$

It follows from (8.231), (8.196), (8.227) and (8.212) that

$$||A_0 x - y_0|| \le ||A_0 x - Ax|| + ||Ax - y|| + ||y - y_0|| \le 2\bar{r} + (r_0 + r_1)/2 + \bar{r} < r_0. \tag{8.233}$$

Relations (8.165), (8.196), (8.232), (8.227) and (8.212) imply that

$$B_0 x - z_0 = B_0 x - Bx + Bx - z + z - z_0$$

$$\le \tilde{d}^{q}_{X,Z}(B, B_0) h_* + ||z - z_0|| h_* \le (2\bar{r} + (r_0 + r_1)/2 + \bar{r}) h_* \le r_0 h_*.$$

Combined with (8.233) and (8.204) this relation implies that $x \in Q_{r_0}$. Together with (8.205), (8.207), (8.208) and (8.203) this inclusion implies that

$$f_0(x) \geq \mu(r_0) \geq \mu(r_1) - \epsilon_1 \geq f_0(\bar{x}) - 2\epsilon_1 \geq f_0(\bar{x}) - \gamma.$$

Thus we have shown that the property (P2) holds.

Now we show that the property (P3) holds. Let Δ be a positive number. Fix

$$\Delta_1 \in (0, \Delta) \tag{8.234}$$

such that

$$||\bar{B}u - \bar{B}\bar{x}|| \leq r_2/8 \text{ for all } u \in B_X(\bar{x}, \Delta_1). \tag{8.235}$$

Put

$$\bar{\Psi}(s, y) = \bar{A}(s, y) - y_0, \ (s, y) \in S \times Y. \tag{8.236}$$

By (8.236), (8.225) and (8.209),

$$\bar{\Psi}(\bar{s}, \bar{y}) = 0. \tag{8.237}$$

It follows from (8.236), (8.226) and (8.209) that

$$(\partial\bar{\Psi}/\partial y)(\bar{s}, \bar{y}) = (\partial\bar{A}/\partial y)(s, \bar{y}) = (\partial A_1/\partial y)(\bar{s}, \bar{y}) + L. \tag{8.238}$$

By (8.221) and (8.238), the operator $(\partial\bar{\Psi}/\partial y)(\bar{s}, \bar{y})$ is invertible. It is easy now to see that the conditions of Theorem 8.13 hold with $X = S$, $Z = Y$, $(x_0, y_0) = (\bar{s}, \bar{y})$ and $W = S \times Y$. Therefore by Theorem 8.13 there exist $\delta > 0$ and $\sigma > 0$ such that the following property holds:

(P4) For each mapping $\Psi \in C^1(X, Y)$ satisfying $\tilde{d}^{(1)}_{X,Y}(\Psi, \bar{\Psi}) \leq 4\sigma$ and each $s \in B_S(\bar{s}, \delta)$ there is $D(s) \in Y$ such that

$$||D(s) - \bar{y}|| \leq \Delta_1 \text{ and } \Psi(s, D(s)) = 0.$$

Fix

$$\bar{\delta} \in (0, 8^{-1} \min\{1, \ \sigma, \ \delta, \ \bar{r}/4\}). \tag{8.239}$$

Assume that $A \in C^{(p)}(X, Y)$, $B \in C^{(q)}(X, Z)$, $y_1 \in Y$, $z_1 \in Z$ satisfy

$$d^{(p)}_{X,Y}(A, \bar{A}) \leq \bar{\delta}, \ d^{(q)}_{X,Z}(B, \bar{B}) \leq \bar{\delta}, \ ||y_1 - y_0|| \leq \bar{\delta}, \ ||z_1 - z_0|| \leq \bar{\delta}. \tag{8.240}$$

It follows from (8.240) and (8.239) that

$$\tilde{d}^{(p)}_{X,Y}(A, \bar{A}) = d^{(p)}_{X,Y}(A, \bar{A})(1 - d^{(p)}_{X,Y}(A, \bar{A}))^{-1} \leq \bar{\delta}(1 - \bar{\delta})^{-1} \leq 2\bar{\delta}, \tag{8.241}$$

$$\tilde{d}^{(q)}_{X,Z}(B, \bar{B}) = d^{(q)}_{X,Z}(\bar{B}, B)(1 - d^{(q)}_{X,Z}(B, \bar{B}))^{-1} \leq \bar{\delta}(1 - \bar{\delta})^{-1} \leq 2\bar{\delta}. \tag{8.242}$$

We show that there exists $x \in X$ such that

$$Ax = y_1, \ Bx \leq z_1, \ ||x - \bar{x}|| \leq \Delta_1. \tag{8.243}$$

Put

$$\Psi(s, y) = A(s, y) - y_1, \ (s, y) \in S \times Y. \tag{8.244}$$

By (8.244), (8.236), (8.241), (8.240) and (8.239),

$$\tilde{d}^{(p)}_{X,Y}(\Psi, \bar{\Psi}) \le \tilde{d}^{(p)}_{X,Y}(A, \bar{A}) + ||y_1 - y_0|| \le 3\bar{\delta} < \sigma. \tag{8.245}$$

By (8.245) and the property (P4), there exists $D(\bar{s}) \in Y$ for which

$$||D(\bar{s}) - \bar{y}|| \le \Delta_1 \text{ and } \Psi(\bar{s}, D(\bar{s})) = 0. \tag{8.246}$$

Put

$$x = (\bar{s}, D(\bar{s})). \tag{8.247}$$

It follows from (8.247), (8.244) and (8.246) that

$$Ax = A(\bar{s}, D(\bar{s})) = \Psi(\bar{s}, D(\bar{s})) + y_1 = y_1. \tag{8.248}$$

By (8.209), (8.247) and (8.246),

$$||\bar{x} - x|| = ||(\bar{s}, \bar{y}) - (\bar{s}, D(\bar{s}))|| = ||\bar{y} - D(\bar{s}))|| \le \Delta_1. \tag{8.249}$$

It follows from (8.165), (8.242), (8.225), (8.235), (8.249) and (8.239) that

$$Bx = \bar{B}x + Bx - \bar{B}x \le \bar{B}x + ||Bx - \bar{B}x||h_*$$

$$\le \bar{B}x + 2\bar{\delta}h_* = \bar{B}x - \bar{B}\bar{x} + \bar{B}\bar{x} + 2\bar{\delta}h_*$$

$$\le \bar{B}x - \bar{B}\bar{x} + z_0 - r_2 h_* + 2\bar{\delta}h_*$$

$$= \bar{B}x - \bar{B}\bar{x} + z_1 + (z_0 - z_1) - r_2 h_* + 2\bar{\delta}h_*$$

$$\le \bar{B}x - \bar{B}\bar{x} + z_1 + 3\bar{\delta}h_* - r_2 h_*$$

$$\le z_1 + (r_2/8)h_* + 3\bar{\delta}h_* - r_2 h_* \le z_1.$$

Combined with (8.249) and (8.248) this relation implies (8.243). Thus the property (P3) holds. This completes the proof of Lemma 8.19.

8.14 Proof of Theorems 8.15 and 8.16

It is easy to see that the set $\mathcal{A}$ is nonempty and that (A1) and (H2) hold. By Theorem 4.1 we need to show that (H1) holds. By Proposition 8.2 it is sufficient to show that (A2), (A3) and (A4) hold. (A2) follows from Lemma 8.7. Lemma 8.18 implies (A3). By Lemma 8.19 (A4) holds. This completes the proof of the theorems.

8.15 Comments

In this chapter we study minimization problems with mixed constraints

$$\text{minimize } f(x) \text{ subject to } G(x) = y,\ H(x) \leq z,$$

where f is a continuous (differentiable) finite-valued function defined on a Banach space X, y is an element of a finite-dimensional Banach space Y, z is an element of a Banach space Z ordered by a convex closed cone and $G : X \to Y$ and $H : X \to Z$ are continuous (differentiable) mappings. We consider two classes of these problems and show that most of the problems (in the Baire category sense) are well-posed. Our result is a generalization of a result of Ioffe, Lucchetti and Revalski [50] obtained for finite-dimensional Banach spaces X and Z. Our first class of problems is identified with the corresponding complete metric space of quintets (f, G, H, y, z). We show that for a generic quintet (f, G, H, y, z) the corresponding minimization problem has a unique solution and is well-posed. Our second class of problems is identified with the corresponding complete metric space of triples (f, G, H) while y and z are fixed. We show that for a generic triple (f, G, H) the corresponding minimization problem has a unique solution and is well-posed. This chapter is based on the works [124, 125].

9

Vector Optimization

9.1 Generic and density results in vector optimization

We use the convention that $\infty/\infty = 1$ and denote by $\mathrm{Card}(E)$ the cardinality of the set E.

Let R^1 be the set of real numbers and let $n \geq 1$ be an integer. Consider the finite-dimensional space R^n with the norm

$$||x|| = ||(x_1, \dots, x_n)|| = \max\{|x_i| : \ i = 1, \dots, n\}, \ x = (x_1, \dots, x_n) \in R^n.$$

Let $\{e_1, \dots, e_n\}$ be the standard basis in R^n:

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1).$$

Let $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$. We equip the space R^n with the natural order and say that

$$x \geq y \text{ if } x_i \geq y_i \text{ for all } i \in \{1, \dots, n\},$$

$$x > y \text{ if } x \geq y \text{ and } x \neq y$$

and

$$x >> y \text{ if } x_i > y_i \text{ for all } i \in \{1, \dots, n\}.$$

We say that $x << y$ (respectively, $x < y$, $x \leq y$) if $y >> x$ (respectively, $y > x$, $y \geq x$).

Let (X, ρ) be a complete metric space such that each of its bounded closed subsets is compact. Fix $\theta \in X$.

Denote by $\mathcal{A}$ the set of all continuous mappings $F = (f_1, \dots, f_n) : X \to R^n$ such that for all $i \in \{1, \dots, n\}$

$$\lim_{\rho(x,\theta) \to \infty} f_i(x) = \infty. \tag{9.1}$$

For each $F = (f_1, \dots, f_n)$, $G = (g_1, \dots, g_n) \in \mathcal{A}$ set

A.J. Zaslavski, *Optimization on Metric and Normed Spaces*, Springer Optimization and Its Applications 44, DOI 10.1007/978-0-387-88621-3_9,

$$\tilde{d}(F,G) = \sup\{|f_i(x) - g_i(x)| : \ x \in X \text{ and } i = 1,\dots,n\},$$

$$d(F,G) = \tilde{d}(F,G)(1+\tilde{d}(F,G))^{-1}.$$

Clearly the metric space $(\mathcal{A}, d)$ is complete.

Note that $\tilde{d}(F,G) = \sup\{||F(x) - G(x)|| : \ x \in X\}$ for all F, $G \in \mathcal{A}$.

Let $A \subset R^n$ be a nonempty set. An element $x \in A$ is called a minimal element of A if there is no $y \in A$ for which $y < x$.

Let $F \in \mathcal{A}$. A point $x \in X$ is called a point of minimum of F if $F(x)$ is a minimal element of $F(X)$. If $x \in X$ is a point of minimum of F, then $F(x)$ is called a minimal value of F. Denote by $M(F)$ the set of all points of minimum of F and put $v(F) = F(M(F))$.

We prove the following proposition obtained in [129].

Proposition 9.1. *Let $F = (f_1,\dots,f_n) \in \mathcal{A}$. Then $M(F)$ is a nonempty bounded subset of (X,ρ) and for each $z \in F(X)$ there is $y \in v(F)$ such that $y \le z$.*

Assume that $n \ge 2$ and that the space (X,ρ) has no isolated points.

We prove the following theorem which was also obtained in [129].

Theorem 9.2. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that for each $F \in \mathcal{F}$ the set $v(F)$ is infinite.*

We also prove the following theorem which was also obtained in [128].

Theorem 9.3. *Suppose that the space (X,ρ) is connected. Let*

$$F = (f_1,\dots,f_n) \in \mathcal{A}$$

and let ϵ be a positive number. Then there exists $G \in \mathcal{A}$ such that $\tilde{d}(F,G) \le \epsilon$ and the set $v(G)$ is not closed.

It is clear that if X is a finite-dimensional Euclidean space, then X is a complete metric space such that all its bounded closed subsets are compact and Theorems 9.2 and 9.3 hold. It is also clear that Theorems 9.2 and 9.3 hold if X is a convex compact subset of a Banach space or if X is a convex closed cone generated by a convex compact subset of a Banach space which does not contain zero.

9.2 Proof of Proposition 9.1

Let $z \in F(X)$. We show that there is $y \in v(F)$ such that $y \le z$. Put

$$\Omega_0 = \{h \in F(X) : \ h \le z\}. \tag{9.2}$$

We consider the set Ω_0 with the natural order and show that Ω_0 has a minimal element by using Zorn's lemma.

Assume that D is a nonempty subset of Ω_0 such that for each $h_1, h_2 \in D$ either $h_1 \geq h_2$ or $h_1 \leq h_2$. Since all bounded closed subsets of X are compact it follows from (9.1) that the set $F(X)$ is bounded from below. Combined with (9.2) this implies that the set D is bounded. For each natural number $i \in \{1, \dots, n\}$ put

$$\bar{h}_i = \inf\{\lambda \in R^1 : \text{ there is } x = (x_1, \dots, x_n) \in D \text{ for which } x_i = \lambda\} \quad (9.3)$$

and put

$$\bar{h} = (\bar{h}_1, \dots, \bar{h}_n). \quad (9.4)$$

It is easy to see that the vector $\bar{h}$ is well defined.

Let $p \geq 1$ be an integer. It follows from (9.3) and (9.4) that for each integer $j \in \{1, \dots, n\}$ there exists

$$z^{(p,j)} = (z_1^{(p,j)}, \dots, z_n^{(p,j)}) \in D \quad (9.5)$$

such that

$$\bar{h}_j \geq z_j^{(p,j)} - 1/p. \quad (9.6)$$

It is easy to see that there exists

$$z^{(p)} \in \{z^{(p,j)} : j = 1, \dots, n\} \quad (9.7)$$

such that

$$z^{(p)} \leq z^{(p,j)} \text{ for all } j = 1, \dots, n. \quad (9.8)$$

By (9.6), (9.8), (9.3), (9.7) and (9.5), for each $j = 1, \dots, n$,

$$\bar{h}_j \leq z_j^{(p)} \leq \bar{h}_j + 1/p. \quad (9.9)$$

It follows from (9.7), (9.5) and (9.2) that for each natural number p there exists $x_p \in X$ such that

$$F(x_p) = z^{(p)}. \quad (9.10)$$

It follows from (9.10), (9.9) and (9.1) that the sequence $\{x_p\}_{p=1}^{\infty}$ is bounded. Since any bounded closed set in (X, ρ) is compact there is a subsequence $\{x_{p_i}\}_{i=1}^{\infty}$ of the sequence $\{x_p\}_{p=1}^{\infty}$ which converges to some point $\bar{x} \in X$. Relations (9.9) and (9.10) imply that

$$F(\bar{x}) = \lim_{i \to \infty} F(x_{p_i}) = \lim_{i \to \infty} z^{(p_i)} = \bar{h}$$

and $\bar{h} \in F(X)$. Combined with (9.2) and (9.3) this implies that $\bar{h} \in \Omega_0$. In view of (9.3), $\bar{h} \leq h$ for all $h \in D$. Zorn's lemma implies that there exists a minimal element $y \in F(X)$ for which $y \leq z$. Proposition 9.1 is proved.

9.3 Auxiliary results

Proposition 9.4. *Let* $F = (f_1, \dots, f_n) \in \mathcal{A}$ *and let* $\mathrm{Card}(v(F)) = p$ *where* $p \geq 1$ *is an integer. Then there exists a neighborhood* W *of* F *in* $(\mathcal{A}, d)$ *such that* $\mathrm{Card}(v(G)) \geq p$ *for each* $G = (g_1, \dots, g_n) \in W$.

Proof: Let

$$y_1, \dots, y_p \in v(F), \tag{9.11}$$

$$y_i \neq y_j \text{ for each } (i,j) \in \Omega := \{1, \dots, p\} \times \{1, \dots, p\} \setminus \{(i,i) : \ i = 1, \dots, p\}. \tag{9.12}$$

For each $i \in \{1, \dots, p\}$ there exists $x_i \in X$ such that

$$F(x_i) = y_i. \tag{9.13}$$

It follows from (9.12) and (9.13) that for each $(i,j) \in \Omega$ there is $p(i,j) \in \{1, \dots, n\}$ such that

$$f_{p(i,j)}(x_i) > f_{p(i,j)}(x_j). \tag{9.14}$$

Fix a positive number ϵ such that

$$f_{p(i,j)}(x_i) > f_{p(i,j)}(x_j) + 4\epsilon \tag{9.15}$$

for all $(i,j) \in \Omega$. Put

$$W = \{G \in \mathcal{A} : \ \tilde{d}(G, F) \leq \epsilon\}. \tag{9.16}$$

Assume that

$$G = (g_1, \dots, g_n) \in W. \tag{9.17}$$

For each $i \in \{1, \dots, p\}$ the inclusion $G(x_i) \in G(X)$ holds and by Proposition 9.1 there exists

$$\bar{y}_i \in v(G) \tag{9.18}$$

such that

$$\bar{y}_i \leq G(x_i). \tag{9.19}$$

Let $i \in \{1, \dots, p\}$. It follows from (9.18) that there exists $\bar{x}_i \in X$ such that

$$G(\bar{x}_i) = \bar{y}_i. \tag{9.20}$$

Relations (9.17) and (9.16) imply that

$$||G(\bar{x}_i) - F(\bar{x}_i)|| \leq \epsilon. \tag{9.21}$$

By (9.11), (9.12), the equality $\mathrm{Card}(v(F)) = p$ and Proposition 9.1, there exists $k(i) \in \{1, \dots, p\}$ such that

$$F(\bar{x}_i) \geq y_{k(i)}. \tag{9.22}$$

It follows from (9.13), (9.22), (9.21), (9.20) and (9.19) that

$$F(x_{k(i)}) = y_{k(i)} \le F(\bar{x}_i) \le G(\bar{x}_i) + \epsilon(1,1,\dots,1)$$

$$\le G(x_i) + \epsilon(1,1,\dots,1) \le F(x_i) + 2\epsilon(1,1,\dots,1). \quad (9.23)$$

By (9.23) and (9.15), $k(i) = i$. Together with (9.23), (9.20) and (9.13) this equality implies that

$$y_i \le \bar{y}_i + \epsilon(1,1,\dots,1) \le y_i + 2\epsilon(1,1,\dots,1).$$

By this inequality, (9.15) and (9.13),

$$\bar{y}_i \ne \bar{y}_j \text{ if } i,j \in \{1,\dots,p\} \text{ satisfy } i \ne j.$$

Proposition 9.4 is proved.

Proposition 9.5. *Assume that* $F = (f_1,\dots,f_n) \in \mathcal{A}$*,* $p \ge 1$ *is an integer,* $\mathrm{Card}(v(F)) = p$ *and that*

$$v(F) = \{y_1,\dots,y_p\},\ x_i \in X,\ F(x_i) = y_i,\ i = 1,\dots,p, \quad (9.24)$$

$$y_i \ne y_j \textit{ for all } i,j \in \{1,\dots,p\} \textit{ satisfying } i \ne j.$$

Then for each $i = 1,\dots,p$ *the inequality* $F(x_i) \le F(x)$ *holds for all* x *belonging to a neighborhood of* x_i*.*

Proof: It is sufficient to consider the case with $i = 1$. It is easy to see that for each $j \in \{2,\dots,n\}$ there exists $s(j) \in \{1,\dots,n\}$ for which $f_{s(j)}(x_1) < f_{s(j)}(x_j)$. Fix a positive number ϵ such that

$$f_{s(j)}(x_1) < f_{s(j)}(x_j) - 2\epsilon \text{ for all } j \in \{2,\dots,n\}. \quad (9.25)$$

There exists a positive number δ such that for each $x \in X$ satisfying $\rho(x,x_1) \le \delta$,

$$||F(x) - F(x_1)|| \le \epsilon/2. \quad (9.26)$$

Let $x \in X$ satisfy $\rho(x,x_1) \le \delta$. Then (9.26) holds. Proposition 9.1 implies that there exists $y \in v(F)$ satisfying

$$y \le F(x). \quad (9.27)$$

In order to complete the proof it is sufficient to show that $y = F(x_1)$.

Assume the contrary. Then there exists $j \in \{2,\dots,n\}$ for which $y = y_j = F(x_j)$. This relation, (9.26) and (9.27) imply that

$$F(x_j) = y_j = y \le F(x) \le F(x_1) + (\epsilon/2)(1,1,\dots,1)$$

and

$$f_{s(j)}(x_j) \le f_{s(j)}(x_1) + \epsilon/2.$$

This contradicts (9.25). The contradiction we have reached proves Proposition 9.5.

Proposition 9.6. *Assume that* $F = (f_1, \ldots, f_n) \in \mathcal{A}$*,* ϵ *is a positive number,* $p \geq 1$ *is an integer and that*

$$\mathrm{Card}(v(F)) = p,\ x_1, \ldots, x_p \in X,\ y_i = F(x_i),\ i = 1, \ldots, p, \tag{9.28}$$

$$v(F) = \{y_i :\ i = 1, \ldots, p\}.$$

Then there exists $G = (g_1, \ldots, g_n) \in \mathcal{A}$ *such that*

$$f_i(x) \leq g_i(x),\ x \in X,\ i = 1, \ldots, n,$$

$$g_i(x_j) = f_i(x_j),\ i = 1, \ldots, n,\ j = 1, \ldots, p, \tag{9.29}$$

$$\tilde{d}(F, G) \leq \epsilon, \tag{9.30}$$

$$v(G) = \{G(x_j) :\ j = 1, \ldots, p\} \tag{9.31}$$

and that for each $x \in X \setminus \{x_1, \ldots, x_p\}$ *there exists an integer* $j \in \{1, \ldots, p\}$ *for which*

$$G(x) \geq G(x_j) + \epsilon \min\{1, \rho(x, x_i) :\ i = 1, \ldots, p\}(1, 1, \ldots, 1). \tag{9.32}$$

Proof: For each $x \in X$ and $i = 1, \ldots, n$ put

$$g_i(x) = f_i(x) + \epsilon \min\{1, \rho(x, x_j) :\ j = 1, \ldots, p\} \tag{9.33}$$

and set $G = (g_1, \ldots, g_n)$. It is easy to see that $G \in \mathcal{A}$,

$$g_i(x) \geq f_i(x),\ x \in X,\ i = 1, \ldots, n,$$

$$g_i(x_j) = f_i(x_j) \text{ for each } i \in \{1, \ldots, n\} \text{ and each } j \in \{1, \ldots, p\}$$

and that $\tilde{d}(F, G) \leq \epsilon$. Hence (9.29) and (9.30) are true.

Let $j \in \{1, \ldots, p\}$. We will show that $G(x_j) \in v(G)$. Assume that $x \in X$ satisfies

$$G(x) \leq G(x_j). \tag{9.34}$$

Relations (9.29), (9.33) and (9.34) imply that

$$F(x) \leq F(x) + \epsilon \min\{1, \rho(x, x_i) :\ i = 1, \ldots, p\}(1, 1, \ldots, 1) = G(x)$$

$$\leq G(x_j) = F(x_j).$$

Combined with (9.28) this relation implies that

$$F(x) = F(x_j),\ x \in \{x_i :\ i = 1, \ldots, p\} \text{ and } x = x_j.$$

Hence

$$\{G(x_j) :\ j = 1, \ldots, p\} \subset v(G). \tag{9.35}$$

Let

$$x \in X \setminus \{x_1, \ldots, x_p\}. \tag{9.36}$$

It follows from Proposition 9.1 and (9.28) that there exists $j \in \{1, \ldots, p\}$ for which

$$F(x_j) \le F(x). \tag{9.37}$$

In view of (9.29), (9.37), (9.33) and (9.36),

$$G(x_j) = F(x_j) \le F(x)$$

$$< F(x) + \epsilon \min\{1, \rho(x, x_i) : \; i = 1, \ldots, p\}(1, \ldots, 1) \le G(x).$$

This relation implies that

$$G(x) > G(x_j) + \epsilon \min\{1, \rho(x, x_i) : \; i = 1, \ldots, p\}(1, 1, \ldots, 1)$$

and $G(x) \notin v(G)$. Combined with (9.35) this relation implies (9.31). Proposition 9.6 is proved.

Proposition 9.7. *Assume that $F = (f_1, \ldots, f_n) \in \mathcal{A}$, $p \ge 1$ is an integer,*

$$\mathrm{Card}(v(F)) = p \tag{9.38}$$

and that ϵ is a positive number. Then there exists $G \in \mathcal{A}$ such that $\tilde{d}(F, G) \le \epsilon$ and $\mathrm{Card}(v(G)) = p + 1$.

Proof: Let

$$v(F) = \{y_1, \ldots, y_p\} \tag{9.39}$$

where $y_1, \ldots, y_p \in R^n$. It is easy to see that

$$y_i \ne y_j \text{ for each } i, j \in \{1, \ldots, p\} \text{ such that } i \ne j. \tag{9.40}$$

For each $i \in \{1, \ldots, p\}$ there exists $x_i \in X$ such that

$$F(x_i) = y_i. \tag{9.41}$$

Proposition 9.6 implies that there exists $F^{(1)} = (f_1^{(1)}, \ldots, f_1^{(n)}) \in \mathcal{A}$ which satisfies

$$f_i^{(1)}(x) \ge f_i(x) \text{ for all } x \in X \text{ and all } i = 1, \ldots, n, \tag{9.42}$$

$$f_i^{(1)}(x_j) = f_i(x_j), i = 1, \ldots, n, \; j = 1, \ldots, p, \tag{9.43}$$

$$\tilde{d}(F, F^{(1)}) \le \epsilon/4, \tag{9.44}$$

$$v(F^{(1)}) = \{F^{(1)}(x_j) : \; j = 1, \ldots, p\} \tag{9.45}$$

and that for each $x \in X \setminus \{x_1, \ldots, x_p\}$ there is $j \in \{1, \ldots, p\}$ such that

$$F^{(1)}(x) \ge F^{(1)}(x_j) + \epsilon \min\{1, \rho(x, x_i) : \; i = 1, \ldots, p\}(1, 1, \ldots, 1). \tag{9.46}$$

It is easy to see that there exists

$$\epsilon_0 \in (0, \min\{1, \epsilon\}/8) \tag{9.47}$$

and that for each $i, j \in \{1, \dots, p\}$ satisfying $i \neq j$ there exists $s(i, j) \in \{1, \dots, n\}$ such that

$$f^{(1)}_{s(i,j)}(x_i) < f^{(1)}_{s(i,j)}(x_j) - 8\epsilon_0. \tag{9.48}$$

Fix a positive number $\delta_0 < 1/8$ such that

$$\rho(x_i, x_j) \geq 8\delta_0 \text{ for each } i, j \in \{1, \dots, p\} \text{ satisfying } i \neq j. \tag{9.49}$$

There exists a positive number $\delta_1 < \delta_0/2$ such that for each $x \in X$ satisfying $\rho(x_1, x) \leq 2\delta_1$

$$||F^{(1)}(x_1) - F^{(1)}(x)|| \leq \epsilon_0/8. \tag{9.50}$$

Set

$$\epsilon_1 = \epsilon_0/4. \tag{9.51}$$

There exists $x_0 \in X$ such that

$$0 < \rho(x_0, x_1) < \delta_1. \tag{9.52}$$

Relations (9.52) and (9.49) imply that

$$x_0 \notin \{x_i : \ i = 1, \dots, p\}. \tag{9.53}$$

Fix

$$\epsilon_2 \in (0, \min\{\epsilon_0\rho(x_0, x_1)/4, \ \epsilon_1\}). \tag{9.54}$$

Fix $\delta_2 > 0$ for which

$$4\delta_2 < \rho(x_0, x_1), \tag{9.55}$$

$$|f^{(1)}_i(x_0) - f^{(1)}_i(x)| \leq \epsilon_2/4 \text{ for each } i \in \{1, \dots, n\}$$
$$\text{and each } x \in X \text{ satisfying } \rho(x, x_0) \leq 4\delta_2. \tag{9.56}$$

Fix $\lambda > 0$ such that

$$\lambda\delta_2 > 2\epsilon_1 + 2\epsilon_2. \tag{9.57}$$

Put

$$g_1(x) = f^{(1)}_1(x) \text{ for each } x \in X \text{ satisfying } \rho(x, x_0) > 2\delta_2, \tag{9.58}$$

$$g_1(x) = \min\{f^{(1)}_1(x), \ f^{(1)}_1(x_0) - \epsilon_1 + \lambda\rho(x, x_0)\} \tag{9.59}$$
$$\text{for each } x \in X \text{ satisfying } \rho(x, x_0) \leq 2\delta_2.$$

For $i \in \{2, \dots, n\}$ put

$$g_i(x) = f^{(1)}_i(x) \text{ for each } x \in X \text{ satisfying } \rho(x, x_0) > 2\delta_2, \tag{9.60}$$

$$g_i(x) = \min\{f^{(1)}_i(x), \ f^{(1)}_i(x_0) - \epsilon_2 + \lambda\rho(x, x_0)\} \tag{9.61}$$

for each $x \in X$ satisfying $\rho(x, x_0) \le 2\delta_2$.

Set $G = (g_1, \dots, g_n)$. It follows from (9.56) and (9.57) that for each $i \in \{1, \dots, n\}$ and each $x \in X$ satisfying $\delta_2 \le \rho(x, x_0) \le 2\delta_2$,

$$f_i^{(1)}(x_0) - \epsilon_1 + \lambda\rho(x, x_0) \ge f_i^1(x_0) - \epsilon_1 + \lambda\delta_2$$

$$\ge f_i^1(x_0) + \epsilon_1 + 2\epsilon_2 \ge f_i^{(1)}(x) - \epsilon_2/4 + \epsilon_1 + 2\epsilon_2. \tag{9.62}$$

Relations (9.62), (9.59) and (9.61) imply that for each $i \in \{1, \dots, n\}$ and each $x \in X$ satisfying $\delta_2 \le \rho(x, x_0) \le 2\delta_2$,

$$g_i(x) = f_i^{(1)}(x). \tag{9.63}$$

By (9.63) and (9.58)–(9.61), G is continuous. It follows from (9.58) and (9.60) that $G \in \mathcal{A}$. By (9.63), (9.58) and (9.60), for each $x \in X$ satisfying $\rho(x_0, x) \ge \delta_2$

$$F^{(1)}(x) = G(x). \tag{9.64}$$

Relations (9.59) and (9.61) imply that for each $x \in X$ satisfying $\rho(x_0, x) \le 2\delta_2$,

$$G(x) \le F^1(x). \tag{9.65}$$

Let $x \in X$ satisfy

$$\rho(x, x_0) \le \delta_2. \tag{9.66}$$

It follows from (9.58)–(9.61), (9.54), (9.66) and (9.56) that for each $i \in \{1, \dots, n\}$,

$$f_i^{(1)}(x) \ge g_i(x) \ge \min\{f_i^{(1)}(x),\ f_i^{(1)}(x_0) - \epsilon_1\}$$

$$\ge \min\{f_i^{(1)}(x), f_i^{(1)}(x) - \epsilon_2/4 - \epsilon_1\} \ge f_i^{(1)}(x) - (5/4)\epsilon_1$$

and

$$F^{(1)}(x) \ge G(x) \ge F^{(1)}(x) - (\epsilon/2)(1, 1, \dots, 1).$$

Combined with (9.64) this inequality implies that $\tilde{d}(F^{(1)}, G) \le \epsilon/2$. Together with (9.44) this implies that

$$\tilde{d}(F, G) \le \tilde{d}(F, F^{(1)}) + \tilde{d}(F^{(1)}, G) < \epsilon/4 + \epsilon/2. \tag{9.67}$$

Let $x \in X$. We show that there exists $j \in \{0, \dots, p\}$ such that $G(x) \ge G(x_j)$. There are two cases:

$$\rho(x, x_0) \ge \delta_2; \tag{9.68}$$

$$\rho(x, x_0) < \delta_2. \tag{9.69}$$

Assume that (9.68) holds. Then (9.64) implies that $G(x) = F^{(1)}(x)$. Proposition 9.1 and (9.45) imply that there exists $j \in \{1, \dots, p\}$ for which

$$F^{(1)}(x_j) \le F^{(1)}(x) = G(x). \tag{9.70}$$

If $j = 1$, then (9.55) implies that

$$\rho(x_j, x_0) = \rho(x_0, x_1) > 4\delta_2.$$

If $j \neq 1$, then it follows from (9.49) and (9.52) that

$$\rho(x_j, x_0) \geq \rho(x_j, x_1) - \rho(x_1, x_0) \geq 8\delta_0 - \delta_1 \geq 7\delta_0 > 4\delta_2.$$

Thus in both cases $\rho(x_j, x_0) > 4\delta_2$. It follows from this inequality and (9.64) that

$$F^{(1)}(x_j) = G(x_j).$$

Combined with (9.70) this equality implies that $G(x_j) \leq G(x)$. Assume that (9.69) holds. We show that $G(x_0) \leq G(x)$. In view of (9.59) and (9.61),

$$G(x_0) = (f_1^{(1)}(x_0) - \epsilon_1, f_2^{(1)}(x_0) - \epsilon_2, \dots, f_n^{(1)}(x_0) - \epsilon_2)$$

$$= F^{(1)}(x_0) - (\epsilon_1, \epsilon_2, \dots, \epsilon_2). \tag{9.71}$$

By (9.69)

$$F^{(1)}(x) \geq F^{(1)}(x_0) - (\epsilon_2/4)(1, 1, \dots, 1). \tag{9.72}$$

It follows from (9.69), (9.59), (9.61), (9.72) and (9.54) that

$$g_1(x) \geq \min\{f_1^{(1)}(x_0) - \epsilon_2/4, f_1^{(1)}(x_0) - \epsilon_1\} = f_1^{(1)}(x_0) - \epsilon_1$$

and for $i \in \{1, \dots, p\} \setminus \{1\}$

$$g_i(x) \geq \min\{f_i^{(1)}(x_0) - \epsilon_2/4, f_i^{(1)}(x_0) - \epsilon_2\} = f_i^{(1)}(x_0) - \epsilon_2.$$

Combined with (9.71) these inequalities imply that $G(x) \geq G(x_0)$. Thus we have shown that for each $x \in X$ there is $j \in \{0, \dots, p\}$ such that $G(x) \geq G(x_j)$.

Now assume that $j_1, j_2 \in \{0, \dots, p\}$ satisfy

$$G(x_{j_1}) \leq G(x_{j_2}). \tag{9.73}$$

We will show that $j_1 = j_2$. Relations (9.49) and (9.52) imply that for each $i \in \{2, \dots, p\}$

$$\rho(x_i, x_0) \geq \rho(x_i, x_1) - \rho(x_0, x_1) \geq 8\delta_0 - \delta_1 > 7\delta_0 > 4\delta_2.$$

By (9.55) $\rho(x_0, x_1) > 4\delta_2$. Thus for all $i \in \{1, \dots, p\}$

$$\rho(x_i, x_0) > 4\delta_2. \tag{9.74}$$

In view of (9.74), (9.58) and (9.60),

$$G(x_i) = F^{(1)}(x_i),\ i = 1, \dots, p. \tag{9.75}$$

If $j_1, j_2 \in \{1, \ldots, p\}$, then (9.73), (9.75) and (9.48) imply that $F^{(1)}(x_{j_1}) = F^{(1)}(x_{j_2})$ and $j_1 = j_2$. Therefore we may consider only the case with $0 \in \{j_1, j_2\}$. Let $i \in \{1, \ldots, p\} \setminus \{1\}$. In view of (9.75),

$$G(x_i) = F^{(1)}(x_i). \tag{9.76}$$

It follows from (9.48) that

$$f^{(1)}_{s(i,1)}(x_i) < f^{(1)}_{s(i,1)}(x_1) - 8\epsilon_0,\ f^{(1)}_{s(i,1)}(x_1) < f^{(1)}_{s(1,i)}(x_i) - 8\epsilon_0. \tag{9.77}$$

By (9.76), (9.77), (9.50), (9.52), (9.51), (9.59) and (9.61),

$$g_{s(i,1)}(x_i) = f^{(1)}_{s(i,1)}(x_i) < f^{(1)}_{s(i,1)}(x_1) - 8\epsilon_0 \le f^{(1)}_{s(i,1)}(x_0) + \epsilon_0/4 - 8\epsilon_0$$

$$< f^{(1)}_{s(i,1)}(x_0) - 2\epsilon_1 \le g_{s(i,1)}(x_0) - \epsilon_1,$$

$$g_{s(i,1)}(x_i) = f^{(1)}_{s(i,1)}(x_i) > f^{(1)}_{s(1,i)}(x_1) + 8\epsilon_0$$

$$\ge f^{(1)}_{s(i,1)}(x_0) - \epsilon_0/4 + 8\epsilon_0 > g_{s(1,i)}(x_0) + 7\epsilon_0.$$

It follows from these inequalities that the inequality $G(x_i) \le G(x_0)$ does not hold and that the inequality $G(x_0) \le G(x_i)$ does not hold too. Combined with (9.73) and the inclusion $0 \in \{j_1, j_2\}$ this implies that

$$\{j_1, j_2\} \subset \{0, 1\}. \tag{9.78}$$

It follows from (9.75) that

$$G(x_1) = F^{(1)}(x_1) \text{ and } g_s(x_1) = f^1_s(x_1),\ s = 1, \ldots, n. \tag{9.79}$$

By (9.74) and (9.46), there exists an integer $q \in \{1, \ldots, p\}$ such that

$$F^{(1)}(x_0) \ge F^{(1)}(x_q) + \epsilon \min\{1, \rho(x_0, x_i) :\ i = 1, \ldots, p\}(1, 1, \ldots, 1). \tag{9.80}$$

By (9.52), (9.50) and (9.80),

$$F^{(1)}(x_q) \le F^{(1)}(x_0) \le F^{(1)}(x_1) + (\epsilon_0/8)(1, \ldots, 1).$$

Combined with (9.48) this implies that $q = 1$. Together with (9.80) this equality implies that

$$F^{(1)}(x_0) \ge F^{(1)}(x_1) + \epsilon \min\{1, \rho(x_0, x_i) :\ i = 1, \ldots, p\}(1, 1, \ldots, 1).$$

The inequality above, (9.49), (9.52) and (9.79) imply that for all $i \in \{1, \ldots, p\} \setminus \{1\}$

$$\rho(x_0, x_i) \ge \rho(x_1, x_i) - \rho(x_1, x_0) \ge 8\delta_0 - \delta_1 \ge 7\delta_0,$$

$$\min\{1, \rho(x_0, x_i) :\ i = 1, \ldots, p\} = \rho(x_0, x_1)$$

and

$$F^{(1)}(x_0) \geq F^{(1)}(x_1) + \epsilon\rho(x_0, x_1)(1, 1, \dots, 1)$$
$$= G^{(1)}(x_1) + \epsilon\rho(x_0, x_1)(1, 1, \dots, 1). \quad (9.81)$$

By (9.81), (9.61) and (9.54), for $i \in \{1, \dots, p\} \setminus \{1\}$,

$$g_i^{(1)}(x_1) + \epsilon\rho(x_0, x_1) \leq f_i^{(1)}(x_0) = g_i(x_0) + \epsilon_2,$$

$$g_i^{(1)}(x_1) \leq g_i(x_0) + \epsilon_2 - \epsilon\rho(x_0, x_1) \leq g_i(x_0) - \epsilon_2.$$

In view of (9.81) and (9.59),

$$g_1(x_1) + \epsilon\rho(x_0, x_1) \leq f_1^{(1)}(x_0) = g_1(x_0) + \epsilon_1.$$

It follows from this relation, (9.52), (9.50), (9.51) and (9.79) that

$$g_1(x_0) = f_1^{(1)}(x_0) - \epsilon_1 \leq f_1^{(1)}(x_1) + \epsilon_0/8 - \epsilon_1 = f_1^{(1)}(x_1) - \epsilon_0/8 = g_1(x_1) - \epsilon_0/8.$$

Then each of the inequalities $G(x_0) \leq G(x_1)$, $G(x_1) \leq G(x_0)$ does not hold. Combined with (9.78), the inclusion $0 \in \{j_1, j_2\}$ and (9.73) this implies that $j_1 = j_2 = 0$. Thus we have shown that if $j_1, j_2 \in \{0, \dots, p\}$ and if $G(x_{j_1}) \leq G(x_{j_2})$, then $j_1 = j_2$. Therefore $\mathrm{Card}(v(G)) = p + 1$. This completes the proof of Proposition 9.7.

9.4 Proof of Theorem 9.2

Lemma 9.8. *Let $F \in \mathcal{A}$, p be a natural number and let ϵ be a positive number. Then there exists an open nonempty set*

$$\mathcal{U} \subset \{H \in \mathcal{A} : \tilde{d}(F, H) \leq \epsilon\}$$

such that for each $G \in \mathcal{U}$ the inequality $\mathrm{Card}(v(G)) \geq p + 1$ holds.

Proof: If for each $G \in \mathcal{A}$ satisfying $\tilde{d}(F, G) < \epsilon$ we have $\mathrm{Card}(v(G)) \geq p + 1$, then set

$$\mathcal{U} = \{H \in \mathcal{A} : \tilde{d}(F, H) < \epsilon\}.$$

Assume that there is $G_0 \in \mathcal{A}$ such that

$$\tilde{d}(F, G_0) < \epsilon \text{ and } \mathrm{Card}(v(G_0)) \leq p.$$

Proposition 9.7 implies that there exists $G_1 \in \mathcal{A}$ for which

$$\tilde{d}(F, G_1) < \epsilon \text{ and } \mathrm{Card}(v(G_1)) = p + 1.$$

Proposition 9.4 implies that there exists an open neighborhood $\mathcal{U}$ of G_1 in $\mathcal{A}$ for which

$$\mathcal{U} \subset \{H \in \mathcal{A} : \tilde{d}(H, G) < \epsilon\}$$

and that for each $G \in \mathcal{U}$ the inequality $\mathrm{Card}(v(G)) \geq p+1$ holds. Lemma 9.8 is proved.

Proof of Theorem 9.2: Let $p \geq 1$ be an integer. It follows from Lemma 9.8 that for each $F \in \mathcal{A}$ and each natural number i there exists an open nonempty set

$$\mathcal{U}(F, i, p) \subset \{H \in \mathcal{A} : \tilde{d}(F, H) \leq (2i)^{-1}\}$$

such that for each $H \in \mathcal{U}(F, i, p)$ the inequality $\mathrm{Card}(v(H)) \geq p$ holds. Define

$$\mathcal{F} = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(F, i, p) : F \in \mathcal{A} \text{ and } i \text{ is a natural number}\}.$$

Evidently, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ and for each $G \in \mathcal{F}$ the set $v(G)$ is infinite. Theorem 9.2 is proved.

9.5 Proof of Theorem 9.3

Proposition 9.1 implies that there exists $x_* \in X$ for which

$$F(x_*) \in v(F). \tag{9.82}$$

There exists a positive number $\delta < 1/4$ such that

$$||F(x) - F(x_*)|| \leq \epsilon/8 \text{ for each } x \in X \text{ such that } \rho(x, x_*) \leq 2\delta, \tag{9.83}$$

$$\{z \in X : \rho(z, x_*) = 8\delta\} \neq \emptyset. \tag{9.84}$$

Since the metric space X is connected for each $t \in (0, 8\delta]$ there is $z \in X$ such that $\rho(z, x_*) = t$.

Evidently, there exists a continuous function $\psi : X \to [0, 1]$ such that

$$\psi(x) = 1 \text{ for each } x \in X \text{ satisfying } \rho(x, x_*) \leq \delta, \tag{9.85}$$

$$\psi(x) = 0 \text{ for each } x \in X \text{ satisfying } \rho(x, x_*) \geq 2\delta.$$

For $x \in X$ and $i = 1, \dots, n$ define

$$f_i^{(1)}(x) = \psi(x) f_i(x_*) + (1 - \psi(x)) f_i(x), \tag{9.86}$$

$$F^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)}). \tag{9.87}$$

It is easy to see that $F^{(1)} \in \mathcal{A}$ and that

$$F^{(1)}(x) = F(x_*) \text{ for all } x \in X \text{ satisfying } \rho(x, x_*) \leq \delta. \tag{9.88}$$

We will show that $\tilde{d}(F, F^{(1)}) \leq \epsilon/8$. Let $x \in X$. If $\rho(x, x_*) \geq 2\delta$, then $\psi(x) = 0$ and $F^{(1)}(x) = F(x)$. If $\rho(x, x_*) \leq 2\delta$, then it follows from (9.83) that

$$||F(x) - F(x_*)|| \le (\epsilon/8).$$

Together with (9.86) the inequality above implies that

$$||F^{(1)}(x) - F(x)|| = ||\psi(x)(F(x_*) - F(x))|| \le ||F(x_*) - F(x)|| \le \epsilon/8.$$

Thus

$$\tilde{d}(F, F^{(1)}) \le \epsilon/8. \tag{9.89}$$

We will show that $F^{(1)}(x_*) \in v(F^{(1)})$. Assume that $x \in X$ and that

$$F^{(1)}(x) \le F^{(1)}(x_*) = F(x_*). \tag{9.90}$$

It follows from (9.90), (9.86) and (9.87) that

$$F(x_*) \ge F^{(1)}(x) = \psi(x)F(x_*) + (1 - \psi(x))F(x)). \tag{9.91}$$

If $\psi(x) = 1$, then $F^{(1)}(x) = F(x_*)$. If $\psi(x) < 1$, then (9.91) and (9.82) imply that $F(x_*) \ge F(x)$ and $F(x_*) = F(x)$ and it follows from (9.86) and (9.87) that $F^{(1)}(x) = F(x_*)$. Hence $F(x_*) \in v(F^{(1)})$. For $x \in X$ and $i \in \{1, \dots, n\}$ put

$$f_i^{(2)}(x) = f_i^{(1)}(x) + \min\{1, \max\{\rho(x, x_*) - \delta/2, 0\}\}(\epsilon/8), \tag{9.92}$$

$$F^{(2)} = (f_1^{(2)}, \dots, f_n^{(2)}).$$

It is easy to see that

$$F^{(2)} \in \mathcal{A} \text{ and } F^{(2)}(x) \ge F^{(1)}(x) \text{ for all } x \in X. \tag{9.93}$$

Relations (9.92) and (9.88) imply that

$$F^{(2)}(x) = F^{(1)}(x) = F(x_*) \text{ for all } x \in X \text{ satisfying } \rho(x, x_*) \le \delta/2. \tag{9.94}$$

It follows from (9.92) and (9.88) that for each $x \in X$ satisfying $\rho(x, x_*) \in [\delta/2, \delta]$

$$F^{(2)}(x) = F^{(1)}(x_*) + (\epsilon/8)[\rho(x, x_*) - \delta/2](1, 1, \dots, 1). \tag{9.95}$$

In view of (9.92),

$$\tilde{d}(F^{(2)}, F^{(1)}) \le \epsilon/8.$$

By inequality and (9.89),

$$\tilde{d}(F, F^{(2)}) \le \epsilon/4. \tag{9.96}$$

Evidently, the inclusion $F(x_*) \in v(F^{(1)})$, (9.93) and (9.94) imply that

$$F(x_*) \in v(F^{(2)}). \tag{9.97}$$

We will show that

$$\text{if } x \in X \text{ satisfies } F^{(2)}(x) = F(x_*), \text{ then } \rho(x, x_*) \le \delta/2. \tag{9.98}$$

Assume that

$$x \in X \text{ and } F^{(2)}(x) = F(x_*). \tag{9.99}$$

It follows from (9.93) and (9.99) that $F^{(1)}(x) \le F^{(2)}(x) \le F(x_*)$. Combined with the inclusion $F(x_*) \in v(F^{(1)})$ the inequality above implies that $F^{(1)}(x) = F(x_*) \ge F^{(2)}(x)$. Together with (9.92) this relation implies that

$$\rho(x, x_*) \le \delta/2.$$

Hence (9.98) is proved.

Since $F(x_*) \in v(F^{(1)})$ it follows from (9.92) that the following property holds:

(P1) For each $x \in X$ satisfying $\rho(x, x_*) > \delta/2$ there exists an integer $i \in \{1, \dots, n\}$ such that

$$f_i^{(1)}(x) \ge f_i^{(1)}(x_*) = f_i(x_*)$$

and that

$$\begin{aligned} f_i^{(2)}(x) &= f_i^{(1)}(x) + (\epsilon/8)\min\{1, \rho(x, x_*) - \delta/2\} \\ &\ge f_i^{(1)}(x_*) + (\epsilon/8)\min\{1, \rho(x, x_*) - \delta/2\} \\ &= f_i(x_*) + (\epsilon/8)\min\{1, \rho(x, x_*) - \delta/2\}. \end{aligned}$$

Fix

$$\delta_0 \in (0, \delta/8) \text{ and } \lambda_0 \in (0, \epsilon/16). \tag{9.100}$$

Define functions $\phi_1, \phi_2 : [0, \infty) \to R^1$ as follows:

$$\phi_1(x) = x,\ x \in [0, 1],\ \phi_1(x) = 1,\ x \in (1, 2], \tag{9.101}$$

$$\phi_1(x) = x - 1,\ x \in (2, 8],\ \phi_1(x) = 15 - x,\ x \in (8, 14],$$

$$\phi_1(x) = 1,\ x \in (14, 15],\ \phi_1(x) = 16 - x,\ x \in (15, 16],\ \phi_1(x) = 0,\ x \in (16, \infty),$$

$$\phi_2(x) = -x,\ x \in [0, 2],\ \phi_2(x) = x - 4,\ x \in (2, 8], \tag{9.102}$$

$$\phi_2(x) = 12 - x,\ x \in (8, 14],\ \phi_2(x) = -16 + x,\ x \in (14, 16],$$

$$\phi_2(x) = 0,\ x \in (16, \infty).$$

Evidently, ϕ_1, ϕ_2 are continuous functions and

$$\sup\{|\phi_i(x)| :\ x \in R^1 \text{ and } i = 1, 2\} \le 8. \tag{9.103}$$

Define a function $G = (g_1, \dots, g_n) : X \to R^n$ as follows:

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \phi_1(16\rho(x, x_*)\delta_0^{-1}),\ x \in X; \tag{9.104}$$

for $i \in \{2, \dots, n\}$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0 \phi_2(16\rho(x, x_*)\delta_0^{-1}), \ x \in X. \tag{9.105}$$

It is easy to see that $G \in \mathcal{A}$. It follows from (9.105), (9.104), (9.103) and (9.100) that $\tilde{d}(G, F^{(2)}) \le \epsilon/2$. Combined with (9.96) this implies that

$$\tilde{d}(G, F) < \epsilon. \tag{9.106}$$

By (9.104), (9.105) and (9.102), for each $x \in X$ satisfying $\rho(x, x_*) \ge \delta_0$,

$$G(x) = F^{(2)}(x). \tag{9.107}$$

It follows from (9.107) and (9.94) that for each $x \in X$ satisfying $\rho(x, x_*) \in [\delta_0, \delta/2]$ we have

$$G(x) = F(x_*). \tag{9.108}$$

By (9.107) and (9.95) for each $x \in X$ satisfying $\rho(x, x_*) \in [\delta/2, \delta]$,

$$G(x) = F(x_*) + (\epsilon/8)[\rho(x, x_*) - \delta/2](1, 1, \dots, 1). \tag{9.109}$$

Relations (9.104), (9.105), (9.94), (9.101) and (9.102) imply that for each $x \in X$ satisfying $\rho(x, x_*) \le \delta_0/16$ we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \rho(x, x_*)\delta_0^{-1}16 = f_1(x_*) + \lambda_0 \rho(x, x_*)\delta_0^{-}16;$$

for $i = 2, \dots, n$

$$g_i(x) = f_i^{(2)}(x_*) - \lambda_0 \rho(x, x_*)\delta_0^{-1}16 = f_i(x_*) - \lambda_0 \rho(x, x_*)\delta_0^{-1}16$$

and

$$G(x) = F(x_*) + \lambda_0 \rho(x, x_*)\delta_0^{-1}16(1, -1, \dots, -1). \tag{9.110}$$

In view of (9.101), (9.104), (9.105), (9.94) and (9.102), for each $x \in X$ satisfying $\rho(x, x_*) = \delta_0/8$ we have

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \phi_1(2) = f_1(x_*) + \lambda_0;$$

for $i = 2, \dots, n$

$$g_i(x) = f_i^{(2)}(x) + \lambda_0 \phi_2(2) = f_i(x_*) - 2\lambda_0$$

and

$$G(x) = F(x_*) + \lambda_0(1, -2, -2, \dots, -2). \tag{9.111}$$

We will show that for each $x \in X$ satisfying $\rho(x, x_*) \le \delta_0$ the following property holds:

(P2) There exists $z \in X$ such that $\rho(z, x_*) \in [0, \delta_0/16] \cup \{\delta_0/8\}$ and $G(z) \le G(x)$.

Let $x \in X$ satisfy

$$\rho(x, x_*) \le \delta_0. \tag{9.112}$$

It is easy to see that if $\rho(x, x_*) \le \delta_0/16$, then (P2) holds with $z = x$. We consider the following cases:

$$\rho(x, x_*) \in [\delta_0/16, \delta_0/8]; \tag{9.113}$$

$$\rho(x, x_*) \in (\delta_0/8, \delta_0/2]; \tag{9.114}$$

$$\rho(x, x_*) \in (\delta_0/2, (7/8)\delta_0]; \tag{9.115}$$

$$\rho(x, x_*) \in ((7/8)\delta_0, (15/16)\delta_0]; \tag{9.116}$$

$$\rho(x, x_*) \in ((15/16)\delta_0, \delta_0]. \tag{9.117}$$

Let (9.113) hold. Then (9.113), (9.104), (9.105), (9.101), (9.102) and (9.94) imply that

$$g_1(x) = f_1^{(2)}(x) + \lambda_0 \phi_1(\rho(x, x_*)\delta_0^{-1}16) = f_1(x_*) + \lambda_0;$$

for $i = 2, \dots, n$

$$\begin{aligned} g_i(x) &= f_i^{(2)}(x) + \lambda_0 \phi_2(\rho(x, x_*)\delta_0^{-1}16) \\ &= f_i(x_*) - \lambda_0(\rho(x, x_*)\delta_0^{-1}16) \ge f_i(x_*) - 2\lambda_0. \end{aligned}$$

Combined with (9.111) these relations imply that

$$G(x) \ge G(z) \text{ if } z \in X \text{ satisfies } \rho(z, x_*) = \delta_0/8.$$

Thus property (P2) holds if (9.113) is valid.

Assume that (9.114) holds. It follows from (9.114), (9.104), (9.105), (9.101), (9.102) and (9.94) that

$$g_1(x) = f_1(x_*) + \lambda_0(\rho(x, x_*)\delta_0^{-1}16 - 1) \ge f_1(x_*) + \lambda_0;$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(\rho(x, x_*)\delta_0^{-1}16 - 4) \ge f_i(x_*) + \lambda_0(-2).$$

Combined with (9.111) these relations imply that for each $z \in X$ satisfying $\rho(z, x_*) = \delta_0/8$ we have $G(z) \le G(x)$. Thus property (P2) holds if (9.114) is valid.

Assume that (9.115) holds. It follows from (9.115), (9.104), (9.105), (9.101), (9.102) and (9.94) that

$$g_1(x) = f_1(x_*) + \lambda_0(15 - \rho(x, x_*)\delta_0^{-1}16) \ge f_1(x_*) + \lambda_0;$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(12 - \rho(x, x_*)\delta_0^{-1}16) \ge f_i(x_*) - 2\lambda_0.$$

Combined with (9.111) these relations imply that for each $z \in X$ satisfying $\rho(z, x_*) = \delta_0/8$ we have $G(z) \le G(x)$. Thus property (P2) holds if (9.115) is valid.

Assume that (9.116) holds. In view of (9.116), (9.104), (9.105), (9.101), (9.102) and (9.94) we have

$$g_1(x) = f_1(x_*) + \lambda_0;$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x) + \lambda_0(-16 + \rho(x, x_*)\delta_0^{-1}16) \geq f_i(x_*) - 2\lambda_0.$$

Combined with (9.111) these relations imply that for each $z \in X$ satisfying $\rho(z, x_*) = \delta_0/8$ we have $G(z) \leq G(x)$. Thus property (P2) holds if (9.116) is valid.

Assume that (9.117) holds. In view of (9.117), (9.104), (9.105), (9.101), (9.102) and (9.94) we have

$$g_1(x) = f_1(x_*) + \lambda_0(16 - \rho(x, x_*)\delta_0^{-1}16); \tag{9.118}$$

for $i = 2, \dots, n$

$$g_i(x) = f_i(x_*) + \lambda_0(\rho(x, x_*)\delta_0^{-1}16 - 16). \tag{9.119}$$

Since the space X is connected relations (9.84) and (9.117) imply that there is $z \in X$ for which

$$\rho(z, x_*) = \delta_0 - \rho(x, x_*) \in [0, \delta_0/16]. \tag{9.120}$$

It follows from (9.120), (9.110), (9.118) and (9.119) that

$$G(z) = F(x_*) + \lambda_0\rho(z, x_*)\delta_0^{-1}16(1, -1, \dots, -1) =$$

$$F(x_*) + \lambda_0(\delta_0 - \rho(x, x_*))\delta_0^{-1}16(1, -1, -1, \dots, -1) = G(x).$$

Thus property (P2) holds if (9.117) is valid.

We have shown that (P2) holds in all the cases. We have also shown that the following property holds:

(P3) For each $x \in X$ satisfying $\rho(x, x_*) \leq \delta_0$ there exists $z \in X$ such that

$$\rho(z, x_*) \in [0, \delta_0/16] \cup \{\delta_0/8\},\ G(z) \leq G(x).$$

We will show that the following property holds:

(P4) If $x \in X$ satisfies $\rho(x, x_*) \geq \delta/2$ and $z \in X$ satisfies $\rho(z, x_*) \in \{\delta_0/8\} \cup [0, \delta_0/16]$, then the inequality $G(x) \leq G(z)$ does not hold.

Assume that

$$x \in X,\ z \in X,\ \rho(x, x_*) > \delta/2,\ \rho(z, x_*) \in \{\delta_0/8\}[0, \delta_0/16]. \tag{9.121}$$

It follows from property (P1) and (9.121) that there exists $j \in \{1, \dots, n\}$ for which

$$f_j^{(2)}(x) \geq f_j(x_*) + (\epsilon/8)\min\{1, \rho(x, x_*) - \delta/2\}.$$

Relations (9.92) and (9.121) imply that

$$f_i^{(2)}(x) > f_i(x_*),\ i = 1, \dots, n.$$

Combined with (9.107) and (9.100) this implies that

$$g_i(x) > f_i(x_*),\ i = 1, \dots, n. \tag{9.122}$$

In view of (9.121),

$$G(z) \in \{F(x_*) + \lambda_0(1, -1, -1, \dots, -1)t :\ t \in [0, 1]\}$$

$$\cup\{F(x_*) + \lambda_0(1, -2, -2, \dots, -2)\}.$$

By this inclusion and (9.122), the inequality $G(x) \le G(z)$ does not hold. Therefore property (P4) holds.

Let $t \in [0, 1)$. We show that $F(x_*) + \lambda_0 t(1, -1, -1, \dots, -1) \in v(G)$. Since the space X is connected it follows from (9.84) that there is $z \in X$ such that

$$\rho(z, x_*) = (\delta_0/16)t. \tag{9.123}$$

Relations (9.123) and (9.110) imply that

$$G(z) = F(x_*) + \lambda_0\rho(z, x_*)\delta_0^{-1}16(1, -1 - 1, \dots, -1) \tag{9.124}$$

$$= F(x_*) + t\lambda_0(1, -1, -1, \dots, -1) \in G(X).$$

Assume that

$$x \in X \text{ and } G(x) \le G(z). \tag{9.125}$$

We will show that $G(x) = G(z)$. If $\rho(x, x_*) > \delta/2$, then it follows from (9.92), (9.107) and (9.100) that the relation (9.122) is true and combined with (9.125) this implies that $G(z) >> F(x_*)$. This contradicts (9.124). Therefore

$$\rho(x, x_*) \le \delta/2. \tag{9.126}$$

If $\rho(x, x_*) \in [\delta_0, \delta/2]$, then (9.108), (9.125), (9.124) and (9.123) imply that

$$G(x) = F(x_*),\ t = 0,\ z = x_*,\ G(z) = G(x). \tag{9.127}$$

Assume that

$$\rho(x, x_*) \le \delta_0. \tag{9.128}$$

In view of (9.128) and property (P3) there exists $y \in X$ such that

$$G(y) \le G(x),\ \rho(y, x_*) \in [0, \delta_0/16] \cup \{\delta_0/8\}.$$

It follows from the relations above, (9.125) and (9.124) that

$$G(y) \le G(x) \le G(z) = F(x_*) + \lambda_0 t(1, -1, -1, \dots, -1). \tag{9.129}$$

If $\rho(y, x_*) \in \{\delta_0/8\}$, then in view of (9.111)

$$G(y) = F(x_*) + \lambda_0(1, -2, -2, \dots, -2)$$

and since $t \in [0, 1)$ the equality above contradicts (9.129). Therefore in view of the choice of y,

$$\rho(y, x_*) \in [0, \delta_0/16].$$

It follows from the inclusion above and (9.110) that

$$G(y) = F(x_*) + \lambda_0 \rho(x, x_*) \delta_0^{-1}(16)(1, -1, -1, \dots, -1).$$

Combined with (9.129) this equality implies that $t = \rho(x, x_*)\delta_0^{-1}16$ and $G(y) = G(x) = G(z)$. Thus we have shown that (9.125) implies that $G(x) = G(z)$. Therefore

$$F(x_*) + \lambda_0 t(1, -1, -1, \dots, -1) \in v(G) \text{ for all } t \in [0, 1). \tag{9.130}$$

Let $x \in X$ satisfy $\rho(x, x_*) = \delta_0/8$. (Note that by (9.84) such x exists.) It follows from (9.111) that

$$G(x) = F(x_*) + \lambda_0(1, -2, -2, \dots, -2) < F(x_*) + \lambda_0(1, -1, -1, \dots, -1).$$

Thus

$$F(x_*) + \lambda_0(1, -1, -1, \dots, -1) \notin v(G).$$

Combined with (9.130) this implies that $v(G)$ is not closed. This completes the proof of Theorem 9.3.

9.6 Vector optimization with continuous objective functions

We study a class of vector minimization problems on a complete metric space X without compactness assumptions. This class of problems is associated with a complete metric space of continuous bounded from below vector functions $\mathcal{A}$ which is defined below. Let $F \in \mathcal{A}$. An element $z \in X$ is a called a solution of the vector minimization problem $F(x) \to \min$, $x \in X$ if $F(z)$ is a minimal element of the image $F(X) = \{F(x) : \ x \in X\}$. The set of all solutions of the minimization problem above is denoted by K_F. We show that for most (in the sense of Baire category) functions $F \in \mathcal{A}$ the set K_F is nonempty and compact.

Let (X, ρ) be a complete metric space and n be a natural number. For each vector $x = (x_1, \dots, x_n)$ of the n-dimensional Euclidean space R^n set $||x|| = \max\{|x_i| : \ i = 1, \dots, n\}$. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in R^n$. We say that $x \le y$ if $x_i \le y_i$ for all $i = 1, \dots, n$. We say that $x < y$ if $x \le y$ and $x \ne y$, and say that $x << y$ if $x_i < y_i$ for all $i = 1, \dots, n$.

Let $D \subset R^n$ be a nonempty set. Recall that an element $x \in D$ is called a minimal element of D if there is no $y \in D$ such that $y < x$. A function $F : X \to R^n$ is called bounded from below if there is $a \in R^n$ such that $F(x) \geq a$ for all $x \in X$. Denote by $\mathcal{A}$ the set of all continuous functions $F = (f_1, \dots, f_n) : X \to R^n$ which are bounded from below. For each $F = (f_1, \dots, f_n)$, $G = (g_1, \dots, g_n) \in \mathcal{A}$ define

$$\tilde{d}(F, G) = \sup\{||F(x) - G(x)|| : \ x \in X\},$$
$$d(F, G) = \tilde{d}(F, G)(1 + \tilde{d}(F, G))^{-1}. \tag{9.131}$$

(Here we use the convention that $\infty/\infty = 1$.) It is not difficult to see that the metric space $(\mathcal{A}, d)$ is complete. For each $x \in X$ and each $r > 0$ set

$$B_X(x, r) = \{y \in X : \ \rho(x, y) \leq r\},$$
$$B_X^o(x, r) = \{y \in X : \ \rho(x, y) < r\}.$$

For each $x \in R^n$ and each $r > 0$ set

$$B_{R^n}(x, r) = \{y \in R^n : \ ||x - y|| \leq r\},$$
$$B_{R^n}^o(x, r) = \{y \in R^n : \ ||x - y|| < r\}.$$

Set $R_+^n = \{x = (x_1, \dots, x_n) \in R^n : \ x \geq 0\}$ and let $e = (1, 1, \dots, 1)$ be an element of R^n all of whose coordinates are unity. Denote by $\mathrm{cl}(E)$ the closure of a set $E \subset R^n$.

In this chapter we prove the following result obtained in [137].

Theorem 9.9. *There exists a set $\mathcal{F} \subset (\mathcal{A}, d)$ which is a countable intersection of open everywhere dense subsets of $(\mathcal{A}, d)$ such that for each $G \in \mathcal{F}$ the following assertions hold:*

1. There is a nonempty compact set $K_G \subset X$ such that $G(K_G)$ is the set of all minimal elements of cl$(G(X))$.

2. For each $\epsilon > 0$ there is $\delta > 0$ such that for each $H \in \mathcal{A}$ satisfying $\tilde{d}(H, G) \leq \delta$, each minimal element a of cl$(H(X))$ and each $z \in X$ satisfying $H(z) \leq a + \delta e$ there exists $x \in K_G$ such that $\rho(x, z) \leq \epsilon$, $||H(z) - G(x)|| \leq \epsilon$, $||a - G(x)|| \leq \epsilon$.

3. For each $\epsilon > 0$ there is $\delta > 0$ such that for each $x \in K_G$ and each $z \in X$ satisfying $G(z) \leq G(x) + \delta e$ the inequalities $\rho(z, x) \leq \epsilon$ and $||G(z) - G(x)|| \leq \epsilon$ hold.

The proof of Theorem 9.9 consists of three parts. First we show (see Lemma 9.12) that for each $F \in \mathcal{A}$ and each $\epsilon > 0$ there exists $G \in \mathcal{A}$ such that $\tilde{d}(F, G) \leq \epsilon$ and the set of minimal elements of $\mathrm{cl}(G(X))$ is finite. Then we show (see Lemma 9.13) that for each $F \in \mathcal{A}$ with a finite set of minimal elements of $\mathrm{cl}(F(X))$ and each $\gamma > 0$ there exists $F_\gamma \in \mathcal{A}$ such that $\tilde{d}(F, F_\gamma) \leq \epsilon$, the set K_{F_γ} is finite and if G is close enough to F_γ in $(\mathcal{A}, d)$ and z is an approximate solution of the problem $G(x) \to \min$, $x \in X$, then z belongs to a small neighborhood of K_{F_γ}. Finally using Lemmas 9.12 and 9.13 we construct the set $\mathcal{F}$ and complete the proof of Theorem 9.9.

9.7 Preliminaries

Zorn's lemma implies the following result.

Proposition 9.10. *Let $E \subset R^n$ be a nonempty closed subset of R^n. Assume that there is $z_0 \in R^n$ such that $z_0 \le y$ for all $y \in E$. Then for each $z \in E$ there is a minimal element x of E such that $x \le z$.*

Proposition 9.11. *Let Ω be a nonempty closed subset of R^n, let $z_* \in R^n$ satisfy $z_* \le z$ for all $z \in \Omega$ and let ϵ be a positive number. Then there exists $M > 0$ such that for each $x \in \Omega$ there is $y \in \Omega$ satisfying $||y|| \le M$ and $y \le x + \epsilon e$.*

Proof: Let us assume the contrary. Then for each natural number q there exists $x^{(q)} = (x_1^{(q)}, \dots, x_n^{(q)}) \in \Omega$ such that

$$\Omega \cap ((x^{(q)} + \epsilon e) - R^n_+) \cap B_{R^n}(0, q) = \emptyset. \tag{9.132}$$

Clearly we may assume without loss of generality that $n > 1$. It is easy to see that for all natural numbers q

$$x^{(q)} \ge z_*, \ ||x^{(q)}|| \ge q. \tag{9.133}$$

Extracting a subsequence and reindexing we may assume without loss of generality that there exist subsets $I_1, I_2 \subset \{1, \dots, n\}$ such that

$$I_1 \cup I_2 = \{1, \dots, n\}, \ I_1 \cap I_2 = \emptyset,$$

$$\{x_i^{(q)}\}_{q=1}^{\infty} \text{ is a bounded sequence for all } i \in I_1, \ \lim_{q \to \infty} x_i^{(q)} = \infty \text{ for all } i \in I_2. \tag{9.134}$$

In view of (9.133), $I_2 \neq \emptyset$. If $I_1 = \emptyset$, then $\lim_{q\to\infty} x_i^{(q)} = \infty$ for all $i \in \{1, \dots, n\}$ and $x^{(q)} \ge z_0$ for all sufficiently large natural numbers q where $z_0 \in \Omega$. This contradicts (9.132). Therefore $I_1 \neq \emptyset$. Clearly the set $\{x_i^{(q)} : i \in I_1, \ q \text{ is a natural number}\}$ is bounded. There exists a natural number q_0 such that the following property holds:

(P1) For each integer $q \ge 1$ there is $j \in \{1, \dots, q_0\}$ such that

$$|x_i^{(q)} - x_i^{(j)}| \le \epsilon/4 \text{ for all } i \in I_1. \tag{9.135}$$

By (9.134) there exists a natural number $q_1 > q_0$ such that

$$x_i^{(q)} > x_i^{(j)} \text{ for each } i \in I_2, \text{ each integer}$$

$$q \ge q_1 \text{ and each positive integer } j \le q_0. \tag{9.136}$$

We show that the following property holds:

(P2) For each integer $q \geq 1$ there is $j \in \{1, \dots, q_1\}$ such that $x^{(j)} \leq x^{(q)} + \epsilon e$.

Let an integer $q \geq 1$. Clearly we may show that (P2) holds only in the case $q > q_1$. By property (P1) there is $j \in \{1, \dots, q_0\}$ such that (9.135) holds. Since $q > q_1$ and $j \leq q_0$ it follows from the choice of q_1 (see (9.136)) that $x_i^{(q)} > x_i^{(j)}$ for all $i \in I_2$. Combined with (9.135) this inequality implies that $x^{(j)} \leq x^{(q)} + \epsilon e$. Thus we have shown that property (P2) holds. Now it is not difficult to see that property (P2) contradicts (9.132). The contradiction we have reached proves Proposition 9.11.

9.8 Auxiliary results

Lemma 9.12. *Let $F = (f_1, \dots, f_n) \in \mathcal{A}$ and $\epsilon > 0$. Then there exist $G = (g_1, \dots, g_n) \in \mathcal{A}$, $x_i \in X$, $i = 1, \dots, p$ where p is a natural number such that $\tilde{d}(F, G) \leq \epsilon$ and the following property holds:*

For each $x \in X$ there is $i \in \{1, \dots, p\}$ for which $G(x) \geq G(x_i)$.

Proof: Since $F \in \mathcal{A}$ there is $a \in R^n$ such that

$$a \leq y \text{ for all } y \in F(X). \tag{9.137}$$

Denote by M_0 the set of all minimal elements of the set $\text{cl}(F(X))$. By (9.137) and Proposition 9.10 the set M_0 is nonempty. Choose positive constants

$$\gamma < \min\{1, \epsilon/8\}, \ \delta_0 < \gamma. \tag{9.138}$$

By Proposition 9.11 and (9.137) there is $c_0 > 0$ such that the following property holds:

For each $x \in \text{cl}(F(X))$ there exists $y \in \text{cl}(F(X))$ such that $||y|| \leq c_0$, $y \leq x + (\delta_0/64)e$.

This property implies that there are a natural number p and $y_1, \dots, y_p \in \text{cl}(F(X))$ such that the following property holds:

For each $x \in \text{cl}(F(X))$ there exists $i \in \{1, \dots, p\}$ such that $y_i \leq x + (\delta_0/32)e$.

Clearly, for each $i \in \{1, \dots, p\}$ there is $x_i \in X$ such that $||F(x_i) - y_i|| \leq \delta_0/32$. Now it is not difficult to see that the following property holds:

(P3) For each $x \in \text{cl}(F(X))$ there is $i \in \{1, \dots, p\}$ such that $F(x_i) \leq x + (\delta_0/16)e$.

We may assume without loss of generality that

$$Fx_i \neq Fx_j \text{ for each } i, j \in \{1, \dots, p\} \text{ satisfying } i \neq j. \tag{9.139}$$

Relation (9.139) implies that there exists a number ϵ_0 such that

$$\epsilon_0 \in (0, \epsilon/4), \tag{9.140}$$

$$||F(x_i) - F(x_j)|| \geq 16\epsilon_0 \text{ for each } i, j \in \{1, \dots, p\} \text{ satisfying } i \neq j. \quad (9.141)$$

By (9.139) and the continuity of F there exists a number δ_1 such that

$$\delta_1 \in (0, \delta_0/4), \quad (9.142)$$

$$\rho(x_i, x_j) \geq 16\delta_1 \text{ for each } i, j \in \{1, \dots, p\} \text{ satisfying } i \neq j, \quad (9.143)$$

$$||F(x_i) - F(z)|| \leq \gamma/8 \text{ for all } z \in B_X(x_i, \delta_1) \quad (9.144)$$

and all $i = 1, \dots, p$. Choose

$$\delta_2 \in (0, \delta_1) \quad (9.145)$$

and define a continuous function $\phi : [0, \infty) \to [0, 1]$ as follows:

$$\phi(t) = 0,\ t \in [\delta_1, \infty),\ \phi(t) = 1,\ t \in [0, \delta_2],$$

$$\phi(t) = (\delta_1 - t)(\delta_1 - \delta_2)^{-1},\ t \in (\delta_2, \delta_1). \quad (9.146)$$

Define a function $G : X \to R^n$ as follows:

$$G(z) = F(z),\ z \in X \setminus \cup_{i=1}^{p} B_X(x_i, 2\delta_1), \quad (9.147)$$

$$G(z) = (1 - \phi(\rho(z, x_i)))F(z) + \phi(\rho(z, x_i))[F(x_i) + (\rho(z, x_i) - \gamma)e],$$

$$z \in B_X(x_i, 2\delta_1),\ i = 1, \dots, p. \quad (9.148)$$

In view of (9.143) G is well defined. It follows from (9.143), (9.146)–(9.148) that $G \in \mathcal{A}$. By (9.138), (9.142), (9.144) and (9.146)–(9.148),

$$\tilde{d}(F, G) = \sup\{||F(z) - G(z)|| :\ z \in \cup_{i=1}^{p} B_X(x_i, \delta_1)\}$$

$$\leq \sup_{i=1,\dots,p} \sup\{||F(x_i) + (\rho(z, x_i) - \gamma)e - F(z)|| :\ z \in B_X(x_i, \delta_1)\}$$

$$\leq \sup_{i=1,\dots,p} \sup\{\gamma + \delta_1 + ||F(x_i) - F(z)|| :\ z \in B_X(x_i, \delta_1)\} \leq \gamma + \delta_1 + \gamma/8 < \epsilon. \quad (9.149)$$

Assume that $x \in X$. We show that there is $j \in \{1, \dots, p\}$ such that $G(x_j) \leq G(x)$. There are two cases:

$$x \in X \setminus \cup_{i=1}^{p} B_X(x_i, \delta_1), \quad (9.150)$$

$$x \in \cup_{i=1}^{p} B_X(x_i, \delta_1). \quad (9.151)$$

Assume that (9.150) is valid. By (9.150) and (9.146)–(9.148)

$$G(x) = F(x). \quad (9.152)$$

In view of (9.152), (P3), (9.148) and (9.138) there exists $j \in \{1, \dots, p\}$ such that

$$G(x) \geq Fx_j - (\delta_0/16)e = G(x_j) + (\gamma - \delta_0/16)e > G(x_j).$$

Therefore if (9.150) is valid, then there is $j \in \{1, \dots, p\}$ such that $G(x) > Gx_j$.

Now assume that (9.151) is true. Then there is $j \in \{1, \dots, p\}$ such that

$$\rho(x, x_j) \le \delta_1. \tag{9.153}$$

It follows from (9.146), (9.148), (9.153) and the choice of δ_1 (see (9.144)) that

$$G(x) - G(x_j) = (1 - \phi(\rho(x, x_j)))F(x) + \phi(\rho(x, x_j))[F(x_j) + (\rho(x, x_j) - \gamma)e]$$

$$-[F(x_j) - \gamma e] = \phi(\rho(x, x_j))\rho(x, x_j)e + (1 - \phi(\rho(x, x_j)))[F(x) - F(x_j) + \gamma e]$$

$$\ge (1 - \phi(\rho(x, x_j)))[F(x) - F(x_j) + \gamma e] \ge (1 - \phi(\rho(x, x_j)))[(-\gamma/8)e + \gamma e] \ge 0.$$

Thus we have shown that if (9.151) is true, then there is $j \in \{1, \dots, p\}$ such that $G(x) \ge G(x_j)$. This completes the proof of the lemma.

Lemma 9.13. *Assume that $F = (f_1, \dots, f_n) \in \mathcal{A}$, p is a natural number and that $x_1, \dots, x_p \in X$ satisfy the following conditions:*

(C1) For each $x \in X$ there is $i \in \{1, \dots, p\}$ such that $F(x) \ge F(x_i)$.

(C2) If $i, j \in \{1, \dots, p\}$ and $F(x_i) \le F(x_j)$, then $i = j$.

Let $\gamma \in (0, 1)$ and let

$$F_\gamma(x) := F(x) + \gamma \min\{1, \rho(x, x_i) : \ i = 1, \dots, p\}e, \ x \in X. \tag{9.154}$$

Then $F_\gamma \in \mathcal{A}$, $\tilde{d}(F_\gamma, F) \le \gamma$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that the following assertion holds:

If $G \in \mathcal{A}$ satisfies $\tilde{d}(F_\gamma, G) \le \delta$, a is a minimal element of $\mathrm{cl}(G(X))$ and $x \in X$ satisfies $G(x) \le a + \delta e$, then there is $j \in \{1, \dots, p\}$ such that $\rho(x, x_j) \le \epsilon$, $||G(x) - F(x_j)|| \le \epsilon$, $||a - F(x_j)|| \le \epsilon$.

Proof: Clearly $F_\gamma \in \mathcal{A}$ and $\tilde{d}(F, F_\gamma) \le \gamma$. Let $\epsilon > 0$. In view of (C2) we may assume without loss of generality that $\epsilon < 1$ and that the following condition holds:

(C3) For each $i, j \in \{1, \dots, p\}$ satisfying $i \ne j$ the relations

$$B_X(x_i, 4\epsilon) \cap B_X(x_j, 4\epsilon) = \emptyset, \ ||F(x_i) - F(x_j)|| \ge 8\epsilon$$

hold and there is $s(i, j) \in \{1, \dots, n\}$ such that $f_{s(i,j)}(x_i) < f_{s(i,j)}(x_j) - 8\epsilon$.

There exists

$$\delta_1 \in (0, \epsilon/8) \tag{9.155}$$

such that for each $i \in \{1, \dots, p\}$

$$||F(x) - F(x_i)|| \le \epsilon/2 \text{ for each } x \in B_X(x_i, 2\delta_1). \tag{9.156}$$

Choose a positive number

$$\epsilon_1 < \gamma\delta_1/2. \tag{9.157}$$

There exists a positive number

$$\delta_0 < \min\{\delta_1, \epsilon_1\}/4 \tag{9.158}$$

such that for each $i \in \{1, \dots, p\}$

$$||F(x) - F(x_i)|| \le \epsilon_1 \text{ for each } x \in B_X(x_i, 2\delta_0). \tag{9.159}$$

Choose a positive number

$$\delta < (\gamma\delta_0)/8. \tag{9.160}$$

Let $G \in \mathcal{A}$ satisfy

$$\tilde{d}(F_\gamma, G) \le \delta, \tag{9.161}$$

a be a minimal element of $\mathrm{cl}(G(X))$ and let $x \in X$ satisfy

$$G(x) \le a + \delta e. \tag{9.162}$$

There exists a sequence $\{y_i\}_{i=1}^{\infty} \subset X$ such that

$$a = \lim_{i \to \infty} G(y_i). \tag{9.163}$$

We show that for all large enough natural numbers i

$$y_i \in \cup_{j=1}^{p} B_X(x_j, \delta_0). \tag{9.164}$$

Let us assume the contrary. Then extracting a subsequence and reindexing we may assume without loss of generality that

$$y_i \in X \setminus (\cup_{j=1}^{p} B_X(x_j, \delta_0)) \text{ for all natural numbers } i. \tag{9.165}$$

(C1) implies that for each natural number i there is $q(i) \in \{1, \dots, p\}$ such that

$$F(y_i) \ge F(x_{q(i)}). \tag{9.166}$$

Extracting a subsequence and reindexing we may assume that

$$q(i) = q(1) \text{ for all natural numbers } i. \tag{9.167}$$

It follows from (9.154), (9.155), (9.158), (9.160), (9.161) and (9.165)–(9.167) that for each natural number i

$$G(y_i) \ge F_\gamma(y_i) - \delta e = F(y_i) + \gamma \min\{1, \rho(x_j, y_i) :\ j = 1, \dots, p\}e - \delta e$$

$$\ge F(y_i) + \gamma\delta_0 e - \delta e \ge F(x_{q(1)}) + (\gamma\delta_0 - \delta)e \ge F_\gamma(x_{q(1)}) + (\gamma\delta_0 - \delta)e$$

$$\ge G(x_{q(1)}) - \delta e + (\gamma\delta_0 - \delta)e = G(x_{q(1)}) + (\gamma\delta_0 - 2\delta)e \ge G(x_{q(1)}) + (\gamma\delta_0/2)e.$$

Combined with (9.163) this relation implies that

$$a = \lim_{i \to \infty} G(y_i) \ge G(x_{q(1)}) + (\gamma\delta_0/2)e.$$

Since a is a minimal element of $\mathrm{cl}(G(X))$ we obtained a contradiction. The contradiction we have reached proves that (9.164) is true for all sufficiently large natural numbers i. Extracting a subsequence and reindexing we may assume without loss of generality that there is $q \in \{1, \dots, p\}$ such that

$$\rho(y_i, x_q) \le \delta_0 \text{ for all integers } i \ge 1. \tag{9.168}$$

Relations (9.168) and (9.159) imply that

$$||F(y_i) - F(x_q)|| \le \epsilon_1 \text{ for all integers } i \ge 1. \tag{9.169}$$

By (9.154), (9.161) and (9.169) for each natural number i

$$G(y_i) \le F_\gamma(y_i) + \tilde{d}(G, F_\gamma)e \le F_\gamma(y_i) + \delta e$$

$$= F(y_i) + \gamma \min\{1, \rho(y_i, x_j) : \ j = 1, \dots, p\}e + \delta e$$

$$\le F(x_q) + (\epsilon_1 + \delta)e + \gamma \min\{1, \rho(y_i, x_j) : \ j = 1, \dots, p\}e. \tag{9.170}$$

In view of (9.168), (C3), (9.158) and (9.155) for each natural number i

$$\rho(y_i, x_q) = \min\{1, \rho(y_i, x_j) : \ j = 1, \dots, p\}. \tag{9.171}$$

It follows from (9.161), (9.170) and (9.171) that for each natural number i

$$F_\gamma(y_i) - \delta e \le G(y_i) \le F(x_q) + (\epsilon_1 + \delta + \gamma\rho(y_i, x_q))e. \tag{9.172}$$

Relations (9.171), (9.154) and (9.172) imply that for each natural number i

$$F(y_i) + \gamma\rho(y_i, x_q)e = F(y_i) + \gamma \min\{1, \rho(x_j, y_i) : \ j = 1, \dots, p\}e$$

$$= F_\gamma(y_i) \le F(x_q) + (\epsilon_1 + 2\delta + \gamma\rho(y_i, x_q))e$$

and

$$F(y_i) \le F(x_q) + (\epsilon_1 + 2\delta)e. \tag{9.173}$$

Let $i \ge 1$ be an integer. By (C1) there exists $j \in \{1, \dots, p\}$ such that $F(x_j) \le F(y_i)$. Combined with (C3), (9.173), (9.160), (9.158), (9.157) and (9.155) this inequality implies that $j = q$ and

$$F(x_q) \le F(y_i) \text{ for all natural numbers } i. \tag{9.174}$$

Relations (9.163), (9.170) and (9.168) imply that

$$a = \lim_{i \to \infty} G(y_i) \le F(x_q) + (\epsilon_1 + \delta)e + \gamma\delta_0 e. \tag{9.175}$$

It follows from (9.154), (9.161), (9.162) and (9.165) that

$$F(x) + \gamma \min\{1, \rho(x, x_i) : \ i = 1, \dots, p\}e = F_\gamma(x) \le G(x) + \delta e$$

$$\le a + 2\delta e \le F(x_q) + (\epsilon_1 + 3\delta + \gamma\delta_0)e. \tag{9.176}$$

By (C1) there is $m \in \{1, \dots, p\}$ such that

$$F(x_m) \le F(x). \tag{9.177}$$

In view of this inequality and (9.176), (9.160), (9.158) and (9.155)

$$F(x_m) \le F(x_m) + \gamma \min\{1, \rho(x, x_i) : \ i = 1, \dots, p\}e$$

$$\le F(x_q) + (\epsilon_1 + 3\delta + \gamma\delta_0)e \le F(x_q) + \epsilon. \tag{9.178}$$

Combined with (C3) and (9.177) this inequality implies that $m = q$ and

$$F(x) \ge F(x_q). \tag{9.179}$$

It follows from (9.178), the equality $m = q$, (9.160) and (9.158) that

$$F(x_q) + \gamma \min\{1, \rho(x, x_i) : \ i = 1, \dots, p\}e$$

$$\le F(x_q) + (\epsilon_1 + 3\delta + \gamma\delta_0)e \le F(x_q) + 2\epsilon_1 e,$$

$$\min\{1, \rho(x, x_i) : \ i = 1, \dots, p\} \le 2\epsilon_1\gamma^{-1}.$$

Together with (9.157) and (9.155) this inequality implies that there is $l \in \{1, \dots, p\}$ such that

$$\rho(x, x_l) \le 2\epsilon_1\gamma^{-1} \le \delta_1. \tag{9.180}$$

By (9.180) and the choice of δ_1 (see (9.156))

$$||F(x) - F(x_l)|| \le \epsilon/2. \tag{9.181}$$

In view of (9.179) and (9.181) $F(x_q) \le F(x) \le F(x_l) + 2^{-1}\epsilon e$. Combined with (C3) this inequality implies that $l = q$. Together with (9.180) and (9.155) this equality implies that

$$\rho(x, x_q) \le \delta_1 < \epsilon. \tag{9.182}$$

In view of (9.154), (9.161), (9.181), equality $l = q$, (9.182), (9.160), (9.158) and (9.155)

$$||G(x) - F(x_q)|| = ||G(x) - F_\gamma(x_q)|| \le ||G(x) - F_\gamma(x)|| + ||F_\gamma(x) - F_\gamma(x_q)||$$

$$\le \delta + ||F_\gamma(x) - F(x)|| + ||F(x) - F(x_q)|| \le \delta + \gamma\rho(x, x_q) + \epsilon/2 \le \delta + \gamma\delta_1 + \epsilon/2 < \epsilon. \tag{9.183}$$

It follows from (9.179), (9.161), (9.162) and (9.175) that

$$F(x_q) - 2\delta e \le F(x) - 2\delta e \le G(x) - \delta e \le a \le F(x_q) + (\epsilon_1 + \delta + \gamma\delta_0)e.$$

Together with (9.160), (9.158), (9.157) and (9.155) this inequality implies that $||a - F(x_q)|| \le \epsilon$. This completes the proof of the lemma.

9.9 Proof of Theorem 9.9

Denote by E_0 the set of all $F = (f_1, \dots, f_n) \in \mathcal{A}$ for which the following property holds:

(P4) There exist a natural number p_F and $x_{F,i} \in X$, $i = 1, \dots, p_F$ such that: if $i, j \in \{1, \dots, p_F\}$ and $F(x_{F,i}) \leq F(x_{F,j})$, then $i = j$; for each $x \in X$ there is $j \in \{1, \dots, p_F\}$ such that $F(x_{F,j}) \leq F(x)$.

By Lemma 9.12 E_0 is an everywhere dense subset of $(\mathcal{A}, d)$. Let $F = (f_1, \dots, f_n) \in E_0$, $\gamma \in (0,1)$ and let

$$F_\gamma(x) := F(x) + \gamma \min\{1, \rho(x, x_{F,i}) : \ i = 1, \dots, p_F\}e, \ x \in X. \tag{9.184}$$

Lemma 9.13 implies that $F_\gamma \in \mathcal{A}$ and

$$\tilde{d}(F, F_\gamma) \leq \gamma. \tag{9.185}$$

In view of property (P4) there exists $\delta_F \in (0,1)$ such that the following condition holds:

(C4) For each $i, j \in \{1, \dots, p_F\}$ satisfying $i \neq j$ we have $\rho(x_{F,i}, x_{F,j}) \geq 8\delta_F$ and there is $s(i,j) \in \{1, \dots, n\}$ for which

$$f_{s(i,j)}(x_{F,i}) < f_{s(i,j)}(x_{F,j}) - 8\delta_F.$$

Let $q \geq 1$ be an integer. By Lemma 9.13 there exists

$$\delta(F, \gamma, q) \in (0, (4q)^{-1}\delta_F) \tag{9.186}$$

such that the following property holds:

(P5) If $G \in \mathcal{A}$ satisfies $\tilde{d}(F_\gamma, G) < \delta(F, \gamma, q)$, a is a minimal element of $\mathrm{cl}(G(X))$ and $x \in X$ satisfies $G(x) \leq a + \delta(F, \gamma, q)e$, then there is $j \in \{1, \dots, p_F\}$ such that

$$\max\{\rho(x, x_{F,j}), \ ||G(x) - F_\gamma(x_{F,j})||, \ ||a - F_\gamma(x_{F,j})||\} \leq \delta_F(4q)^{-1}.$$

Define

$$\mathcal{F} = \cap_{q=1}^{\infty} \cup \{\{G \in \mathcal{A} : \ \tilde{d}(F_\gamma, G) < \delta(F, \gamma, q)\} : \ F \in E_0, \ \gamma \in (0,1)\}. \tag{9.187}$$

In view of (9.187) and (9.185) $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\mathcal{A}, d)$. Let

$$G \in \mathcal{F}. \tag{9.188}$$

Relations (9.188) and (9.187) imply that for each natural number q there exist

$$\gamma_q \in (0,1), \ F^{(q)} = (f_1^{(q)}, \dots, f_n^{(q)}) \in E_0 \tag{9.189}$$

such that

$$\tilde{d}((F^{(q)})_{\gamma_q}, G) < \delta(F^{(q)}, \gamma_q, q). \tag{9.190}$$

For each natural number q set

$$\mathcal{U}_q = \{H \in \mathcal{A} : \ \tilde{d}(H, (F^{(q)})_{\gamma_q}) < \delta(F^{(q)}, \gamma_q, q)\}. \tag{9.191}$$

By (P5), (9.189) and (9.191) the following property holds:

(P6) If q is a natural number, $H \in \mathcal{U}_q$, a is a minimal element of $\mathrm{cl}(H(X))$ and $x \in X$ satisfies $H(x) \le a + \delta(F^{(q)}, \gamma_q, q)e$, then there is $j \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\max\{\rho(x, x_{F^{(q)},j}), ||H(x) - (F^{(q)})_{\gamma_q}(x_{F^{(q)},j})||, ||a - (F^{(q)})_{\gamma_q}(x_{F^{(q)},j})||\}$$
$$\le \delta_{F^{(q)}}(4q)^{-1}.$$

Proposition 9.10 implies that the set of minimal elements of $\mathrm{cl}(G(X))$ is nonempty.

Assume that

$$a \text{ is a minimal element of } \mathrm{cl}(G(X)), \ \{y_i\}_{i=1}^{\infty} \subset X, \ \lim_{i \to \infty} G(y_i) = a. \tag{9.192}$$

Let $q \ge 1$ be an integer. It follows from (9.192) that there is a natural number i_0 such that for each integer $i \ge i_0$

$$G(y_i) \le a + \delta(F^{(q)}, \gamma_q, q)e. \tag{9.193}$$

Combined with property (P6) and (9.190)–(9.192) the inequality (9.193) implies that for each integer $i \ge i_0$ there is $k(i) \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\rho(y_i, x_{F^{(q)},k(i)}) \le \delta_{F^{(q)}}(4q)^{-1}, \ ||G(y_i) - (F^{(q)})_{\gamma_q}(x_{F^{(q)},k(i)})|| \le \delta_{F^{(q)}}(4q)^{-1},$$
$$||a - (F^{(q)})_{\gamma_q}(x_{F^{(q)},k(i)})|| \le \delta_{F^{(q)}}(4q)^{-1}. \tag{9.194}$$

In view of (9.194), (9.184) and condition (C4)

$$k(i) = k(i_0) \text{ for all integers } i \ge i_0. \tag{9.195}$$

Set

$$k = k(i_0). \tag{9.196}$$

Relations (9.194)–(9.196) imply that for all integers $i \ge i_0$

$$\rho(y_i, x_{F^{(q)},k}) \le \delta_{F^{(q)}}(4q)^{-1}, \ ||G(y_i) - (F^{(q)})_{\gamma_q}(x_{F^{(q)},k})|| \le \delta_{F^{(q)}}(4q)^{-1},$$
$$||a - (F^{(q)})_{\gamma_q}(x_{F^{(q)},k})|| \le \delta_{F^{(q)}}(4q)^{-1}. \tag{9.197}$$

Since q is any natural number it follows from (9.197) that $\{y_i\}_{i=1}^{\infty}$ is a Cauchy sequence. Therefore there exists

$$y = \lim_{i \to \infty} y_i. \tag{9.198}$$

By (9.198), (9.197), (9.192) and (9.186)

$$\rho(y, x_{F^{(q)},k}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(y) - (F^{(q)})_{\gamma_q}(x_{F^{(q)},k})|| \le \delta_{F^{(q)}}(4q)^{-1},$$

$$G(y) = \lim_{i\to\infty} G(y_i) \le a. \tag{9.199}$$

Since a is a minimal element of $\mathrm{cl}(G(X))$ we conclude that $G(y) = a$. Thus we have shown that for any minimal element a of $\mathrm{cl}(G(X))$ there is $y \in X$ such that $G(y) = a$.

Denote by K_G the set of all $y \in X$ for which $G(y)$ is a minimal element of $\mathrm{cl}(G(X))$. We have shown that

$$G(K_G) \text{ is the set of all minimal elements of } \mathrm{cl}(G(X)). \tag{9.200}$$

It follows from property (P6), (9.190), (9.191) and (9.200) that the following property holds:

(P7) For each $x \in K_G$ and each integer $q \ge 1$ there exists $k_q \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\rho(x, x_{F^{(q)},k_q}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(x) - (F^{(q)})_{\gamma_q} x_{F^{(q)},k_q}|| \le \delta_{F^{(q)}}(4q)^{-1}.$$

Property (P7) implies that any sequence of elements of K_G has a convergent subsequence. Thus the closure of K_G is compact. Now we show that the following property holds:

(P8) For each integer $q \ge 1$ and each $j \in \{1, \dots, p_{F^{(q)}}\}$ there is $x \in K_G$ such that

$$\rho(x, x_{F^{(q)},j}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(x) - F^{(q)}(x_{F^{(q)},j})|| \le \delta_{F^{(q)}}(4q)^{-1}.$$

Assume that q is a natural number and $j \in \{1, \dots, p_{F^{(q)}}\}$. In view of Proposition 9.10, (9.190) and (9.184) there is a minimal element a of $\mathrm{cl}(G(X))$ such that

$$a \le G(x_{F^{(q)},j}) \le F^{(q)}(x_{F^{(q)},j}) + \delta(F^{(q)}, \gamma_q, q)e. \tag{9.201}$$

By (9.190) there is $x \in K_G$ such that $G(x) = a$. Together with (9.201) and (9.186) this equality implies that

$$G(x) \le F^{(q)}(x_{F^{(q)},j}) + \delta_{F^{(q)}}(4q)^{-1}e. \tag{9.202}$$

Since $x \in K_G$ it follows from property (P7) that there exists $k \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\rho(x, x_{F^{(q)},k}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(x) - (F^{(q)})_{\gamma_q}(x_{F^{(q)},k})|| \le \delta_{F^{(q)}}(4q)^{-1}.$$

Combined with (9.202), (9.184) and property (C4) these inequalities imply that $k = j$. Therefore we have shown that property (P8) holds.

Let us show that the set K_G is compact. Since we have already shown that the closure of K_G is compact it is sufficient to verify that K_G is closed.

Assume that

$$\{y_i\}_{i=1}^{\infty} \subset K_G,\ y = \lim_{i\to\infty} y_i. \tag{9.203}$$

We show that $y \in K_G$. Let us assume the contrary. Then by Proposition 9.10 there is $y_* \in K_G$ such that

$$Gy_* < Gy. \tag{9.204}$$

Let $\epsilon > 0$. Choose a natural number q such that

$$q^{-1} < \epsilon. \tag{9.205}$$

By property (P7), (9.200) and (9.203) extracting a subsequence and reindexing we may assume without loss of generality that there is $j \in \{1, \dots, p_{F^{(q)}}\}$ such that for each natural number i

$$\rho(y_i, x_{F^{(q)},j}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(y_i) - F^{(q)}(x_{F^{(q)},j})|| \le \delta_{F^{(q)}}(4q)^{-1}. \tag{9.206}$$

In view of (P7) and the inclusion $y_* \in K_G$ there is $l \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\rho(y_*, x_{F^{(q)},l}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(y_*) - F^{(q)}(x_{F^{(q)},l})|| \le \delta_{F^{(q)}}(4q)^{-1}. \tag{9.207}$$

It follows from (9.203) and (9.206) that

$$\rho(y, x_{F^{q)},j}) = \lim_{i\to\infty} \rho(y_i, x_{F^{(q)},j}) \le \delta_{F^{(q)}}(4q)^{-1}, \tag{9.208}$$

$$G(y) = \lim_{i\to\infty} G(y_i) \le F^{(q)}(x_{F^{(q)},j}) + \delta_{F^{(q)}}(4q)^{-1}e. \tag{9.209}$$

Relations (9.204), (9.207) and (9.209) imply that

$$F^{(q)}(x_{F^{(q)},l}) \le G(y_*) + \delta_{F^{(q)}}(4q)^{-1}e < G(y) + \delta_{F^{(q)}}(4q)^{-1}e$$

$$\le F^{(q)}(x_{F^{(q)},j}) + \delta_{F^{(q)}}(2q)^{-1}e.$$

Combined with (C4) this inequality implies that $l = j$. By this equality, (9.205), (9207) and (9.208)

$$\rho(y, y_*) \le \rho(y, x_{F^{(q)},l}) + \rho(x_{F^{(q)},l}, y_*) \le \delta_{F^{(q)}}(2q)^{-1} < q^{-1} < \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that $y = y_*$. This contradicts (9.204). The contradiction we have reached proves that $y \in K_G$, the set K_G is closed and therefore compact. Thus the first assertion of Theorem 9.9 is proved.

Let $\epsilon > 0$. Choose a natural number $q > 1/\epsilon$. Assume that $H \in \mathcal{U}_q$, a is a minimal element of $\mathrm{cl}(H(X))$ and $z \in X$ satisfies $Hz \le a + \delta(F^{(q)}, \gamma_q, q)e$. Combined with property (P6) this implies that there is $j \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\rho(z, x_{F^{(q)},j}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||Hz - F^{(q)}(x_{F^{(q)},j})||$$

$$\le \delta_{F^{(q)}}(4q)^{-1},\ ||a - F^{(q)}(x_{F^{(q)},j})|| \le \delta_{F^{(q)}}(4q)^{-1}. \tag{9.210}$$

By property (P8) there is $x \in K_G$ such that

$$\rho(x, x_{F^{(q)},j}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(x) - F^{(q)}(x_{F^{(q)},j})|| \le \delta_{F^{(q)}}(4q)^{-1}.$$

In view of these inequalities and (9.210)

$$\rho(z, x) \le q^{-1} < \epsilon,\ ||H(z) - G(x)|| \le q^{-1} < \epsilon,$$

$$||a - G(x)|| \le q^{-1} < \epsilon.$$

Thus Assertion 2 holds.

We prove Assertion 3. Let $\epsilon > 0$. Choose a natural number $q > 1/\epsilon$. Let

$$x \in K_G,\ z \in X,\ G(z) \le G(x) + \delta(F^{(q)}, \gamma_q, q)e. \tag{9.211}$$

By (9.211), (9.190), (9.191), (9.200) and (P6) there exist $k, j \in \{1, \dots, p_{F^{(q)}}\}$ such that

$$\rho(z, x_{F^{(q)},j}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(z) - F^{(q)}(x_{F^{(q)},j})|| \le \delta_{F^{(q)}}(4q)^{-1}, \tag{9.212}$$

$$\rho(x, x_{F^{(q)},k}) \le \delta_{F^{(q)}}(4q)^{-1},\ ||G(x) - F^{(q)}(x_{F^{(q)},k})|| \le \delta_{F^{(q)}}(4q)^{-1}. \tag{9.213}$$

Relations (9.186), (9.211)–(9.213) imply that

$$F^{(q)}(x_{F^{(q)},j}) \le G(z) + \delta_{F^{(q)}}(4q)^{-1}e \le G(x) + \delta_{F^{(q)}}e \le F_q(x_{F^{(q)},k}) + 2\delta_{F^{(q)}}e.$$

Combined with property (C4) this inequality implies that $k = j$. In view of this equality, (9.212) and (9.213)

$$\rho(z, x) \le q^{-1} < \epsilon,\ ||G(z) - G(x)|| \le 1/q < \epsilon.$$

Thus Assertion 3 is proved. This completes the proof of the theorem.

9.10 Vector optimization with semicontinuous objective functions

We study a class of vector minimization problems on a complete metric space X which is identified with the corresponding complete metric space of lower semicontinuous bounded from below objective functions $\mathcal{A}$. We establish the existence of a G_δ everywhere dense subset $\mathcal{F}$ of $\mathcal{A}$ such that for any objective function belonging to $\mathcal{F}$ the corresponding minimization problem possesses a solution.

Let (X, ρ) be a complete metric space. A function $f : X \to R^1 \cup \{\infty\}$ is called lower semicontinuous if for each convergent sequence $\{x_k\}_{k=1}^{\infty}$ in (X, ρ) the inequality

$$f(\lim_{k\to\infty} x_k) \le \liminf_{k\to\infty} f(x_k)$$

holds. For each function $f : X \to R^1 \cup \{\infty\}$ set

$$\mathrm{dom}(f) = \{x \in X : \ f(x) < \infty\}.$$

We use the convention that $\infty + \infty = \infty$, $\infty - \infty = 0$, $\infty/\infty = 1$, $x + \infty = \infty$ and $x - \infty = -\infty$ for all $x \in R^1$, $\lambda \cdot \infty = \infty$ for all $\lambda > 0$ and that $\lambda \cdot \infty = -\infty$ for all $\lambda < 0$. We also assume that $-\infty < x < \infty$ for all $x \in R^1$.

Set $\bar{R} = R^1 \cup \{\infty, \ -\infty\}$ and let n be a natural number. For each $x = (x_1, \dots, x_n) \in \bar{R}^n$ set

$$||x|| = \max\{|x_i| : \ i = 1, \dots, n\}.$$

Let $x = (x_1, \dots, x_n), \ (y_1, \dots, y_n) \in \bar{R}^n$. We say that $x \le y$ if $x_i \le y_i$ for all $i = 1, \dots, n$. We say that $x < y$ if $x \le y$ and $x \ne y$ and say that $x << y$ if $x_i < y_i$ for all $i = 1, \dots, n$. Set

$$R^n_+ = \{x = (x_1, \dots, x_n) \in R^n : \ x \ge 0\}$$

and let $e = (1, 1, \dots, 1)$ be an element of R^n all of whose coordinates are unity.

Let $D \subset R^n$ be a nonempty set. Recall that an element $x \in D$ is called a minimal element of D if there is no $y \in D$ such that $y < x$. A function $F : X \to (R^1 \cup \{\infty\})^n$ is called bounded from below if there is $a \in R^n$ such that $F(x) \ge a$ for all $x \in X$.

Denote by $\mathcal{A}$ the set of all functions $F = (f_1, \dots, f_n) : X \to (R^1 \cup \{\infty\})^n$ which are bounded from below and such that the function f_i is lower semicontinuous for all $i = 1, \dots, n$ and such that the following condition holds:

(C1) For each $z \in F(X)$ and each $\epsilon > 0$ there exists $y \in F(X) \cap R^n$ such that $y - \epsilon e \le z$.

For each $F = (f_1, \dots, f_n)$, $G = (g_1, \dots, g_n) \in \mathcal{A}$ define

$$\tilde{d}(F, G) = \sup\{||F(x) - G(x)|| : \ x \in X\},$$

$$d(F, G) = \tilde{d}(F, G)(1 + \tilde{d}(F, G))^{-1}. \tag{9.214}$$

It is not difficult to see that the metric space $(\mathcal{A}, d)$ is complete. The topology induced by d in $\mathcal{A}$ is called the strong topology. For each $x \in X$ and each $r > 0$ set

$$B_X(x, r) = \{y \in X : \ \rho(x, y) \le r\},$$

$$B^o_X(x, r) = \{y \in X : \ \rho(x, y) < r\}$$

and for each $x \in R^n$ and each $r > 0$ put

$$B_{R^n}(x, r) = \{y \in R^n : \ ||x - y|| \le r\},$$

$$B^o_{R^n}(x, r) = \{y \in R^n : \ ||x - y|| < r\}.$$

Denote by $\mathrm{cl}(E)$ the closure of a set $E \subset R^n$.

Let $F = (f_1, \ldots, f_n) \in \mathcal{A}$. Set

$$\mathrm{epi}(F) = \{(y, a) \in X \times R^n : \ a \geq F(y)\}.$$

Clearly, $\mathrm{epi}(F)$ is a closed subset of $X \times R^n$. Define a function $\Delta_F : X \times R^n \to R^1$ by

$$\Delta_F(x, a) = \inf\{\rho(x, y) + ||a - b|| : \ (y, b) \in \mathrm{epi}(F)\}, \ (x, a) \in X \times R^n. \quad (9.215)$$

For each integer $q \geq 1$ put

$$\mathcal{E}(q) = \{(F, G) \in \mathcal{A} \times \mathcal{A} :$$

$$|\Delta_F(x, u) - \Delta_G(x, u)| \leq 1/q, \ (x, a) \in X \times R^n\}. \quad (9.216)$$

We equip the space $\mathcal{A}$ with the uniformity determined by the base $\mathcal{E}(q)$, $q = 1, 2, \ldots$. The topology in $\mathcal{A}$ induced by this uniformity is called the weak topology. It is clear that this topology is weaker than the strong topology.

Let $G = (g_1, \ldots, g_n) \in \mathcal{A}$. A sequence $\{z_i\}_{i=1}^{\infty} \subset X$ is called (G)-minimizing if there exist a sequence $\{a_i\}_{i=1}^{\infty} \subset R^n$ of minimal elements of $\mathrm{cl}(G(X) \cap R^n)$ and a sequence $\{\Delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that

$$\lim_{i \to \infty} \Delta_i = 0, \quad (9.217)$$

$$G(z_i) \leq a_i + \Delta_i e \text{ for all integers } i \geq 1.$$

For each $F \in \mathcal{A}$ denote by $\Omega(F)$ the set of all $x \in X$ such that $F(x)$ is a minimal element of $\mathrm{cl}(F(X) \cap R^n)$.

We prove the following theorem which was obtained in [144].

Theorem 9.14. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $F = (f_1, \ldots, f_n) \in \mathcal{F}$ the following assertions hold.*

1. Any (F)-minimizing sequence of elements of X possesses a convergent subsequence.

2. For each minimal element a of $\mathrm{cl}(F(x) \cap R^n)$ there is $x \in X$ such that $Fx = a$.

3. $F(\Omega(F))$ is the set of all minimal elements of $\mathrm{cl}(F(X) \cap R^n)$.

4. Any sequence of elements of $\Omega(F)$ has a convergent subsequence.

5. Let $\epsilon > 0$. Then there are $\delta > 0$ and an open neighborhood $\mathcal{U}$ of F in $\mathcal{A}$ with the weak topology such that the following properties hold:

(a) for each $G \in \mathcal{U}$, each minimal element a of $\mathrm{cl}(G(X) \cap R^n)$ and each $x \in X$ satisfying $G(x) \leq a + \delta e$ the inequality $\inf\{\rho(x, z) : \ z \in \Omega(F)\} < \epsilon$ holds;

(b) for each $G \in \mathcal{U}$ and each $z \in \Omega(F)$ there exists a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ such that $\rho(z_i, z) < \epsilon$ for all natural numbers i and that there is $\lim_{i \to \infty} G(z_i)$ which is a minimal element of $\mathrm{cl}(G(X) \cap R^n)$.

6. For each $x \in X$ there is $y \in \Omega(F)$ such that $F(y) \leq F(x)$.

Remark 9.15. Assume that $F = (f_1, \dots, f_n) : X \to (R^1 \cup \{\infty\})^n$ is bounded from below and that f_i is lower semicontinuous for all $i = 1, \dots, n$. Then (C1) holds if the following condition holds:

(C2) For each $z \in F(X)$ there is $y \in F(X) \cap R^n$ such that $y \le z$.

Note that if the space X is compact, then the conditions (C1) and (C2) are equivalent and if they hold, then the sets $F(X)$, $F(X) \cap R^n$, the closure of $F(X)$ and its intersection with R^n have the same minimal points. We are interested in solutions $x \in X$ of the vector minimization problem with the objective function F such that $F(x)$ is finite-valued. By this reason we consider functions F satisfying condition (C1) and solutions of the minimization problems which are minimal points of the closure of $F(X) \cap R^n$. By assertion 6 of Theorem 9.14, a generic function $F \in \mathcal{A}$ satisfies condition (C2).

Now we present an example of a function $F \in \mathcal{A}$ with a noncompact space X which does not satisfy (C2) and such that the set $F(X)$ possesses a minimal point which does not belong to R^n.

Assume that X is the set of nonnegative integers, $\rho(x, y) = |x - y|$, $x, y \in X$, $n = 2$, $F = (f_1, f_2)$ where

$$f_1(0) = 0, \; f_1(i) = -1/i, \; i = 1, 2, \dots,$$

$$f_2(0) = \infty, \; f_2(i) = i, \; i = 1, 2, \dots.$$

Clearly, $F \in \mathcal{A}$, F does not satisfy (C2), $F(0)$ is a minimal point of $F(X)$ which does not belong to R^2 and the set $F(X) \cap R^2$ is closed.

9.11 Auxiliary results for Theorem 9.14

Lemma 9.16. *Let $F = (f_1, \dots, f_n) \in \mathcal{A}$ and let $\epsilon > 0$. Then there exist $G = (g_1, \dots, g_n) \in \mathcal{A}$, $x_i \in X$, $i = 1, \dots, p$, where p is a natural number, and $\delta_1 > 0$ such that $\tilde{d}(F, G) \le \epsilon$, $G(x_i) \in R^n$, $i = 1, \dots, p$ and that for each $x \in X \setminus \{x_1, \dots, x_p\}$ there is $i \in \{1, \dots, p\}$ which satisfies $G(x) \ge G(x_i) + \delta_1 e$.*

Proof: Since $F \in \mathcal{A}$ there is $a \in R^n$ such that

$$a \le y \text{ for all } y \in F(X). \tag{9.218}$$

Denote by M_0 the set of all minimal elements of the set $\mathrm{cl}(F(X) \cap R^n)$. By (9.218), (C1) and Proposition 9.10 the set M_0 is nonempty. Choose positive constants

$$\delta_0 < \epsilon, \; \delta_1 \le \delta_0/16. \tag{9.219}$$

By (9.218), (C1) and Proposition 9.11 there is $c_0 > 0$ such that the following property holds:

For each $x \in \mathrm{cl}(F(X) \cap R^n)$ there is $y \in \mathrm{cl}(F(X) \cap R^n)$ such that $||y|| \le c_0$ and $y \le x + (\delta_0/64)e$.

This property implies that there are a natural number p and $y_1, \dots, y_p \in \mathrm{cl}(F(x) \cap R^n)$ such that the following property holds:

For each $x \in \mathrm{cl}(F(X) \cap R^n)$ there exists $i \in \{1, \dots, p\}$ such that $y_i \le x + (\delta_0/32)e$.

Clearly, for each $i \in \{1, \dots, p\}$ there is $x_i \in X$ such that $||F(x_i) - y_i|| \le \delta_0/64$. Now it is not difficult to see that the following property holds:

(P1) For each $x \in \mathrm{cl}(F(X) \cap R^n)$ there is $i \in \{1, \dots, p\}$ such that $F(x_i) \le x + [\delta_0/32 + \delta_0/64]e$.

(P1) and (C1) imply that the following property holds:

(P2) For each $x \in F(X)$ there is $i \in \{1, \dots, p\}$ such that $F(x_i) \le x + (\delta_0/16)e$.

Clearly,

$$F(x_i) \in R^n, \; i = 1, \dots, p. \tag{9.220}$$

Define $G = (g_1, \dots, g_n) : X \to (R^1 \cup \{\infty\})^n$ as follows:

$$G(x) = F(x) \text{ for all } x \in X \setminus \{x_1, \dots, x_p\}, \tag{9.221}$$

$$G(x_i) = F(x_i) - (\delta_0/8)e, \; i = 1, \dots, p.$$

Clearly, $G \in \mathcal{A}$. In view of (9.211) and (9.219) $\tilde{d}(F, G) \le \epsilon$. By (9.221) and (9.220), $G(x_i) \in R^n$, $i = 1, \dots, p$. It follows from (P2) and (9.221) and (9.219) that for each $x \in X \setminus \{x_1, \dots, x_p\}$ there is $i \in \{1, \dots, p\}$ such that $G(x) = F(x) \ge F(x_i) - (\delta_0/16)e \ge G(x_i) + \delta_1 e$. Lemma 9.16 is proved.

Lemma 9.17. *Let $F = (f_1, \dots, f_n) \in \mathcal{A}$, $\gamma > 0$, p be a natural number and $x_1, \dots, x_p \in X$ satisfy $F(x_i) \in R^n$, $i = 1, \dots, p$ and the following conditions:*

(P3) For each $x \in X \setminus \{x_1, \dots, x_p\}$ there is $i \in \{1, \dots, p\}$ such that $F(x) \ge F(x_i) + \gamma e$.

(P4) If $i, j \in \{1, \dots, p\}$ and $F(x_i) \le F(x_j)$, then $i = j$.

Then for each $\epsilon > 0$ there exists an integer $q \ge 1$ such that the following assertions hold.

1. If $G \in \mathcal{A}$ satisfies $(G, F) \in \mathcal{E}(q)$, a is a minimal element of $\mathrm{cl}(G(X) \cap R^n)$ and $x \in X$ satisfies $G(x) \le a + q^{-1}e$, then there is $j \in \{1, \dots, p\}$ such that $\rho(x, x_j) \le \epsilon$.

2. If $G \in \mathcal{A}$, $(G, F) \in \mathcal{E}(q)$ and $j \in \{1, \dots, p\}$, then there is $\{z_k\}_{k=1}^{\infty} \subset X$ such that $\rho(z_k, x_j) < \epsilon$ for all integers $k \ge 1$ and there exists $\lim_{k \to \infty} G(z_k)$ which is a minimal element of $\mathrm{cl}(G(X) \cap R^n)$.

Proof: Let $\epsilon > 0$. By (P4) there is $\epsilon_0 \in (0, \epsilon)$ such that

$$\rho(x_i, x_j) > 4\epsilon_0 \text{ for each pair } i, j \in \{1, \dots, p\} \text{ such that } i \ne j; \tag{9.222}$$

$$\epsilon_0 < \max\{f_s(x_i) - f_s(x_j) : \; s = 1, \dots, n\}$$
$$\text{for all pairs } i, j \in \{1, \dots, p\} \text{ such that } i \ne j. \tag{9.223}$$

Choose an integer $q \ge 1$ such that

$$8q^{-1} < \min\{\gamma/2,\ \epsilon_0\}. \tag{9.224}$$

Assume that

$$G \in \mathcal{A},\ (G, F) \in \mathcal{E}(q). \tag{9.225}$$

Let $j \in \{1, \dots, p\}$. Then $(x_j, F(x_j)) \in \text{epi}(F)$ and in view of (9.225), (9.216) and (9.215), $\Delta_G(x_j, F(x_j)) \le 1/q$. Together with (9.215) this implies that there exists

$$(y_j, b_j) \in \text{epi}(G) \tag{9.226}$$

such that

$$\rho(x_j, y_j) + ||b_j - F(x_j)|| < 2/q. \tag{9.227}$$

By (9.226) and (9.227),

$$G(y_j) \le b_j \le F(x_j) + (2q^{-1})e.$$

Therefore for all $j = 1, \dots, p$ we have defined $y_j \in X$ such that

$$\rho(x_j, y_j) < 2/q,\ G(y_j) \le F(x_j) + (2/q)e. \tag{9.228}$$

Assume that

$$x \in X \text{ and } Gx \in R^n. \tag{9.229}$$

By (9.229), (9.225), (9.215) and (9.216), $\Delta_F(x, G(x)) \le 1/q$. In view of this inequality, (9.229) and (9.215) there exist $Tx \in X$, $b \in R^n$ such that

$$(Tx, b) \in \text{epi}(F),\ \rho(Tx, x) + ||b - Gx|| \le (3/2)/q. \tag{9.230}$$

Relations (9.230) imply that

$$F(Tx) \le b \le G(x) + (3/2)q^{-1}e,\ \rho(Tx, x) \le (3/2)/q. \tag{9.231}$$

Thus we have shown that the following property holds:

(P5) For each $x \in X$ satisfying (9.229) we defined $Tx \in X$ satisfying (9.231).

Let $j \in \{1, \dots, p\}$. Then (9.228) holds. By (9.231)

$$F(Ty_j) \le G(y_j) + (3/2)q^{-1}e,\ \rho(y_j, Ty_j) < (3/2)/q. \tag{9.232}$$

By (9.228) and (9.232),

$$\rho(x_j, Ty_j) < (7/2)/q,\ F(Ty_j) \le F(x_j) + (7/(2q))e. \tag{9.233}$$

We will show that $Ty_j \in \{x_1, \dots, x_p\}$. Assume the contrary. Then by (P3) there is $i \in \{1, \dots, p\}$ such that $F(Ty_j) \ge F(x_i) + \gamma e$. Combined with (9.224) and (9.233) this implies that

$$F(x_i) \le F(x_j) + ((7/2)q^{-1} - \gamma)e \le F(x_j) - (\gamma/2)e << Fx_j.$$

Together with (P4) this implies that $i = j$, a contradiction. The contradiction we have reached proves that $Ty_j \in \{x_1, \dots, x_p\}$. Together with (9.233), (9.224) and (9.223) this implies that $F(Ty_j) << F(x_j) + \epsilon_0 e$ and $Ty_j = x_j$. Thus

$$Ty_j = x_j \text{ for all } j \in \{1, \dots, p\}. \tag{9.234}$$

Relations (9.228), (9.232) and (9.234) imply that

$$||F(x_j) - G(y_j)|| \le 2q^{-1}e \text{ for all } j \in \{1, \dots, p\}. \tag{9.235}$$

Assume that

$$a \text{ is a minimal element of } \mathrm{cl}(G(X) \cap R^n),\ x \in X,\ G(x) \le a + q^{-1}e. \tag{9.236}$$

We will show that there is $j \in \{1, \dots, p\}$ such that $Tx = x_j$. Assume that

$$Tx \notin \{x_1, \dots, x_p\}. \tag{9.237}$$

By (P3) there is $j \in \{1, \dots, p\}$ for which $F(Tx) \ge F(x_j) + \gamma e$. Combined with (9.236), (P5), (9.231), (9.235) and (9.224) this implies that

$$a \ge G(x) - q^{-1}e \ge F(Tx) - (3/2)q^{-1}e - q^{-1}e \ge F(x_j) + \gamma e - 3q^{-1}e$$

$$\ge G(y_j) - (2q)^{-1}e + \gamma e - 3q^{-1}e = G(y_j) + [\gamma - 5q^{-1}]e \ge G(y_j) + 2^{-1}\gamma e.$$

This contradicts (9.236). The contradiction we have reached proves that $Tx \in \{x_1, \dots, x_p\}$ and that there is $j \in \{1, \dots, p\}$ such that $Tx = x_j$. Combined with (P5), (9.236) and (9.231) this implies that $\rho(x, x_j) \le (3/2)q^{-1} < \epsilon$. Now it is clear that assertion 1 holds. Moreover, we showed that the following property holds:

(P6) If $a \in R^n$ and $x \in X$ satisfy (9.236), then $Tx \in \{x_1, \dots, x_p\}$.

Let $j \in \{1, \dots, p\}$. By (C1) and Proposition 9.10 there is a minimal element a of $\mathrm{cl}(G(X) \cap R^n)$ such that $a \le G(y_j)$. Clearly there exists a sequence $\{z_k\}_{k=1}^{\infty} \subset X$ such that $a = \lim_{k \to \infty} G(z_k)$. Then we may assume that

$$G(z_k) \le a + (2q)^{-1}e \text{ for all natural numbers } k. \tag{9.238}$$

In view of (9.238), (P6) and the choice of a, $\{Tz_k\}_{k=1}^{\infty} \subset \{x_1, \dots, x_p\}$.

Extracting a subsequence and reindexing we may assume without loss of generality that there is $s \in \{1, \dots, p\}$ such that

$$Tz_k = x_s \text{ for all natural numbers } k. \qquad (9239)$$

We will show that $s = j$. By (9.235), (9.239), (P5), (9.231), (9.238) and the inequality $a \le G(y_j)$ we have that for all natural numbers k,

$$F(x_s) \le F(Tz_k) \le G(z_k) + (3/2)q^{-1}e \le a + 2q^{-1}e \le G(y_j) + 2q^{-1}e. \tag{9.240}$$

By (9.240), (9.224) and (9.235), $F(x_s) \le G(y_j) + 2q^{-1}e \le F(x_j) + (4q^{-1})e \le F(x_j) + \epsilon_0 e$. Combined with (9.223) this implies that $s = j$. In view of (9.239)

$$Tz_k = x_j \text{ for all natural numbers } k.$$

Together with (P5), (9.224) and (9.231) this implies that

$$\rho(z_k, x_j) \leq (3/2)q^{-1} < \epsilon \text{ for all natural numbers } k.$$

We have shown that for each $j \in \{1, \dots, p\}$ there exists a minimal element a of $\mathrm{cl}(G(X) \cap R^n)$ and a sequence $\{z_k\}_{k=1}^{\infty} \subset X$ such that $\rho(z_k, x_j) < \epsilon$ for all natural numbers k and $a = \lim_{k\to\infty} G(z_k)$. Therefore assertion 2 holds. This completes the proof of Lemma 9.17.

9.12 Proof of Theorem 9.14

Denote by $\mathcal{L}$ the set of all $F = (f_1, \dots, f_n) \in \mathcal{A}$ for which there exist $\gamma > 0$, a natural number p and $x_1, \dots, x_p \in X$ such that $F(x_i) \in R^n$, $i = 1, \dots, p$ and conditions (P3) and (P4) hold. By Lemma 9.16 $\mathcal{L}$ is an everywhere dense subset of $\mathcal{A}$ with the strong topology.

Let $F = (f_1, \dots, f_n) \in \mathcal{L}$. By (P3), (P4) and the definition of $\mathcal{L}$, there exist $\gamma^{(F)} > 0$, a natural number $p^{(F)}$ and $x_1^{(F)}, \dots, x_{p^{(F)}}^{(F)} \in X$ such that

$$F(x_i^{(F)}) \in R^n,\ i = 1, \dots, p^{(F)}; \tag{9.241}$$

$$\text{for each } x \in X \setminus \{x_1^{(F)}, \dots, x_{p^{(F)}}^{(F)}\} \text{ there is } i \in \{1, \dots, p^{(F)}\}$$

$$\text{such that } F(x_i^{(F)}) \leq F(x) - \gamma^{(F)}e; \tag{9.242}$$

$$\text{if } i, j \in \{1, \dots, p^{(F)}\} \text{ and } F(x_i^{(F)}) \leq F(x_j^{(F)}), \text{ then } i = j. \tag{9.243}$$

Let k be a natural number. By (9.241)–(9.243) and Lemma 9.17 there exist $\delta(F, k) > 0$ and an open neighborhood $\mathcal{U}(F, k)$ of F in $\mathcal{A}$ with the weak topology such that:

(P7) If $G \in \mathcal{U}(F, k)$, a is a minimal element of $\mathrm{cl}(G(X) \cap R^n)$ and if $x \in X$ satisfies $G(x) \leq a + \delta(F, k)e$, then

$$\min\{\rho(x, x_i^{(F)}) :\ i = 1, \dots, p^{(F)}\}$$

$$\leq k^{-1}4^{-1} \min\{1, \rho(x_i^{(F)}, x_j^{(F)}) :\ i, j \in \{1, \dots, p^{(F)}\},\ i \neq j\};$$

(P8) If $G \in \mathcal{U}(F, k)$ and $j \in \{1, \dots, p^{(F)}\}$, then there is a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ such that

$$\rho(z_s, x_j^{(F)}) < (4k)^{-1} \min\{1, \rho(x_i^{(F)}, x_j^{(F)}) :\ i, j \in \{1, \dots, p^{(F)}\},\ i \neq j\}$$

for all integers $s \geq 1$ and there is $\lim_{i\to\infty} G(z_i)$ which is a minimal element of $\mathrm{cl}(G(X) \cap R^n)$.

Put

$$\mathcal{F} = \cap_{k=1}^{\infty} \cup \{\mathcal{U}(F,k) : F \in \mathcal{L}\}. \tag{9.244}$$

It is easy to see that $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$.

Let $H = (h_1, \dots, h_n) \in \mathcal{F}$, $\epsilon > 0$. Choose a natural number k such that $16/k < \epsilon$. Assume that $\{z_i\}_{i=1}^{\infty} \subset X$ is an (H)-minimizing sequence. Then there exist a sequence $\{a_i\}_{i=1}^{\infty} \subset R^n$ of minimal elements of $\mathrm{cl}(H(X) \cap R^n)$ and a sequence $\{\Delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that

$$\lim_{i \to \infty} \Delta_i = 0, \; H(z_i) \le a_i + \Delta_i e \text{ for all integers } i \ge 1. \tag{9.245}$$

By (9.244) there exists $F \in \mathcal{L}$ such that

$$H \in \mathcal{U}(F,k). \tag{9.246}$$

It follows from (P7), (9.245) and (9.246) that there exists a subsequence $\{z_{i_j}\}_{j=1}^{\infty}$ of $\{z_i\}_{i=1}^{\infty}$ such that $\rho(z_{i_j}, z_{i_s}) \le 2/k < \epsilon$ for all natural numbers s, j. Since ϵ is an arbitrary positive number and $\{z_i\}_{i=1}^{\infty} \subset X$ is an arbitrary (H)-minimizing sequence we obtain using induction and diagonalization process that there exists a convergent subsequence of $\{z_i\}_{i=1}^{\infty}$ in (X, ρ). Thus any (H)-minimizing sequence possesses a convergent subsequence and assertion 1 holds.

Assume that a is a minimal element of $\mathrm{cl}(H(X) \cap R^n)$. There is an (H)-minimizing sequence $\{z_i\}_{i=1}^{\infty} \subset X$ such that $\lim_{i \to \infty} H(z_i) = a$. By assertion 1 the sequence $\{z_i\}_{i=1}^{\infty}$ possesses a convergent subsequence which converges to $z \in X$. Since the functions h_i, $i = 1, \dots, n$ are lower semicontinuous we have $H(z) \le a$. Since a is a minimal element of $\mathrm{cl}(H(X) \cap R^n)$ we obtain that $H(z) = a$. Therefore for each minimal element a of $\mathrm{cl}(H(X) \cap R^n)$ there is $z \in X$ such that $H(z) = a$ and assertion 2 holds. It is clear that assertion 2 implies assertion 3 and that assertion 4 follows from assertion 1.

By (P7) and (9.246) for each $z \in \Omega(H)$,

$$\min\{\rho(z, x_i^{(F)}) : \; i = 1, \dots, p^{(F)}\} \le (4k)^{-1}. \tag{9.247}$$

Let $j \in \{1, \dots, p^{(F)}\}$. By (9.246) and (P8) there is a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ such that

$$\rho(z_i, x_j^{(F)}) \le (4k)^{-1} \text{ for all integers } i \ge 1 \tag{9.248}$$

and that there exists $\lim_{i \to \infty} H(z_i)$ which is a minimal element of $\mathrm{cl}(H(X) \cap R^n)$. By assertion 1 the sequence $\{z_i\}_{i=1}^{\infty}$ possesses a convergent subsequence. Extracting a subsequence if necessary we may assume without loss of generality that there exists $z = \lim_{i \to \infty} z_i$. Since the functions h_i, $i = 1, \dots, n$ are lower semicontinuous and $\lim_{i \to \infty} H(z_i)$ is a minimal element of $\mathrm{cl}(H(X) \cap R^n)$ we conclude that $H(z) \le \lim_{i \to \infty} H(z_i)$, $H(z)$ is a minimal element of $\mathrm{cl}(H(X) \cap R^n)$ and $z \in \Omega(H)$. In view of (9.248) $\rho(z, x_j^{(F)}) \le (4k)^{-1}$. Thus for each $j \in \{1, \dots, p^{(F)}\}$,

$$\min\{\rho(x_j^{(F)}, z) : \ z \in \Omega(H)\} \le (4k)^{-1}. \tag{9.249}$$

Assume that $G \in \mathcal{U}(F,k)$, a is minimal element of $\mathrm{cl}(G(x) \cap R^n)$ and $x \in X$ satisfies $G(x) \le a + \delta(F,k)e$. By (P7),

$$\min\{\rho(x, x_i^{(F)}) : \ i = 1, \dots, p^{(F)}\} \le (4k)^{-1}.$$

Together with (9.249) this implies that

$$\inf\{\rho(x,z) : \ z \in \Omega(H)\} \le (2k)^{-1} < \epsilon$$

and the property (a) of assertion 5 holds.

Let $z \in \Omega(H)$. In view of (9.247) there is $j \in \{1, \dots, p^{(F)}\}$ such that

$$\rho(z, x_j^{(F)}) < (3k)^{-1}. \tag{9.250}$$

By (P8) there exists a sequence $\{z_i\}_{i=1}^{\infty} \subset X$ such that $\rho(z_i, x_j^{(F)}) < (4k)^{-1}$ for all integers $i \ge 1$ and that there is $\lim_{i\to\infty} G(z_i)$ which is a minimal element of $\mathrm{cl}(G(X) \cap R^n)$. Combined with (9.250) this implies that $\rho(z_i, z) < 1/k < \epsilon$ for all integers $i \ge 1$ and the property (b) of assertion 5 holds.

Now we prove assertion 6. Let $x \in X$. By (C1) for any natural number i there is $x_i \in X$ such that

$$F(x_i) \in R^n, \ F(x_i) - i^{-1}e \le F(x).$$

By Proposition 9.10 for any natural number i there is a minimal element a_i of $\mathrm{cl}(F(X) \cap R^n)$ such that $a_i \le F(x_i)$. In view of assertion 3 for each natural number i there is $y_i \in \Omega(F)$ such that $F(y_i) = a_i$. By assertion 4 we may assume without loss of generality that there is $y_* = \lim_{i\to\infty} y_i$. Now it is not difficult to see that $F(y_*) \in R^n$ and $F(y_*) \le F(x)$. By Proposition 9.10 and assertion 3 there exists $y \in \Omega(F)$ such that $F(y) \le F(y_*) \le F(x)$. Assertion 6 is proved. This completes the proof of Theorem 9.14.

9.13 Density results

We use the notation and definitions introduced in Section 9.10. Suppose that the complete metric space (X, ρ) does not contain isolated points, that $n \ge 2$ and that $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$ is the standard basis in R^n. We prove the following result obtained in [144].

Theorem 9.18. *There exists an everywhere dense (in the weak topology) set $\mathcal{F} \subset \mathcal{A}$ such that for each $F \in \mathcal{F}$ the set of all $x \in X$ such that Fx is a minimal element of cl$(F(X) \cap R^n)$ is nonempty and not closed.*

Proof: Let

$$F = (f_1, \dots, f_n) \in \mathcal{A}, \text{ an integer } q \geq 1, \ \gamma \in (0, (8q)^{-1}). \tag{9.251}$$

By Lemma 9.16 there exist $F^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)}) \in \mathcal{A}$, $x_i \in X$, $i = 1, \dots, p$, where p is a natural number, such that

$$F^{(1)}(x_i) \in R^n, \ i = 1, \dots, p, \tag{9.252}$$

$$\tilde{d}(F, F^{(1)}) \leq \gamma/8 \tag{9.253}$$

and the following property holds:

(D1) For each $x \in X$ there is $i \in \{1, \dots, p\}$ for which $F^{(1)}(x) \geq F^{(1)}(x_i)$.

We may assume without loss of generality that the following property holds:

(D2) If $i, j \in \{1, \dots, p\}$ satisfy $F^{(1)}(x_i) \leq F^{(1)}(x_j)$, then $i = j$.

Clearly,

$$(F, F^{(1)}) \in \mathcal{E}(8q). \tag{9.254}$$

It is easy to see that there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset X$ such that

$$y_i \neq y_j \text{ for all pairs of integers } i, j \in \{1, 2, \dots\} \text{ such that } i \neq j, \tag{9.255}$$

$$x_1 \neq y_i \text{ for all integers } i \geq 1 \text{ and } \lim_{i \to \infty} \rho(x_1, y_i) = 0. \tag{9.256}$$

We may assume without loss of generality that for all integers $i \geq 1$,

$$F^{(1)}(y_i) \geq F^{(1)}(x_1) - (\gamma/16)e, \ \rho(y_i, x_1) \leq \gamma/16. \tag{9.257}$$

Set

$$F^{(2)}(x) = F^{(1)}(x) \text{ for all } x \in X \setminus (\{x_1\} \cup \{y_i : \ i = 1, 2, \dots\}), \tag{9.258}$$

$$F^{(2)}(x_1) = F^{(1)}(x_1) - (\gamma/8)e,$$

$$F^{(2)}(y_i) = F^{(1)}(x_1) - (\gamma/8)e + (16i)^{-1}\gamma(e_1 - e_2), \ i = 2, 3, 4, ...,$$

$$F^{(2)}(y_1) = F^{(1)}(x_1) - (\gamma/8)e - (\gamma/16)e_1.$$

Clearly, $F^{(2)} \in \mathcal{A}$ and $(F^{(1)}, F^{(2)}) \in \mathcal{E}(2q)$. Together with (9.254) this implies that $(F, F^{(2)}) \in \mathcal{E}(q)$. It is not difficult to see that for all integers $i \geq 1$, $F^{(2)}(y_i)$ is a minimal element of $\mathrm{cl}(F^{(2)}(X) \cap R^n)$, $F^{(2)}(y_1) < F^{(2)}(x_1)$ and $F^{(2)}(x_1)$ is not a minimal element of $\mathrm{cl}(F^{(2)}(X) \cap R^n)$. Theorem 9.18 is proved.

Proposition 9.19. *Assume that $F = (f_1, \dots, f_n) \in \mathcal{A}$, $F(x) \in R^n$ for all $x \in X$, the mapping $F : X \to R^n$ is continuous and $\gamma \in (0, 1)$. Then there exists $G = (g_1, \dots, g_n) \in \mathcal{A}$ such that $\tilde{d}(F, G) \leq \gamma$ and that the set*

$$\{x \in X : \ G(x) \textit{ is a minimal element of } \mathrm{cl}(G(X)\}$$

is nonempty and not closed.

Proof: By Lemma 9.16 there exists $F^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)}) \in \mathcal{A}$ such that $F^{(1)}(X) \subset R^n$ and that

$$\tilde{d}(F, F^{(1)}) \leq \gamma/8 \tag{9.259}$$

and there exist $x_i \in X$, $i = 1, \dots, p$, where p is a natural number, such that the property (D1) holds. We may assume without loss of generality that the property (D2) holds. Clearly, there exists a sequence $\{y_i\}_{i=1}^{\infty} \subset X$ such that (9.255) and (9.256) hold. We may assume without loss of generality that for all integers $i \geq 1$,

$$F^{(1)}(y_i) \geq F^{(1)}(x_1) - 64^{-1}\gamma e, \ \rho(y_i, x_1) \leq 64^{-1}\gamma.$$

Define $F^{(2)} : X \to R^n$ by (9.258) and put $G = F^{(2)}$. It is not difficult to see that $G \in \mathcal{A}$, $\tilde{d}(G, F^{(1)}) \leq \gamma/4$, $G(y_i)$ is a minimal element of $\mathrm{cl}(G(X))$ for all integers $i \geq 1$ and that $G(x_1) < G(y_1)$. Proposition 9.19 is proved.

Proposition 9.20. *Assume that* $F = (f_1, \dots, f_n) \in \mathcal{A}$, $\gamma \in (0, 1)$ *and that a natural number* q *satisfies* $\gamma < (4q)^{-1}$. *Then there exists* $F^{(0)} = (f_1^{(0)}, \dots, f_n^{(0)}) \in \mathcal{A}$ *such that* $(F, F^{(0)}) \in \mathcal{E}(q)$, *the set of all minimal elements of* $\mathrm{cl}(F^{(0)}(X) \cap R^n)$ *is a nonempty finite subset of* $F^{(0)}(X) \cap R^n$ *and that the following property holds:*

For any natural number k *satisfying* $1/k < \gamma/64$ *there exist* $G \in \mathcal{A}$ *and* $\bar{x} \in X$ *such that* $(G, F^{(0)}) \in \mathcal{E}(k)$, $G(\bar{x})$ *is a minimal element of* $\mathrm{cl}(G(X) \cap R^n)$ *and*

$$||G(\bar{x}) - z|| \geq \gamma/64 \text{ for all minimal elements } z \text{ of } \mathrm{cl}(F^{(0}(X) \cap R^n).$$

Proof: By Lemma 9.16 there exist $F^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)}) \in \mathcal{A}$, $x_i \in X$, $i = 1, \dots, p$, where p is a natural number, such that $F^{(1)}(x_i) \in R^n$, $i = 1, \dots, p$, $\tilde{d}(F, F^{(1)}) \leq \gamma/8$ and (D1) holds. We may assume without loss of generality that the property (D2) holds. Clearly, $x_i \neq x_j$ for all pairs $i.j \in \{1. \dots, p\}$ such that $i \neq j$. There is $r > 0$ such that

$$F^{(1)}(z) \geq F^{(1)}(x_1) - 64^{-1}\gamma e \text{ for all } z \in X \text{ satisfying } \rho(z, x_1) \leq r. \tag{9.260}$$

Choose $x_0 \in X$ such that

$$0 < \rho(x_0, x_1) < 16^{-1} \min\{\gamma, \ r, \ \min\{\rho(x_i, x_j) : \ i, j \in \{1, \dots, p\}, \ i \neq j\}\}, \tag{9.261}$$

$$x_0 \neq x_i, \ i = 1, \dots, p. \tag{9.262}$$

Define $F^{(0)} = (f_1^{(0)}, \dots, f_n^{(0)}) \in \mathcal{A}$ by

$$F^{(0)}(z) = F^{(1)}(z), \ z \in X \setminus \{x_0, \dots, x_p\}, \tag{9.263}$$

$$F^{(0)}(x_i) = F^{(1)}(x_i) - (\gamma/16)e, \ i \in \{1, \dots, p\} \setminus \{1\},$$

$$F^{(0)}(x_1) = F^{(1)}(x_1) - (\gamma/8)e - (\gamma/8)e_2,$$

$$F^{(0)}(x_0) = F^{(1)}(x_1) - (\gamma/8)e - (\gamma/8)e_1.$$

It is not difficult to see that $(F^{(0)}, F^{(1)}) \in \mathcal{E}(4q)$, $(F^{(0)}, F) \in \mathcal{E}(q)$, the set $\{F^{(0)}(x_i) : \ i = 0, \dots, p\}$ contains the set of all minimal elements of $\mathrm{cl}(F^{(0)}(X) \cap R^n)$ and $\{F^{(0)}(x_i) : \ i = 0, 1\}$ is a subset of the set of all minimal elements of $\mathrm{cl}(F^{(0)}(X) \cap R^n)$.

Let a natural number k satisfy

$$k^{-1} < 64^{-1}\gamma. \tag{9.264}$$

Choose $\bar{x} \in X$ such that

$$\rho(\bar{x}, x_0) < 64^{-1} \min\{r, \gamma, k^{-1}, \ \min\{\rho(x_i, x_j) : \ i, j \in \{0, \dots, p\}, \ i \neq j\}\}. \tag{9.265}$$

Define a mapping $G : X \to (R^1 \cup \{\infty\})^n$ by

$$G(z) = F^{(0)}(z), \ z \in X \setminus \{\bar{x}\}, \tag{9.266}$$

$$G(\bar{x}) = F^{(1)}(x_1) - (\gamma/8)e - (\gamma/8)e_1 + 64^{-1}\gamma e_1 - (4k)^{-1}e_2.$$

It is not difficult to see that $G \in \mathcal{A}$ and $G(\bar{x})$, $G(x_i)$, $i = 0, 1$ are minimal elements of $\mathrm{cl}(G(X) \cap R^n)$. By (9.260), (9.261), (9.265) and (9.266)

$$F^{(0)}(\bar{x}) = F^{(1)}(\bar{x}) \geq G(\bar{x}). \tag{9.267}$$

In view of (9.266) and (9.267)

$$\mathrm{epi}(F^{(0)}) \subset \mathrm{epi}(G), \ \mathrm{epi}(G) \setminus \mathrm{epi}(F^{(0)}) \subset \{(\bar{x}, \alpha) : \ \alpha \in [G(\bar{x}), \ F^{(1)}(\bar{x})]\}. \tag{9.268}$$

By (9.268), (9.266), (9.265) and (9.263) for each $(z, \alpha) \in \mathrm{epi}(G) \setminus \mathrm{epi}(F^{(0)})$ we have $\Delta_{F^{(0)}}(z, \alpha) \leq (2k)^{-1}$. Together with (9.263)–(9.266) and (9.268) this implies that $(G, F^{(0)}) \in \mathcal{E}(k)$. By (9.263) and (9.266),

$$\inf\{||G(\bar{x}) - z|| : \ z \text{ is a minimal element of } \mathrm{cl}(F^{(0)}(X) \cap R^n)\}$$

$$\geq \min\{||G(\bar{x}) - F^{(0)}(x_i)|| : \ i = 0, \dots, p\}$$

$$\geq \min\{\gamma/8, \ 64^{-1}\gamma, \ \min\{||G(\bar{x}) - F^{(0)}(x_i)|| : \ i = 2, \dots, p\}\}. \tag{9.269}$$

By (9.266), $G(\bar{x}) \leq F^{(1)}(x_1) - (\gamma/8)e$. It follows from (D2) that for each $i \in \{2, \dots, p\}$ there is $s(i) \in \{1, \dots, n\}$ such that $f^{(1)}_{s(i)}(x_1) < f^{(1)}_{s(i)}(x_i)$ and in view of (9.263)

$$f^{(0)}_{s(i)}(x_i) = f^{(1)}_{s(i)}(x_i) - (\gamma/16) > f^{(1)}_{s(i)}(x_1) - \gamma/16 \geq g_{s(i)}(\bar{x}) + \gamma/16.$$

Thus for all $i \in \{2, \dots, p\}$ we have $||F^{(0)}(x_i) - G(\bar{x})|| \geq \gamma/16$. Together with (9.269) this implies that

$$\inf\{||G(\bar{x}) - z|| : \ z \text{ is a minimal element of } \mathrm{cl}(F^{(0)}(X) \cap R^n)\} \geq \gamma/64.$$

This completes the proof of the proposition.

9.14 Comments

In this chapter we use the generic approach in order to study vector minimization problems on a complete metric space. We show that solutions of vector minimization problems exist generically for certain classes of problems. Any of these classes of problems is identified with a space of functions equipped with a natural complete metric and it is shown that there exists a G_δ everywhere dense subset of the space of functions such that for any element of this subset the corresponding vector minimization problem possesses solutions. We also study the stability and the structure of a set of solutions of a vector minimization problem. It should be mentioned that recently the existence of minimizers for general multiobjective problems with values in partially ordered spaces was established in [9] using advanced tools of variational analysis and generalized differentiation and a refined version of the subdifferential Palais–Smale condition for set-valued mappings.

10

Infinite Horizon Problems

10.1 Minimal solutions for discrete-time control systems in metric spaces

Let (X, d) be a complete metric space. We equip the set $X \times X$ with the metric d_1 defined by

$$d_1((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2),\ x_i, y_i \in X,\ i = 1, 2.$$

Clearly the metric space $(X \times X, d_1)$ is complete. For $x \in X$ and $A \subset X$ set

$$d(x, A) = \inf\{d(x, y) :\ y \in A\}.$$

For $(x_1, x_2) \in X \times X$, $A \subset X \times X$ set

$$d_1((x_1, x_2), A) = \inf\{d_1((x_1, x_2), (y_1, y_2)) :\ (y_1, y_2) \in A\}.$$

Denote by $\mathcal{A}$ the set of all continuous functions $v : X \times X \to R^1$ which satisfy the following assumptions:

(i) (uniform boundedness)

$$\sup\{|v(x, y)| :\ x, y \in X\} < \infty; \tag{10.1}$$

(ii) (uniform continuity) for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|v(x_1, x_2) - v(y_1, y_2)| \le \epsilon \tag{10.2}$$

for each $x_i, y_i \in X$, $i = 1, 2$ which satisfy $d(x_i, y_i) \le \delta$, $i = 1, 2$.

Define a metric $\rho : \mathcal{A} \times \mathcal{A} \to R^1$ by

$$\rho(v, w) = \sup\{|v(x, y) - w(x, y)| :\ x, y \in X\}. \tag{10.3}$$

Clearly the metric space $(\mathcal{A}, \rho)$ is complete.

We consider the optimization problem

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$$\text{minimize} \sum_{i=k_1}^{k_2-1} v(x_i, x_{i+1}) \text{ subject to } \{x_i\}_{i=k_1}^{k_2} \subset X,\ x_{k_1} = y,\ x_{k_2} = z \quad \text{(P)}$$

where $v \in \mathcal{A}$, $y, z \in X$ and $k_2 > k_1$ are integers.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e.g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [62], the infinite-horizon control problem of minimizing $\int_0^T L(z, z')dt$ as $T \to \infty$ [63] and the analysis of a long slender bar of a polymeric material under tension in [64]. Similar optimization problems are also considered in mathematical economics [67, 130].

If the space of states X is compact, then the problem (P) has a solution for each $v \in \mathcal{A}$, $y, z \in X$ and each pair of integers $k_2 > k_1$. For the noncompact space X the existence of solutions of the problem (P) is not guaranteed.

For each $v \in \mathcal{A}$, each natural number m and each $y_1, y_2 \in X$ we set

$$||v|| = \sup\{|v(x, y)| :\ x, y \in X\}, \tag{10.4}$$

$$\mu(v) = \inf\{\liminf_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} v(x_i, x_{i+1}) :\ \{x_i\}_{i=0}^{\infty} \subset X\}, \tag{10.5}$$

$$\sigma(v, m) = \inf\{\sum_{i=0}^{m-1} v(x_i, x_{i+1}) :\ \{x_i\}_{i=0}^{m} \subset X\}, \tag{10.6}$$

$$\sigma_{per}(v, m) = \inf\{\sum_{i=0}^{m-1} v(x_i, x_{i+1}) :\ \{x_i\}_{i=0}^{m} \subset X,\ x_0 = x_m\} \tag{10.7}$$

and

$$\sigma(v, m, y_1, y_2) = \inf\{\sum_{i=0}^{m-1} v(x_i, x_{i+1}) :\ \{x_i\}_{i=0}^{m} \subset X,\ x_0 = y_1,\ x_m = y_2\}. \tag{10.8}$$

The following optimality criterion for infinite horizon problems was introduced by Aubry and Le Daeron [4] in their study of the discrete Frenkel–Kontorova model related to dislocations in one-dimensional crystals.

Let $v \in \mathcal{A}$. A sequence $\{x_i\}_{i=-\infty}^{\infty} \subset X$ is called (v)-minimal if for each pair of integers $m_2 > m_1$

$$\sum_{i=m_1}^{m_2-1} v(x_i, x_{i+1}) = \sigma(v, m_2 - m_1, x_{m_1}, x_{m_2}).$$

If the space of states X is compact, then a (v)-minimal sequence can be constructed as a limit of a sequence of optimal solutions on finite intervals. For the noncompact space X the problem is more difficult and less understood. The difficulty is that for any problem of type (P) the existence of its solution is not guaranteed and that a (v)-minimal sequence is an exact solution of a countable number of optimization problems of type (P). We show that for a generic function v taken from the space $\mathcal{A}$ there exists a (v)-minimal sequence.

A sequence $\{x_i\}_{i=0}^{\infty} \subset X$ is called (v)-good if there exists a number $M > 0$ such that for each natural number m

$$\sum_{i=0}^{m-1} v(x_i, x_{i+1}) \leq \sigma(v, m, x_0, x_m) + M.$$

It is not difficult to see that the following proposition holds.

Proposition 10.1. *Let $v \in \mathcal{A}$ and $\{z_i\}_{i=0}^{\infty} \subset X$ be a (v)-good sequence. Then for each $x \in X$ there is a (v)-good sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$.*

For each $\{x_i\}_{i=0}^{\infty} \subset X$ denote by $\omega(\{x_i\}_{i=0}^{\infty})$ the set of all $y \in X$ for which there exists a subsequence $\{x_{i_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} x_{i_k} = y$.

We prove the following result obtained in [116].

Theorem 10.2. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that for each $v \in \mathcal{F}$ there exists a nonempty compact set $\Omega(v) \subset X$ which satisfies the following conditions:*

(i) There is a (v)-minimal sequence $\{x_i^{(v)}\}_{i=-\infty}^{\infty} \subset \Omega(v)$.

(ii) For each (v)-good sequence $\{y_i\}_{i=0}^{\infty} \subset X$ there exists a (v)-minimal sequence $\{x_i\}_{i=-\infty}^{\infty} \subset \Omega(v) \cap \omega(\{y_i\}_{i=0}^{\infty})$.

10.2 Auxiliary results

Proposition 10.3. *Assume that $v \in \mathcal{A}$ and ϵ is a positive number. Then there exists a positive number δ such that for each integer $m \geq 1$ and each $y_1, y_2, z_1, z_2 \in X$ which satisfy $d(y_i, z_i) \leq \delta$, $i = 1, 2$ the following inequality holds:*

$$|\sigma(v, m, y_1, y_2) - \sigma(v, m, z_1, z_2)| \leq \epsilon. \tag{10.9}$$

Proof: Since the function v is uniformly continuous (see (10.2)) there is a positive number δ such that for each $y_1, y_2, z_1, z_2 \in X$ satisfying $d(y_i, z_i) \leq \delta$, $i = 1, 2$, the inequality

$$|v(y_1, y_2) - v(z_1, z_2)| \leq \epsilon/8 \tag{10.10}$$

holds.

Let $m \geq 1$ be an integer and let $y_i, z_i \in X$, $i = 1, 2$ satisfy $d(y_i, z_i) \leq \delta$, $i = 1, 2$. We show that (10.9) holds. It is easy to see that we need only to consider the case with $m > 1$. There exists $\{x_i\}_{i=0}^{m} \subset X$ such that

$$x_0 = y_1,\ x_m = y_2$$

and

$$\sum_{i=0}^{m-1} v(x_i, x_{i+1}) \le \sigma(v, m, y_1, y_2) + \epsilon/8.$$

Put

$$\tilde{x}_0 = z_1,\ \tilde{x}_m = z_2,\ \tilde{x}_i = x_i,\ i = 1, \dots, m-1. \tag{10.11}$$

It follows from the definition of $\{\tilde{x}_i\}_{i=0}^m$, the choice of δ, the inequalities $d(y_i, z_i) \le \delta$, $i = 1, 2$ and the choice of $\{x_i\}_{i=0}^m$ that

$$\sigma(v, m, z_1, z_2) \le \sum_{i=0}^{m-1} v(\tilde{x}_i, \tilde{x}_{i+1}) \le \sum_{i=0}^{m-1} v(x_i, x_{i+1})$$

$$+|v(x_0, x_1) - v(\tilde{x}_0, \tilde{x}_1)| + |v(x_{m-1}, x_m) - v(\tilde{x}_{m-1}, \tilde{x}_m)|$$

$$\le \sum_{i=0}^{m-1} v(x_i, x_{i+1}) + \epsilon/8 + \epsilon/8$$

$$\le \sigma(v, m, y_1, y_2) + \epsilon/8 + \epsilon/4.$$

Proposition 10.3 is proved.

Let $v \in \mathcal{A}$. Clearly, for each integer $m \ge 1$,

$$\sigma(v, m) \le \mu(v)m \le \sigma_{per}(v, m) \le \sigma(v, m) + 2||v|| \tag{10.12}$$

and

$$\sigma(v, m, y_1, y_2) \le \sigma(v, m) + 4||v|| \tag{10.13}$$

for each integer $m \ge 1$ and each $y_1, y_2 \in X$.

For each $v \in \mathcal{A}$ and each positive number ϵ put

$$\eta_v(\epsilon) = \sup\{\delta > 0 :\ |v(x_1, x_2) - v(y_1, y_2)| \le \epsilon$$

$$\text{for each } x_1, x_2, y_1, y_2 \in X \text{ such that } d(x_i, y_i) \le \delta,\ i = 1, 2\}. \tag{10.14}$$

For each set A denote by $\mathrm{Card}(A)$ its cardinality.

Let $v \in \mathcal{A}$ and let q be a natural number. Fix

$$\epsilon(v, q) \in (0, 2^{-20} q^{-2} \min\{1, \eta_v(2^{-8} q^{-2})\}) \tag{10.15}$$

and an integer

$$n(v, q) > 64^2 (8||v|| + 48) 4^{2q+1} q \epsilon(v, q)^{-1}. \tag{10.16}$$

There exists a sequence $\{z_i(v, q)\}_{i=-n(v,q)}^{n(v,q)} \subset X$ such that

$$\sum_{i=-n(v,q)}^{n(v,q)-1} v(z_i(v,q), z_{i+1}(v,q)) \le \sigma(v, 2n(v,q)) + 2^{-10}q^{-2}. \tag{10.17}$$

We define a function $u^{(v,q)} : X \times X \to R^1$ by

$$u^{(v,q)}(x,y) = v(x,y)$$

$$+(4q)^{-1} \min\{d_1((x,y), \{(z_i(v,q), z_{i+1}(v,q)) :$$

$$i = -n(v,q), \dots, n(v,q)-1\}), 1\}, \tag{10.18}$$

where $(x,y) \in X \times X$.

Evidently, $u^{(v,q)} \in \mathcal{A}$. Fix

$$\gamma(v,q) \in (0, \epsilon(v,q)(64n(v,q))^{-1}) \tag{10.19}$$

and define

$$\mathcal{U}(v,q) = \{w \in \mathcal{A} : \ \rho(w, u^{(v,q)}) < \gamma(v,q)\}. \tag{10.20}$$

Set

$$\mathcal{F} = \cap_{m=1}^{\infty} \cup \{\mathcal{U}(v,q) : \ v \in \mathcal{A}, \text{ an integer } q \ge m\}. \tag{10.21}$$

It is easy to see that $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{A}$.

Lemma 10.4. *Assume that $v \in \mathcal{A}$, $q \ge 1$ is an integer, $\{z_i\}_{i=-n(v,q)}^{n(v,q)} \subset X$ and*

$$\sum_{i=-n(v,q)}^{n(v,q)-1} u^{(v,q)}(z_i, z_{i+1}) \le \sigma(u^{(v,q)}, 2n(v,q)) + 8(||v|| + 2). \tag{10.22}$$

Then

$$\mathrm{Card}\{i \in \{-n(v,q), \dots, n(v,q)-1\} :$$

$$d_1((z_i, z_{i+1}), \{(z_j(v,q), z_{j+1}(v,q)) : \ j = -n(v,q), \dots, n(v,q)-1\}) \ge \epsilon(v,q)\}$$

$$\le 8(||v|| + 3)4q\epsilon(v,q)^{-1}. \tag{10.23}$$

Proof: It follows from (10.18), (10.22) and (10.17) that

$$\sum_{i=-n(v,q)}^{n(v,q)-1} v(z_i, z_{i+1})$$

$$+(4q)^{-1} \sum_{i=-n(v,q)}^{n(v,q)-1} \min\{1, d_1((z_i, z_{i+1}), \{(z_j(v,q), z_{j+1}(v,q)) :$$

$$j = -n(v,q), \dots, n(v,q)-1\})\}$$

$$= \sum_{i=-n(v,q)}^{n(v,q)-1} u^{(v,q)}(z_i, z_{i+1}) \leq \sigma(u^{(v,q)}, 2n(v,q)) + 8(||v|| + 2)$$

$$\leq \sum_{i=-n(v,q)}^{n(v,q)-1} u^{(v,q)}(z_i(v,q), z_{i+1}(v,q)) + 8(||v|| + 2)$$

$$= \sum_{i=-n(v,q)}^{n(v,q)-1} v(z_i(v,q), z_{i+1}(v,q)) + 8(||v|| + 2)$$

$$\leq \sigma(v, 2n(v,q)) + 2^{-10}q^{-2} + 8(||v|| + 2)$$

$$\leq \sum_{i=-n(v,q)}^{n(v,q)-1} v(z_i, z_{i+1}) + 2^{-10}q^{-2} + 8(||v|| + 2),$$

$$\sum_{i=-n(v,q)}^{n(v,q)-1} \min\{1, d_1((z_i, z_{i+1}), \{(z_j(v,q), z_{j+1}(v,q)) :$$
$$j = -n(v,q), \dots, n(v,q) - 1\})\} \leq 32q(||v|| + 3)$$

and

$$\mathrm{Card}\{i \in \{-n(v,q), \dots, n(v,q) - 1\} :$$
$$d_1((z_i, z_{i+1}), \{(z_j(v,q), z_{j+1}(v,q)) : \ j = -n(v,q), \dots, n(v,q) - 1\}) \geq \epsilon(v,q)\}$$
$$\leq 8(||v|| + 3)4q\epsilon(v,q)^{-1}.$$

This completes the proof of the lemma.

Lemma 10.4, (10.19) and (10.20) imply the following lemma.

Lemma 10.5. *Assume that $v \in \mathcal{A}$, $q \geq 1$ is an integer, $w \in \mathcal{U}(v,q)$,*

$$\{z_i\}_{i=-n(v,q)}^{n(v,q)} \subset X$$

and

$$\sum_{i=-n(v,q)}^{n(v,q)-1} w(z_i, z_{i+1}) \leq \sigma(w, 2n(v,q)) + 8||w|| + 1.$$

Then the inequality (10.23) holds.

Lemma 10.5, (10.23) and (10.16) imply the following result.

Lemma 10.6. *Assume that $v \in \mathcal{A}$, $q \geq 1$ is an integer, $w \in \mathcal{U}(v,q)$ and a sequence*

$$\{z_i\}_{i=-n(v,q)}^{n(v,q)} \subset X$$

satisfies

$$\sum_{i=-n(v,q)}^{n(v,q)-1} w(z_i, z_{i+1}) \le \sigma(w, 2n(v,q)) + 8||w|| + 1.$$

Then there exists an integer i_0 such that

$$|i_0| \le 8(||v|| + 3)4q\epsilon(v,q)^{-1},$$

$$d(z_p, \{z_i(v,q) :\ i = -n(v,q), \dots, n(v,q)\}) \le \epsilon(v,q),$$
$$p = 4^q i_0, \dots, 4^q i_0 + 4^q.$$

For $v \in \mathcal{A}$ and an integer $q \ge 1$ put

$$E(v,q) = \{z_i(v,q) :\ i = -n(v,q), \dots, n(v,q)\}. \tag{10.24}$$

10.3 Proof of Theorem 10.2

Let $w \in \mathcal{F}$. In view of (10.21) there exist a natural number s_1 and $v_1 \in \mathcal{A}$ such that

$$w \in \mathcal{U}(v_1, s_1). \tag{10.25}$$

Using (10.21) we construct by induction a strictly increasing sequence of natural numbers $\{s_k\}_{k=1}^{\infty}$ and a sequence $\{v_k\}_{k=1}^{\infty} \subset \mathcal{A}$ such that for each natural number k

$$s_{k+1} > [16(1 + s_k + n(s_k, v_k))]^2 \text{ and } w \in \mathcal{U}(v_k, s_k). \tag{10.26}$$

Lemma 10.7. *Let k be a natural number integer and let a sequence*

$$\{y_i\}_{i=-n(v_k,s_k)}^{n(v_k,s_k)} \subset X$$

satisfy

$$\sum_{i=-n(v_k,s_k)}^{n(v_k,s_k)-1} w(y_i, y_{i+1}) \le \sigma(w, 2n(v_k, s_k), y_{-n(v_k,s_k)}, y_{n(v_k,s_k)}) + k^{-1}. \tag{10.27}$$

Then there exists an integer i_k such that

$$|i_k| \le 8(||w|| + 4)4s_k\epsilon(v_k, s_k)^{-1}$$

and

$$d(y_p, E(v_k, s_k)) \le \epsilon(v_k, s_k),\ p = 4^{s_k} i_k, \dots, 4^{s_k} i_k + 4^{s_k}. \tag{10.28}$$

Proof: By (10.13) and (10.27),

$$\sum_{i=-n(v_k,s_k)}^{n(v_k,s_k)-1} w(y_i,y_{i+1}) \le \sigma(w,2n(v_k,s_k),y_{-n(v_k,s_k)},y_{n(v_k,s_k)}) + k^{-1}$$

$$\le \sigma(w,2n(v_k,s_k)) + 4||w|| + k^{-1}.$$

It follows from the relation above, (10.26) and Lemma 10.6 that there exists an integer i_k for which

$$|i_k| \le 8(||v_k||+3)4s_k\epsilon(v_k,s_k)^{-1}$$

and (10.28) holds. By the inequality above, (10.26) and (10.20),

$$|i_k| \le 8(||w||+4)4s_k\epsilon(v_k,s_k)^{-1}.$$

This completes the proof of Lemma 10.7.

Lemma 10.8. *Let $q \ge 1$ be an integer, $\{y_i\}_{i=-n(v_q,s_q)}^{n(v_q,s_q)} \subset X$ and let*

$$\sum_{i=-n(v_q,s_q)}^{n(v_q,s_q)-1} w(y_i,y_{i+1}) \le \sigma(w,2n(v_q,s_q),y_{-n(v_q,s_q)},y_{n(v_q,s_q)}) + q^{-1}. \quad (10.29)$$

Then there exists an integer p such that for each natural number $k = 1,\dots,q$,

$$[p-4^{s_k-1},p+4^{s_k-1}] \subset [-n(v_q,s_q),n(v_q,s_q)] \quad (10.30)$$

and

$$d(y_{p+i},E(v_k,s_k)) \le \epsilon(v_k,s_k),\ i=-4^{s_k-1},\dots,4^{s_k-1}. \quad (10.31)$$

Proof: By induction we define a sequence $p_q,\dots,p_1$ such that

$$[p_q-2\cdot 4^{s_q-1},p_q+2\cdot 4^{s_q-1}] \subset [-n(v_q,s_q),n(v_q,s_q)], \quad (10.32)$$

$$d(y_{p_q+i},E(v_q,s_q)) \le \epsilon(v_q,s_q), \quad (10.33)$$

$$i=-2\cdot 4^{s_q-1},\dots,2\cdot 4^{s_q-1};$$

for each integer j satisfying $q \ge j > 1$,

$$|p_j-p_{j-1}| < 2^{s_j}64^{-1} \quad (10.34)$$

and

$$[p_{j-1}-2\cdot 4^{s_{j-1}-1},p_{j-1}+2\cdot 4^{s_{j-1}-1}] \subset [p_j-2\cdot 4^{s_j-1},p_j+2\cdot 4^{s_j-1}]; \quad (10.35)$$

for $j=1,\dots,q$,

$$d(y_{p_j+i},E(v_j,s_j)) \le \epsilon(v_j,s_j), \quad (10.36)$$

$$i = -2 \cdot 4^{s_j - 1}, \ldots, 2 \cdot 4^{s_j - 1}.$$

Let $j = q$. It follows from Lemma 10.7 and (10.29) that there exists an integer i_q which satisfies

$$|i_q| \le 8(||w|| + 4)4s_q \epsilon(v_q, s_q)^{-1}, \tag{10.37}$$

$$d(y_i, E(v_q, s_q)) \le \epsilon(v_q, s_q),\ i = 4^{s_q} i_q, \ldots, 4^{s_q} i_q + 4^{s_q}. \tag{10.38}$$

Put

$$p_q = 4^{s_q} i_q + 2 \cdot 4^{s_q - 1}. \tag{10.39}$$

By (10.39), (10.37), (10.26), (10.20) and (10.16),

$$|p_q| \le 2 \cdot 4^{s_q - 1} + 4^{s_q} \cdot 8(||w|| + 4)4s_q \epsilon(v_q, s_q)^{-1}, \tag{10.40}$$

$$|p_q| + 2 \cdot 4^{s_q - 1} \le 4^{s_q}[8(||w|| + 4)4s_q \epsilon(v_q, s_q)^{-1} + 1] < n(v_q, s_q)/64. \tag{10.41}$$

Inequality (10.41) implies (10.32). It follows from (10.38) and (10.39) that (10.33) holds.

Assume that a natural number k satisfies $q \ge k > 1$ and we have defined integers p_i, $i = q, \ldots, k$ such that (10.32) and (10.33) hold, (10.34) and (10.35) are true for all integers j satisfying $q \ge j > k$ and (10.36) holds for $j = 1, \ldots, k$.

In order to define p_{k-1} we use Lemma 10.7 with the sequence

$$\{y_{p_k + i}\}_{i=-n(v_{k-1}, s_{k-1})}^{n(v_{k-1}, s_{k-1})}.$$

By (10.26),

$$4^{s_k} > s_k > [16(1 + s_{k-1} + n(s_{k-1}, v_{k-1}))]^2. \tag{10.42}$$

By (10.42), (10.29), (10.35) and (10.32),

$$\sum_{i=-n(v_{k-1}, s_{k-1})}^{n(v_{k-1}, s_{k-1}) - 1} w(y_{p_k + i}, y_{p_k + i + 1})$$

$$\le \sigma(w, 2n(v_{k-1}, s_{k-1}), y_{p_k - n(v_{k-1}, s_{k-1})}, y_{p_k + n(v_{k-1}, s_{k-1})}) + q^{-1}. \tag{10.43}$$

Lemma 10.7 and (10.43) imply that there exists an integer i_{k-1} for which

$$|i_{k-1}| \le 8(||w|| + 4)4s_{k-1} \epsilon(v_{k-1}, s_{k-1})^{-1}, \tag{10.44}$$

$$d(y_{p_k + i}, E(v_{k-1}, s_{k-1})) \le \epsilon(v_{k-1}, s_{k-1}), \tag{10.45}$$

$$i = 4^{s_{k-1}} i_{k-1}, \ldots, 4^{s_{k-1}} i_{k-1} + 4^{s_{k-1}}. \tag{10.46}$$

Put

$$p_{k-1} = p_k + 4^{s_{k-1}} i_{k-1} + 2 \cdot 4^{s_{k-1} - 1}. \tag{10.47}$$

By (10.47), (10.44), (10.26), (10.16) and (10.20),

$$|p_{k-1} - p_k| \le 4^{s_{k-1} - 1}(2 + 4|i_{k-1}|) \le 4^{s_{k-1}}(1 + |i_{k-1}|)$$

$$\leq 4^{s_{k-1}}(1 + 8(||w|| + 4)4s_{k-1}\epsilon(v_{k-1}, s_{k-1})^{-1})$$
$$\leq 4^{s_{k-1}}(1 + 8(||v_{k-1}|| + 5)4s_{k-1}\epsilon(v_{k-1}, s_{k-1})^{-1})$$
$$\leq n(v_{k-1}, s_{k-1})4^{-s_{k-1}}64^{-1} < s_k/64$$

and

$$|p_{k-1} - p_k| < s_k/64 < 2^{s_k}/64. \tag{10.48}$$

Therefore (10.34) holds with $j = k$ and this implies that (10.35) holds with $j = k$. It follows from (10.45) and (10.47) that (10.36) is valid with $j = k - 1$.

Therefore by induction we have defined integers $p_q, \ldots, p_1$ such that (10.32) and (10.33) are true, (10.34) and (10.35) hold for each natural number j satisfying $q \geq j > 1$ and (10.36) holds for $j = 1, \ldots, q$.

Assume that an integer k satisfies $1 < k \leq q$. It follows from (10.34), (10.35) and (10.32) that

$$|p_k - p_1| \leq \sum_{j=2}^{k} |p_j - p_{j-1}| < \sum_{j=2}^{k} 2^{s_j}64^{-1} < 64^{-1}2^{s_k+1} \leq 2^{s_k}/32,$$

$$p_k - 2 \cdot 4^{s_k-1} \leq p_1 + 32^{-1}2^{s_k} - 2 \cdot 4^{s_k-1} < p_1 - 2 \cdot 4^{s_k-1}$$
$$+32^{-1}4^{s_k} \leq p_1 - 4^{-1} \cdot 4^{s_k} = p_1 - 4^{s_k-1},$$
$$p_k + 2 \cdot 4^{s_k-1} \geq p_1 - 32^{-1}2^{s_k} + 2 \cdot 4^{s_k-1} \geq p_1 + 4^{s_k-1}$$

and

$$[p_1 - 4^{s_k-1}, p_1 + 4^{s_k-1}] \subset [p_k - 2 \cdot 4^{s_k-1}, p_k + 2 \cdot 4^{s_k-1}]$$
$$\subset [-n(v_q, s_q), n(v_q, s_q)]. \tag{10.49}$$

It follows from (10.49) and (10.36) that for each integer $i \in [p_1 - 4^{s_k-1}, p_1 + 4^{s_k-1}]$,

$$d(y_i, E(v_k, s_k)) \leq \epsilon(v_k, s_k). \tag{10.50}$$

We conclude that for each natural number $k = 1, \ldots, q$,

$$[p_1 - 4^{s_k-1}, p_1 + 4^{s_k-1}] \subset [-n(v_q, s_q), n(v_q, s_q)]$$

and that for each $k = 1, \ldots, q$ the inequality (10.50) is valid for all integers $i \in [p_1 - 4^{s_k-1}, p_1 + 4^{s_k-1}]$. This completes the proof of the lemma.

For each integer $k \geq 1$ define

$$\Omega_k = \{z \in X : \ d(z, E(v_k, s_k)) \leq \epsilon(v_k, s_k)\}, \tag{10.51}$$

$$\Omega_* = \cap_{k=1}^{\infty} \Omega_k. \tag{10.52}$$

It is easy to see that Ω_* is a compact subset of X.

Lemma 10.9. *Assume that for each integer $q \geq 1$ a sequence*

$$\{y_i^{(q)}\}_{i=-n(v_q,s_q)}^{n(v_q,s_q)} \subset X$$

satisfies

$$\sum_{i=-n(v_q,s_q)}^{n(v_q,s_q)-1} w(y_i^{(q)}, y_{i+1}^{(q)}) \leq \sigma(w, 2n(v_q, s_q), y_{-n(v_q,s_q)}, y_{n(v_q,s_q)}) + q^{-1}. \quad (10.53)$$

Then for each integer $q \geq 1$ there exists an integer p_q such that

$$[p_q - 4^{s_q-1}, p_q + 4^{s_q-1}] \subset [-n(v_q, s_q), n(v_q, s_q)] \quad (10.54)$$

and there exist a strictly increasing sequence of natural numbers $\{q_k\}_{k=1}^\infty$ and a (w)-minimal sequence $\{x_i^*\}_{i=-\infty}^\infty \subset \Omega_*$ such that for each integer i,

$$x_i^* = \lim_{k\to\infty} y_{p_{q_k}+i}^{(q_k)}. \quad (10.55)$$

Proof: Let $q \geq 1$ be an integer. In view of Lemma 10.9 and (10.53) there exists an integer p_q such that for each integer $j = 1, \dots, q$,

$$[p_q - 4^{s_j-1}, p_q + 4^{s_j-1}] \subset [-n(v_q, s_q), n(v_q, s_q)],$$

$$d(y_{p_q+i}^{(q)}, E(v_j, s_j)) \leq \epsilon(v_j, s_j),\ i = -4^{s_j-1}, \dots, 4^{s_j-1}. \quad (10.56)$$

Let l be an integer. Fix an integer $k_0 \geq 1$ such that

$$|l| < 4^{s_k-1} \text{ for all } k \geq k_0. \quad (10.57)$$

It follows from (10.56) and (10.57) that for each integer $k \geq k_0$ and each integer $q \geq k$,

$$d(y_{p_q+l}^{(q)}, E(v_k, s_k)) \leq \epsilon(v_k, s_k). \quad (10.58)$$

Let $\{q_i\}_{i=1}^\infty$ be a strictly increasing sequence of natural numbers. Using (10.58) we construct by induction for each integer $k \geq k_0$ a subsequence $\{q_i^{(k)}\}_{i=1}^\infty$ of the sequence $\{q_i\}_{i=1}^\infty$ such that the following properties hold:

(a) for each integer $k \geq k_0$ the sequence $\{q_i^{(k+1)}\}_{i=1}^\infty$ is a subsequence of $\{q_i^{(k)}\}_{i=1}^\infty$;

(b) for each integer $k \geq k_0$ and each pair of natural numbers i, j

$$d(y_{p_{q_i^{(k)}}+l}^{(q_i^{(k)})}, y_{p_{q_j^{(k)}}+l}^{(q_j^{(k)})}) \leq 2\epsilon(v_k, s_k).$$

These properties imply that there exists a subsequence $\{\tilde{q}_i\}_{i=1}^\infty$ of the sequence $\{q_i\}_{i=1}^\infty$ such that the sequence $\{y_{p_{\tilde{q}_i}+l}^{(\tilde{q}_i)}\}_{i=1}^\infty$ is a Cauchy sequence. Therefore there exists

$$\lim_{i\to\infty} y^{(\tilde{q}_i)}_{p_{\tilde{q}_i}+l}.$$

By (10.58) for each natural number $k \ge k_0$ and each $\tilde{q}_i \ge k$,

$$d(y^{(\tilde{q}_i)}_{p_{\tilde{q}_i}+l}, E(v_k, s_k)) \le \epsilon(v_k, s_k)$$

and

$$\lim_{i\to\infty} y^{(\tilde{q}_i)}_{p_{\tilde{q}_i}+l} \in \Omega_*.$$

Thus we have shown that for each strictly increasing sequence of natural numbers $\{q_i\}_{i=1}^{\infty}$ there exists a subsequence $\{\tilde{q}_i\}_{i=1}^{\infty}$ such that there is

$$\lim_{i\to\infty} y^{(\tilde{q}_i)}_{p_{\tilde{q}_i}+l} \in \Omega_*.$$

Since l is an arbitrary integer this fact implies that there exists a strictly increasing sequence of natural numbers $\{q_i\}_{i=1}^{\infty}$ such that for each integer l there is

$$\lim_{i\to\infty} y^{(q_i)}_{p_{q_i}+l} \in \Omega_*. \tag{10.59}$$

For each integer l set

$$x^*_l = \lim_{i\to\infty} y^{(q_i)}_{p_{q_i}+l}. \tag{10.60}$$

It follows from (10.59) and (10.60) that $x^*_l \in \Omega_*$ for all integers l.

Let $j_1 < j_2$ be integers. By (10.60), (10.53) and Proposition 10.3,

$$\sum_{j=j_1}^{j_2-1} w(x^*_j, x^*_{j+1}) = \lim_{i\to\infty} \sum_{j=j_1}^{j_2-1} w(y^{(q_i)}_{p_{q_i}+j}, y^{(q_i)}_{p_{q_i}+j+1})$$

$$\le \limsup_{i\to\infty} [\sigma(w, j_2 - j_1, y^{(q_i)}_{p_{q_i}+j_1}, y^{(q_i)}_{p_{q_i}+j_2}) + q_i^{-1}]$$

$$= \sigma(w, j_2 - j_1, x^*_{j_1}, x^*_{j_2})$$

and

$$\sum_{j=j_1}^{j_2-1} w(x^*_j, x^*_{j+1}) = \sigma(w, j_2 - j_1, x^*_{j_1}, x^*_{j_2}).$$

This completes the proof of the lemma.

Completion of the proof of Theorem 10.2. Put $\Omega(v) = \Omega_*$. Lemma 10.9 implies that $\Omega(v)$ is nonempty. Part (i) of Theorem 10.2 follows from Lemma 10.9.

Now we prove part (ii). Let $\{y_i\}_{i=0}^{\infty} \subset X$ be a (w)-good sequence. Since $\{y_i\}_{i=0}^{\infty}$ is (w)-good for each positive number ϵ there exists an integer $n_\epsilon \ge 1$ such that for each pair of integers m_1, m_2 satisfying $m_2 > m_1 > n_\epsilon$,

$$\sum_{i=m_1}^{m_2-1} w(y_i, y_{i+1}) \le \sigma(w, m_2 - m_1, y_{m_1}, y_{m_2}) + \epsilon.$$

This property implies that there exists a strictly increasing sequence of natural numbers $\{m_q\}_{q=1}^{\infty}$ such that for each integer $q \geq 1$,

$$m_{q+1} > m_q + 4n(v_q, s_q) \tag{10.61}$$

and

$$\sum_{i=m_q}^{m_q+2n(v_q,s_q)-1} w(y_i, y_{i+1}) \leq \sigma(w, 2n(v_q, s_q), y_{m_q}, y_{m_q+2n(v_q,s_q)}) + q^{-1}. \tag{10.62}$$

By (10.62) and Lemma 10.9, for each integer $q \geq 1$ there exists an integer p_q such that

$$[p_q - 4^{s_q-1}, p_q + 4^{s_q-1}] \subset [-n(v_q, s_q), n(v_q, s_q)]$$

and there exist a strictly increasing sequence of natural numbers $\{q_k\}_{k=1}^{\infty}$ and a (w)-minimal sequence $\{\tilde{y}_i\}_{i=-\infty}^{\infty} \subset \Omega_*$ such that for each integer i,

$$\tilde{y}_i = \lim_{k\to\infty} y_{i+p_{q_k}+n(v_q,s_q)+m_q}.$$

It is easy to see that $\{\tilde{y}_i\}_{i=-\infty}^{\infty} \subset \omega(\{y_i\}_{i=0}^{\infty})$. Theorem 10.2 is proved.

10.4 Properties of good sequences

Let $v \in \mathcal{A}$. It follows from (10.12) that for each integer $m \geq 1$

$$\sigma(v, m) \leq \mu(v)m \leq \sigma(v, m) + 2||v||. \tag{10.63}$$

(10.63) implies the following result.

Proposition 10.10. *For each $v \in \mathcal{A}$, each $\{x_i\}_{i=0}^{\infty} \subset X$ and each natural number n*

$$\sum_{i=0}^{n-1} [v(x_i, x_{i+1}) - \mu(v)] \geq \sigma(v, n) - n\mu(v) \geq -2||v||.$$

Proposition 10.10 implies the following proposition.

Proposition 10.11. *For each $v \in \mathcal{A}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ either the sequence $\{\sum_{i=0}^{n-1}[v(x_i, x_{i+1}) - \mu(v)]\}_{n=1}^{\infty}$ is bounded or diverges to ∞.*

Proposition 10.12. *A sequence $\{x_i\}_{i=0}^{\infty} \subset X$ is (v)-good if and only if the sequence $\{\sum_{i=0}^{n-1}[v(x_i, x_{i+1}) - \mu(v)]\}_{n=1}^{\infty}$ is bounded.*

Proof: Assume that $\{x_i\}_{i=0}^{\infty}$ is (v)-good. There exists a positive number M such that for each integer $m \geq 1$,

$$\sum_{i=0}^{m-1} v(x_i, x_{i+1}) \leq \sigma(v, m, x_0, x_m) + M.$$

By this inequality, (10.13) and (10.12), for each integer $m \geq 1$,

$$\sum_{i=0}^{m-1} v(x_i, x_{i+1}) \leq \sigma(v, m, x_0, x_m) + M \leq \sigma(v, m) + M + 4||v||$$

$$\leq m\mu(v) + 4 + 4||v|| + M$$

and by Proposition 10.11 the sequence $\{\sum_{i=0}^{n-1}[v(x_i, x_{i+1}) - \mu(v)]\}_{n=1}^{\infty}$ is bounded.

Assume that the sequence $\{\sum_{i=0}^{n-1}[v(x_i, x_{i+1}) - \mu(v)]\}_{n=1}^{\infty}$ is bounded. There exists a positive number M such that for each natural number n

$$\sum_{i=0}^{n-1}[v(x_i, x_{i+1}) - \mu(v)] \leq M, \ \sum_{i=0}^{n-1} v(x_i, x_{i+1}) \leq M + n\mu(v)$$

and then in view of (10.12) for each natural number n,

$$\sum_{i=0}^{n-1} v(x_i, x_{i+1}) \leq M + n\mu(v) \leq M + \sigma(v, n) + 2||v|| \leq$$

$$M + ||v|| + \sigma(v, n, x_0, x_n).$$

Thus $\{x_i\}_{i=0}^{\infty}$ is (v)-good. The proposition is proved.

10.5 Convex discrete-time control systems in a Banach space

Let $(X, ||\cdot||)$ be a Banach space and let $K \subset X$ be a nonempty closed convex bounded set. Denote by $\mathcal{A}$ the set of all bounded convex functions $v : K \times K \to R^1$ which are continuous at a point (x, x) for any $x \in K$. Denote by $\mathcal{A}_l$ the set of all lower semicontinuous functions $v \in \mathcal{A}$, by $\mathcal{A}_c$ the set of all continuous functions $v \in \mathcal{A}$ and by $\mathcal{A}_u$ the set of all functions $v \in \mathcal{A}$ which satisfy the following assumption:

(uniform continuity) for each positive number ϵ there exists a positive number δ such that for each $x_1, x_2, y_1, y_2 \in K$ satisfying $||x_i - y_i|| \leq \delta$, $i = 1, 2$, the inequality $|v(x_1, x_2) - v(y_1, y_2)| \leq \epsilon$ is true.

We equip the space $\mathcal{A}$ with the metric

$$\rho(u, v) = \sup\{|v(x, y) - u(x, y)| : x, y \in K\}, \ u, v \in \mathcal{A}. \qquad (10.64)$$

Evidently the metric space $(\mathcal{A}, \rho)$ is complete and $\mathcal{A}_l$, $\mathcal{A}_c$ and $\mathcal{A}_u$ are closed subsets of $(\mathcal{A}, \rho)$. We equip the sets $\mathcal{A}_l$, $\mathcal{A}_c$ and $\mathcal{A}_u$ with the metric ρ.

We investigate the structure of "approximate" solutions of optimization problems

$$\text{minimize } \sum_{i=0}^{n-1} v(x_i, x_{i+1}) \tag{P}$$

$$\text{subject to } \{x_i\}_{i=0}^{n} \subset K, \quad x_0 = y, \; x_n = z$$

where $v \in \mathcal{A}$, $y, z \in K$ and n is a natural number.

For each $v \in \mathcal{A}$, each pair of integers $m_1, m_2 > m_1$ and each $y_1, y_2 \in K$ we define

$$\sigma(v, m_1, m_2) = \inf\{\sum_{i=m_1}^{m_2-1} v(z_i, z_{i+1}) : \{z_i\}_{i=m_1}^{m_2} \subset K\}, \tag{10.65}$$

$$\sigma(v, m_1, m_2, y_1, y_2) = \inf\{\sum_{i=m_1}^{m_2-1} v(z_i, z_{i+1}) : \{z_i\}_{i=m_1}^{m_2} \subset K,$$

$$z_{m_1} = y_1, z_{m_2} = y_2\}, \tag{10.66}$$

and the minimal growth rate

$$\mu(v) = \inf\{\liminf_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^{\infty} \subset K\}. \tag{10.67}$$

We show (see Proposition 10.15) that for any $v \in \mathcal{A}$,

$$\mu(v) = \inf\{v(z, z) : \; z \in K\}. \tag{10.68}$$

We construct a set $\mathcal{F}$ ($\mathcal{F}_l, \mathcal{F}_c, \mathcal{F}_u$, respectively) which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ ($\mathcal{A}_l, \mathcal{A}_c, \mathcal{A}_u$, respectively) and such that

$$\mathcal{F}_l \subset \mathcal{A}_l \cap \mathcal{F}, \; \mathcal{F}_c \subset \mathcal{A}_c \cap \mathcal{F}, \; \mathcal{F}_u \subset \mathcal{A}_u \cap \mathcal{F}. \tag{10.69}$$

We prove the following two theorems obtained in [117].

Theorem 10.13. *Let $v \in \mathcal{F}$. Then there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$ and the following assertion holds:*

For each positive number ϵ there exist a neighborhood $\mathcal{U}$ of v in $\mathcal{A}$ and a positive number δ such that for each $u \in \mathcal{U}$ and each $y \in K$ satisfying $u(y, y) \le \mu(u) + \delta$ the inequality $||y - y_v|| \le \epsilon$ holds.

Theorem 10.14. *Let $w \in \mathcal{F}$ and ϵ be a positive number. Then there exist $\delta \in (0, \epsilon)$, a neighborhood $\mathcal{U}$ of w in $\mathcal{A}$ and a natural number N such that for each $u \in \mathcal{U}$, each natural number $n \ge 2N$ and each sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfying*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + \delta \tag{10.70}$$

there exist $\tau_1 \in \{0, \dots, N\}$ *and* $\tau_2 \in \{n - N, \dots, n\}$ *such that*

$$||x_t - y_w|| \leq \epsilon, \ t = \tau_1, \dots, \tau_2.$$

Moreover, if $||x_0 - y_w|| \leq \delta$*, then* $\tau_1 = 0$*, and if* $||y_w - x_n|| \leq \delta$*, then* $\tau_2 = n$*.*

10.6 Preliminary results

Put

$$D_0 = \sup\{||x|| : x \in K\}. \tag{10.71}$$

For each bounded function $v : K \times K \to R^1$ we set

$$||v|| = \sup\{|v(x, y)| : x, y \in K\}. \tag{10.72}$$

Proposition 10.15. *Let* $v \in \mathcal{A}$*. Then*

$$\mu(v) = \inf\{v(z, z) : z \in K\}.$$

Proof: It is easy to see that

$$\inf\{v(x, x) : x \in K\} \geq \mu(v) \tag{10.73}$$

and that for each natural number m and each $y_1, y_2 \in K$,

$$\sigma(v, 0, m, y_1, y_2) - 4||v|| \leq \sigma(v, 0, m) \leq m\mu(v). \tag{10.74}$$

Let ϵ be a positive number. Fix an integer $m \geq 4$ such that

$$(4||v|| + 1)m^{-1} < \epsilon. \tag{10.75}$$

There exists a sequence $\{y_i\}_{i=0}^{m} \subset K$ such that

$$y_0 = y_m \text{ and } \sigma(v, 0, m, y_0, y_0) + 1 \geq \sum_{i=0}^{m-1} v(y_i, y_{i+1}). \tag{10.76}$$

It follows from (10.76) and (10.74) that

$$\sum_{i=0}^{m-1} v(y_i, y_{i+1}) \leq m\mu(v) + 4||v|| + 1. \tag{10.77}$$

Put

$$z_0 = m^{-1} \sum_{i=0}^{m-1} y_i. \tag{10.78}$$

It follows from (10.75)–(10.78) and the convexity of v that

$$v(z_0, z_0) = v(m^{-1} \sum_{i=0}^{m-1} (y_i, y_{i+1})) \le m^{-1} \sum_{i=0}^{m-1} v(y_i, y_{i+1}) \le$$

$$m^{-1}(m\mu(v) + 4||v|| + 1) = \mu(v) + (4||v|| + 1)m^{-1} < \mu(v) + \epsilon$$

and

$$v(z_0, z_0) < \mu(v) + \epsilon.$$

Therefore

$$\inf\{v(z, z) : \ z \in K\} < \mu(v) + \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that

$$\inf\{v(z, z) : \ z \in K\} \le \mu(v).$$

Combined with (10.73) this completes the proof of Proposition 10.15.

In [104] Proposition 10.15 was proved for $v \in \mathcal{A}_u$. In the proof we used uniform continuity of v.

Proposition 10.16. *Let $v \in \mathcal{A}$, $\epsilon \in (0, 1)$. Then there exist $\delta \in (0, \epsilon)$, $u \in \mathcal{A}$ and $z_0 \in K$ such that*

$$0 \le u(x, y) - v(x, y) \le \epsilon, \ x, y \in K, \ \mu(v) + \delta \ge v(z_0, z_0) \tag{10.79}$$

and for each $y \in K$ satisfying $u(y, y) \le \mu(u) + \delta$ the inequality $||y - z_0|| \le \epsilon$ holds.

Moreover, if $v \in \mathcal{A}_l$ ($\mathcal{A}_c$, $\mathcal{A}_u$, respectively), then $u \in \mathcal{A}_l$ ($\mathcal{A}_c$, $\mathcal{A}_u$, respectively).

Proof: Fix a pair of positive numbers δ, γ such that

$$\gamma(8D_0 + 4) \le \epsilon, \quad \delta < 8^{-1}\gamma\epsilon. \tag{10.80}$$

In view of Proposition 10.15 there exists $z_0 \in K$ such that $v(z_0, z_0) < \mu(v) + \delta$. Define $u : K \times K \to R^1$ as

$$u(x, y) = v(x, y) + \gamma(||x - z_0|| + ||y - z_0||), \ x, y \in K. \tag{10.81}$$

Clearly, (10.79) is valid.

Assume that $y \in K$ and

$$u(y, y) \le \mu(u) + \delta. \tag{10.82}$$

By Proposition 10.15, (10.81), (10.82), (10.80) and the choice of z_0,

$$\mu(v) \le \mu(u) \le u(z_0, z_0) = v(z_0, z_0) \le \mu(v) + \delta,$$

$$2\gamma||y - z_0|| + v(y, y) = u(y, y) \le \mu(u) + \delta$$

$$\le \mu(v) + 2\delta \le v(y, y) + 2\delta, \quad ||y - z_0|| \le \delta\gamma^{-1} < \epsilon.$$

The proposition is proved.

Proposition 10.17. *There exist sets $\mathcal{F}^{(0)} \subset \mathcal{A}$, $\mathcal{F}_l^{(0)} \subset \mathcal{A}_l \cap \mathcal{F}^{(0)}$, $\mathcal{F}_c^{(0)} \subset \mathcal{A}_c \cap \mathcal{F}^{(0)}$ and $\mathcal{F}_u^{(0)} \subset \mathcal{A}_u \cap \mathcal{F}^{(0)}$ such that $\mathcal{F}^{(0)}$ ($\mathcal{F}_l^{(0)}, \mathcal{F}_c^{(0)}$ and $\mathcal{F}_u^{(0)}$, respectively) is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ ($\mathcal{A}_l$, $\mathcal{A}_c$ and $\mathcal{A}_u$, respectively) and such that for each $v \in \mathcal{F}^{(0)}$ the following assertions hold:*

(i) there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$;

(ii) for each positive number ϵ there exist a neighborhood $\mathcal{U}$ of v in $\mathcal{A}$ and a positive number δ such that for each $u \in \mathcal{U}$ and each $y \in K$ satisfying $u(y, y) \le \mu(u) + \delta$ the inequality $||y - y_v|| \le \epsilon$ holds.

Proof: Let $w \in \mathcal{A}$ and i be a natural number. In view of Proposition 10.16 there exist $\delta(w, i) \in (0, 4^{-i})$, $u^{(w,i)} \in \mathcal{A}$ and $z(w, i) \in K$ such that

$$0 \le u^{(w,i)}(x, y) - w(x, y) \le 4^{-i}, \ x, y \in K, \tag{10.83}$$

$$w(z(w, i), z(w, i)) \le \mu(w) + \delta(w, i),$$

if $w \in \mathcal{A}_l$ ($\mathcal{A}_c$, $\mathcal{A}_u$, respectively), then $u^{(w,i)} \in \mathcal{A}_l$ ($\mathcal{A}_c$, $\mathcal{A}_u$, respectively), and for each $z \in K$ satisfying

$$u^{(w,i)}(z, z) \le \mu(u^{(w,i)}) + \delta(w, i)$$

the inequality $||z - z(w, i)|| \le 4^{-i}$ is true.

Define

$$\mathcal{U}(w, i) = \{u \in \mathcal{A} : \rho(u, u^{(w,i)}) < 8^{-1}\delta(w, i)\}. \tag{10.84}$$

It is not difficult to see that the following property holds:

(a) For each $u \in \mathcal{U}(w, i)$ and for each $z \in K$ satisfying

$$u(z, z) \le \mu(u) + 8^{-1}\delta(w, i)$$

the inequality $||z - z(w, i)|| \le 4^{-i}$ holds.

Define

$$\mathcal{F}_0 = \cap_{q=1}^{\infty} \cup \{\mathcal{U}(w, i) : w \in \mathcal{A}, \ i = q, q + 1, \dots\}. \tag{10.85}$$

It is clear that $\mathcal{F}_0$ is a countable intersection of open everywhere dense subsets of $\mathcal{A}$. Set

$$\mathcal{F}_l^{(0)} = \mathcal{A}_l \cap \mathcal{F}^{(0)} \ \mathcal{F}_c^{(0)} = \mathcal{A}_c \cap \mathcal{F}^{(0)}, \ \mathcal{F}_u^{(0)} = \mathcal{A}_u \cap \mathcal{F}^{(0)}. \tag{10.86}$$

Assume that $v \in \mathcal{F}_0$. We show that assertions (i) and (ii) are valid. There exists a sequence $\{x_j\}_{j=1}^{\infty} \subset K$ such that

$$\lim_{j \to \infty} v(x_j, x_j) = \mu(v). \tag{10.87}$$

By the definition of $\mathcal{F}_0$ and property (a), $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Therefore there exists $\lim_{j \to \infty} x_j$ and

$$v(\lim_{j \to \infty} x_j, \lim_{j \to \infty} x_j) = \mu(v). \tag{10.88}$$

Since any sequence $\{x_j\}_{j=1}^{\infty} \subset K$ satisfying (10.87), converges in K, we conclude that there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$.

Let ϵ be a positive number. Fix a natural number q such that

$$4^{-q} < 8^{-1}\epsilon. \tag{10.89}$$

In view of the definition of $\mathcal{F}_0$ there exists $w \in \mathcal{A}$ and a natural number $i \geq q$ such that $v \in \mathcal{U}(w, i)$. It follows from property (a) that

$$||y_v - z(w, i)|| \leq 4^{-i}. \tag{10.90}$$

By (10.90), property (a) and (10.89), for each $u \in \mathcal{U}(w, i)$ and each $y \in K$ satisfying

$$u(y, y) \leq \mu(u) + 8^{-1}\delta(w, i)$$

the inequality $||y - y_v|| \leq \epsilon$ is true. This completes the proof of the proposition.

10.7 Proofs of Theorems 10.13 and 10.14

Let the sets $\mathcal{F}^{(0)}$, $\mathcal{F}_l^{(0)}$, $\mathcal{F}_c^{(0)}$ and $\mathcal{F}_u^{(0)}$ be as guaranteed in Proposition 10.17. For each $w \in \mathcal{F}^{(0)}$ there exists a unique $y_w \in K$ such that

$$w(y_w, y_w) = \mu(w). \tag{10.91}$$

Let $v \in \mathcal{F}^{(0)}$ and $\gamma \in (0, 1)$. Define

$$v_\gamma(x, y) = v(x, y) + \gamma(||x - y_v|| + ||y - y_v||), \; x, y \in K. \tag{10.92}$$

It is easy to see that $v_\gamma \in \mathcal{A}$, if $v \in \mathcal{A}_l$ ($\mathcal{A}_c$, $\mathcal{A}_u$, respectively), then $v_\gamma \in \mathcal{A}_l$ ($\mathcal{A}_c$, $\mathcal{A}_u$, respectively).

Lemma 10.18. *Let $\epsilon \in (0, 1)$. Then there exists a natural number n such that for each sequence $\{x_i\}_{i=0}^{n} \subset K$ which satisfies*

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + 4$$

there exists $j \in \{0, \ldots, n-1\}$ such that

$$||x_j - y_v||, \quad ||x_{j+1} - y_v|| \leq \epsilon.$$

Proof: Fix a natural number

$$n > (\epsilon\gamma)^{-1}(5 + 4(||v_\gamma|| + ||v||)).$$

Clearly,

$$n\mu(v) \leq \sigma(v, 0, n) + 2||v||.$$

Assume that $\{x_i\}_{i=0}^n \subset K$ satisfies

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, 0, n, x_0, x_n) + 4.$$

Define $\{y_i\}_{i=0}^n \subset K$ by

$$y_i = x_i,\ i = 0, n,\ y_i = y_v,\ i = 1, \dots n-1.$$

By the relations above,

$$\sigma(v, 0, n) + \gamma \sum_{i=0}^{n-1} (||x_i - y_v|| + ||x_{i+1} - y_v||) \le \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1})$$

$$\le \sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1}) + 4 \le 4||v_\gamma|| + 4 + nv(y_v, y_v)$$

$$= 4 + 4||v_\gamma|| + n\mu(v) \le 4 + 4||v_\gamma|| + \sigma(v, 0, n) + 2||v||,$$

$$\inf\{||x_i - y_v|| + ||x_{i+1} - y_v|| : i = 0, \dots, n-1\}$$
$$\le (n\gamma)^{-1}(4 + 4||v_\gamma|| + 2||v||) < \epsilon.$$

This completes the proof of the lemma.

Lemma 10.18 implies the following auxiliary result.

Lemma 10.19. *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathcal{U}$ of v_γ in $\mathcal{A}$ and an integer $n \ge 1$ such that for each $u \in \mathcal{U}$ and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \le \sigma(u, 0, n, x_0, x_n) + 3$$

there is $j \in \{0, \dots, n-1\}$ such that $||x_j - y_v||, ||x_{j+1} - y_v|| \le \epsilon$.

It is not difficult to see that

$$\sigma(v, 0, m, y_v, y_v) = m\mu(v) \text{ for all integers } m \ge 1, \tag{10.93}$$

$$\mu(v_\gamma) = \mu(v) = v(y_v, y_v) = v_\gamma(y_v, y_v) \tag{10.94}$$

and

$$\sigma(v_\gamma, 0, m, y_v, y_v) = m\mu(v_\gamma) \text{ for all integers } m \ge 1. \tag{10.95}$$

Lemma 10.20. *Let $\epsilon \in (0,1)$. Then there exists $\delta \in (0, \epsilon)$ such that for each natural number n and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$x_0 = x_n = y_v,\ \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, 0, n, y_v, y_v) + \delta, \tag{10.96}$$

the following inequality is true:

$$||x_i - y_v|| \le \epsilon,\ i = 0, \dots, n. \tag{10.97}$$

Proof: Fix
$$\delta \in (0, \epsilon\gamma). \tag{10.98}$$
Assume that $n \geq 1$ is an integer and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (10.96). Then it follows from (10.94)–(10.96), the definition of v_γ (see (10.92)) and (10.93) that

$$n\mu(v) = n\mu(v_\gamma) = \sigma(v_\gamma, 0, n, y_v, y_v) \geq$$
$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) - \delta = -\delta + \sum_{i=0}^{n-1} v(x_i, x_{i+1}) +$$
$$\gamma \sum_{i=0}^{n-1} (||x_i - y_v|| + ||x_{i+1} - y_v||) \geq$$
$$-\delta + \sigma(v, 0, n, y_v, y_v) + \gamma \sum_{i=0}^{n} ||x_i - y_v|| = -\delta + \gamma \sum_{i=0}^{n} ||x_i - v_v|| + n\mu(v)$$

and
$$\delta \geq \gamma \sum_{i=0}^{n} ||x_i - y_v||.$$

Together with (10.98) this implies that

$$||x_i - y_v|| \leq \delta\gamma^{-1} < \epsilon, \ i = 0, \dots, n.$$

This completes the proof of Lemma 10.20.

Lemma 10.21. *Let $\epsilon \in (0, 1)$. Then there exists $\delta \in (0, \epsilon)$ such that for each natural number n and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$||x_i - y_v|| \leq \delta, \ i = 0, n, \ \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + \delta, \tag{10.99}$$

the following inequality is true:

$$||x_i - y_v|| \leq \epsilon, \ i = 0, \dots, n. \tag{10.100}$$

Proof: In view of Lemma 10.20 there exists $\delta_1 \in (0, 2^{-1}\epsilon)$ such that for each integer $q \geq 1$ and each sequence $\{x_i\}_{i=0}^q \subset K$ which satisfies

$$x_0 = x_q = y_v, \ \sum_{i=0}^{q-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, q, y_v, y_v) + 2\delta_1 \tag{10.101}$$

the following inequality holds:

$$||x_i - y_v|| \leq \epsilon, \ i = 0, \dots, q. \tag{10.102}$$

Since v_γ is continuous at the point (y_v, y_v) there exists a positive number

$$\delta < \delta_1/4 \tag{10.103}$$

such that

$$|v_\gamma(z_1, z_2) - v_\gamma(y_v, y_v)| \le \delta_1/4 \tag{10.104}$$

for any $(z_1, z_2) \in K \times K$ satisfying

$$||z_i - y_v|| \le \delta, \ i = 1, 2. \tag{10.105}$$

Assume that $n \ge 1$ is an integer and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (10.99). We will show that (10.100) is valid. Define

$$y_0 = x_0, \ y_n = x_n, \ y_i = y_v \text{ for all } i \in \{0, \dots, n\} \setminus \{0, n\}, \tag{10.106}$$

$$z_i = y_v, \ i = 0, \dots, n. \tag{10.107}$$

It follows from (10.107), (10.91), (10.94), (10.95) and (10.93) that

$$\sum_{i=0}^{n-1} v_\gamma(z_i, z_{i+1}) = n\mu(v_\gamma) = n\mu(v) = \tag{10.108}$$

$$\sigma(v_\gamma, 0, n, y_v, y_v) = \sigma(v, 0, n, y_v, y_v).$$

By (10.106), (10.107), (10.99) and the definition of δ (see (10.103), (10.105)),

$$|\sum_{i=0}^{n-1} v_\gamma(z_i, z_{i+1}) - \sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1})| \le 2^{-1}\delta_1.$$

Combined with (10.108) this implies that

$$\sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1}) \le 2^{-1}\delta_1 + n\mu(v). \tag{10.109}$$

It follows from (10.99), (10.106) and (10.109) that

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, 0, n, x_0, x_n) + \delta \le$$

$$\delta + \sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1}) \le 2^{-1}\delta_1 + \delta + n\mu(v),$$

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le 2^{-1}\delta_1 + \delta + n\mu(v). \tag{10.110}$$

Define

$$\bar{y}_0 = y_v,\ \bar{y}_i = x_{i-1},\ i = 1, \dots, n+1,\ \bar{y}_{n+2} = y_v. \tag{10.111}$$

By (10.111), (10.99) and the definition of δ (see (10.103), (10.105)),

$$|v_\gamma(\bar{y}_0, \bar{y}_1) - v_\gamma(y_v, y_v)|,\ |v_\gamma(\bar{y}_{n+1}, \bar{y}_{n+2}) - v_\gamma(y_v, y_v)| \le \delta_1/4. \tag{10.112}$$

In view of (10.111), (10.112), (10.94), (10.110) and (10.103),

$$\sum_{i=0}^{n+1} v_\gamma(\bar{y}_i, \bar{y}_{i+1}) = v_\gamma(\bar{y}_0, \bar{y}_1) + v_\gamma(\bar{y}_{n+1}, \bar{y}_{n+2}) +$$

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le 2(\delta_1/4 + \mu(v)) + \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1})$$

$$\le n\mu(v) + \delta_1/2 + \delta + \delta_1/2 + 2\mu(v) =$$

$$(n+2)\mu(v) + \delta + \delta_1 < 2\delta_1 + (n+2)\mu(v)$$

and

$$\sum_{i=0}^{n+1} v_\gamma(\bar{y}_i, \bar{y}_{i+1}) \le (n+2)\mu(v) + 2\delta_1. \tag{10.113}$$

Relations (10.113), (10.111), (10.94) and (10.95) imply that

$$\bar{y}_i = y_v,\ i = 0, n+2,$$

$$\sum_{i=0}^{n+1} v_\gamma(\bar{y}_i, \bar{y}_{i+1}) \le 2\delta_1 + \sigma(v_\gamma, 0, n+2, y_v, y_v).$$

By these relations and the definition of δ_1 (see (10.101), (10.102)),

$$||\bar{y}_i - y_v|| \le \epsilon,\ i = 0, \dots, n+2.$$

Combined with (10.111) this implies that

$$||x_i - y_v|| \le \epsilon,\ i = 0, \dots, n.$$

This completes the proof Lemma 10.21.

Lemma 10.22. *Let $\epsilon \in (0,1)$. Then there exist $\delta \in (0, \epsilon)$, a neighborhood $\mathcal{U}$ of v_γ in $\mathcal{A}$ and a natural number N such that for each $u \in \mathcal{U}$, each integer $n \ge 2N$ and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \le \sigma(u, 0, n, x_0, x_n) + \delta \tag{10.114}$$

there exist $\tau_1 \in \{0, \dots, N\}$, $\tau_2 \in \{-N+n, n\}$ such that

$$||x_i - y_v|| \le \epsilon,\ t = \tau_1, \dots \tau_2,$$

and moreover, if $||x_0 - y_v|| \le \delta$, then $\tau_1 = 0$, and if $||x_n - y_v|| \le \delta$, then $\tau_2 = n$.

Proof: In view of Lemma 10.21 there exists $\delta \in (0, 4^{-1}\epsilon)$ such that for each natural number n and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies

$$||x_i - y_v|| \le 4\delta,\ i = 0, n,\ \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, 0, n, x_0, x_n) + 4\delta \quad (10.115)$$

the following inequality holds:

$$||x_i - y_v|| \le \epsilon,\ i = 0, \dots, n.$$

It follows from Lemma 10.19 that there exist a natural number N and a neighborhood $\mathcal{U}_1$ of v_γ in $\mathcal{A}$ such that for each $u \in \mathcal{U}_1$ and each sequence $\{x_i\}_{i=0}^N \subset K$ satisfying

$$\sum_{i=0}^{N-1} u(x_i, x_{i+1}) \le \sigma(u, 0, N, x_0, x_N) + 3 \quad (10.116)$$

there exists $j \in \{0, \dots, N-1\}$ for which

$$||x_j - y_v||,\ ||x_{j+1} - y_v|| \le \delta. \quad (10.117)$$

Define

$$\mathcal{U} = \{u \in \mathcal{U}_1 : \rho(u, v_\gamma) \le (16N)^{-1}\delta\}. \quad (10.118)$$

Assume that $u \in \mathcal{U}$, an integer $n \ge 2N$ and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (10.114). By (10.114) and the definition of $\mathcal{U}_1$, N (see (10.116), (10.117)) there exist integers τ_1, τ_2 such that

$$\tau_1 \in \{0, \dots, N\},\ \tau_2 \in \{n - N, \dots, n\},\ ||x_{\tau_i} - y_v|| \le \delta,\ i = 1, 2; \quad (10.119)$$

if $||x_0 - y_v|| \le \delta$, then $\tau_1 = 0$; if $||x_n - y_v|| \le \delta$, then $\tau_2 = n$.

We show that

$$||x_i - y_v|| \le \epsilon,\ t = \tau_1, \dots, \tau_2.$$

Assume the contrary. Then there exists an integer $s \in (\tau_1, \tau_2)$ such that

$$||x_s - y_v|| > \epsilon. \quad (10.120)$$

In view of (10.114), (10.119) and the definition of $\mathcal{U}_1$, N (see (10.116), (10.117)) there exist integers t_1, t_2 such that

$$\sup\{\tau_1, s - N\} \le t_1 < s,\ s < t_2 \le \inf\{\tau_2, s + N\},\ ||x_{t_i} - y_v|| \le \delta,\ i = 1, 2. \quad (10.121)$$

It follows from (10.118), (10.121) and (10.114) that

$$\sum_{i=t_1}^{t_2-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, t_1, t_2, x_{t_1}, x_{t_2}) + 2\delta. \quad (10.122)$$

By (10.121), (10.122) and the definition of δ (see (10.115)),

$$||x_t - y_t|| \leq \epsilon,\ t = t_1, \ldots, t_2.$$

This is contradictory to (10.120). The contradiction we have reached proves the lemma.

Let $v \in \mathcal{F}^{(0)}$, $\gamma \in (0,1)$ and let j be a natural number. There exist a natural number $N(v,\gamma,j)$, an open neighborhood $\mathcal{U}_0(v,\gamma,j)$ of v_γ in $\mathcal{A}$ and $\delta(v,\gamma,j) \in (0, 2^{-j})$ such that Lemma 10.22 holds with v, γ and $\epsilon = 2^{-j}$, $\delta = \delta(v,\gamma,j)$, $\mathcal{U} = \mathcal{U}_0(v,\gamma,j)$, $N = N(v,\gamma,j)$.

There are an open neighborhood $\mathcal{U}(v,\gamma,j)$ of v_γ in $\mathcal{A}$ and a natural number $N_1(v,\gamma,j)$ such that $\mathcal{U}(v,\gamma,j) \subset \mathcal{U}_0(v,\gamma,j)$ and Lemma 10.19 holds with v,γ and $\mathcal{U} = \mathcal{U}(v,\gamma,j)$, $n = N_1(v,\gamma,j)$, $\epsilon = 4^{-j}\delta(v,\gamma,j)$.

Define

$$\mathcal{F} = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(v,\gamma,j) : v \in \mathcal{F}^{(0)}, \gamma \in (0,1), j = q, q+1, \ldots\}] \cap \mathcal{F}^{(0)}, \quad (10.123)$$

$$\mathcal{F}_l = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(v,\gamma,j) : v \in \mathcal{F}_l^{(0)}, \gamma \in (0,1), j = q, q+1, \ldots\}] \cap \mathcal{F}_l^{(0)},$$

$$\mathcal{F}_c = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(v,\gamma,j) : v \in \mathcal{F}_c^{(0)}, \gamma \in (0,1), j = q, q+1, \ldots\}] \cap \mathcal{F}_c^{(0)}$$

and

$$\mathcal{F}_u = [\cap_{q=1}^{\infty} \cup \{\mathcal{U}(v,\gamma,j) : v \in \mathcal{F}_u^{(0)}, \gamma \in (0,1), j = q, q+1, \ldots\}] \cap \mathcal{F}_u^{(0)}.$$

It is easy to see that $\mathcal{F}$ ($\mathcal{F}_l$, $\mathcal{F}_c$, $\mathcal{F}_u$, respectively) is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ ($\mathcal{A}_l$, $\mathcal{A}_c$, $\mathcal{A}_u$, respectively).

Theorem 10.13 follows from Proposition 10.17 and the definition of $\mathcal{F}$.

Proof of Theorem 10.14: Let $w \in \mathcal{F}$ and ϵ be a positive number. We may assume that $\epsilon < 1$. Fix a natural number q such that

$$64 \cdot 2^{-q} < \epsilon. \quad (10.124)$$

There exist $v \in \mathcal{F}^{(0)}$, $\gamma \in (0,1)$ and an integer $j \geq q$ such that

$$w \in \mathcal{U}(v,\gamma,j). \quad (10.125)$$

It follows from (10.125), Lemma 10.19 which holds with $\mathcal{U} = \mathcal{U}(v,\gamma,j)$, $n = N_1(v,\gamma,j)$, $\epsilon = 4^{-j}\delta(v,\gamma,j)$, v, γ, and the equality

$$\sigma(w, 0, N_1(v,\gamma,j), y_w, y_w) = N_1(v,\gamma,j)\mu(w)$$

that

$$||y_w - y_v|| \leq 4^{-j}\delta(v,\gamma,j). \quad (10.126)$$

Put

$$\mathcal{U} = \mathcal{U}(v,\gamma,j),\ N = N(v,\gamma,j),\ \delta = 4^{-j}\delta(v,\gamma,j). \quad (10.127)$$

Assume that $u \in \mathcal{U}$, an integer $n \geq 2N$ and a sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfies

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + \delta. \tag{10.128}$$

By (10.128), (10.127), the definition of $\mathcal{U}_0(v, \gamma, j)$, $N(v, \gamma, j)$, $\delta(v, \gamma, j)$ and Lemma 10.22, there exist $\tau_1 \in \{0, \dots, N\}$, $\tau_2 \in \{n - N, \dots, n\}$ such that

$$||x_i - y_v|| \leq 2^{-j}, \ t = \tau_1, \dots, \tau_2.$$

Moreover if $||x_0 - y_v|| \leq \delta(v, \gamma, j)$, then $\tau_1 = 0$, and if $||x_n - y_v|| \leq \delta(v, \gamma, j)$, then $\tau_2 = n$. Combined with (10.126), (10.124) and (10.127) this implies that

$$||x_i - y_w|| \leq 2^{1-j} < \epsilon, \ i = \tau_1, \dots, \tau_2;$$

if $||x_0 - y_w|| \leq \delta$, then $||x_0 - y_v|| \leq \delta(v, \gamma, j)$ and $\tau_1 = 0$; if $||x_n - y_w|| \leq \delta$, then $||x_n - y_v|| \leq \delta(v, \gamma, j)$ and $\tau_2 = n$. This completes the proof of Theorem 10.14.

10.8 Control systems on metric spaces

Let (X, ρ) be a metric space. Denote by $\mathcal{A}$ the set of all bounded functions $v : X \times X \to R^1$. Set $\Delta = X \times X$.

We equip the set $\mathcal{A}$ with the metric

$$d(u, v) = \sup\{|v(x, y) - u(x, y)| : \ x, y \in K\}, \ u, v \in \mathcal{A}. \tag{10.129}$$

Evidently $(\mathcal{A}, d)$ is a complete metric space. Denote by $\mathcal{A}_l$ the set of all lower semicontinuous functions $v \in \mathcal{A}$, by $\mathcal{A}_c$ the set of all continuous functions $v \in \mathcal{A}$ and by $\mathcal{A}_u$ the set of all uniformly continuous functions $v \in \mathcal{A}$. Clearly $\mathcal{A}_l, \mathcal{A}_c$ and $\mathcal{A}_u$ are closed subsets of the complete metric space $(\mathcal{A}, d)$.

Let $v \in \mathcal{A}$. Define a minimal growth rate

$$\mu(v) = \inf\{\liminf_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \ \{z_i\}_{i=0}^{\infty} \subset X\}. \tag{10.130}$$

Clearly

$$\mu(v) \leq \inf\{v(x, x) : \ x \in X\}. \tag{10.131}$$

Denote by $\mathcal{A}_*$ the set of all $v \in \mathcal{A}$ such that

$$\mu(v) = \inf\{v(x, x) : \ x \in X\}. \tag{10.132}$$

Clearly $\mathcal{A}_*$ is a closed subset of $(\mathcal{A}, d)$. Set

$$\mathcal{A}_{*l} = \mathcal{A} \cap \mathcal{A}_l, \ \mathcal{A}_{*c} = \mathcal{A}_* \cap \mathcal{A}_c, \ \mathcal{A}_{*u} = \mathcal{A}_* \cap \mathcal{A}_u. \tag{10.133}$$

Clearly $\mathcal{A}_* \neq \emptyset$. For example, if $v(x, y) = c$ for all $(x, y) \in X \times X$ where c is a constant, then $v \in \mathcal{A}_{*c}$.

The following proposition was obtained in [131].

Proposition 10.23. *Let $v \in \mathcal{A}_*$. Then for each $x \in X$ there is a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that $x_0 = x$ and*

$$\limsup_{N\to\infty} N^{-1} \sum_{t=0}^{N-1} v(x_i, x_{i+1}) = \mu(v).$$

For each $v \in \mathcal{A}$ set

$$||v|| = \sup\{|v(x,y)| :\ x, y \in X\}.$$

For each $x \in X$ and each subset $B \subset X$ set

$$\rho(x, B) = \inf\{\rho(x,y) :\ y \in B\}.$$

Denote by Card(A) the cardinality of a set A.

Denote by $\mathcal{F}$ the set of all $v \in \mathcal{A}_*$ which have the following property:

(P) For each $\epsilon > 0$ there exists $\delta > 0$ and a neighborhood $\mathcal{V}$ of v in $\mathcal{A}$ such that for each $u \in \mathcal{V}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$\limsup_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} u(x_i, x_{i+1}) \le \mu(u) + \delta$$

the following relation holds:

$$\limsup_{N\to\infty} N^{-1} \mathrm{Card}\{i \in \{0, \dots, N-1\} :\ \rho(x_i, x_{i+1}) \ge \epsilon\} \le \epsilon.$$

If $v \in \mathcal{A}_*$ has property (P), then all good programs spend most of the time in a small neighborhood of the diagonal Δ.

We show that most elements of $\mathcal{A}_*$ (in the sense of Baire's categories) have property (P). Moreover, we show that the complement of the set of all functions which have property (P) is not only of the first category, but also σ-porous.

We prove the following result obtained in [131].

Theorem 10.24. *The set $\mathcal{A}_* \subset \mathcal{F}$ ($\mathcal{A}_{*l} \setminus \mathcal{F}$, $\mathcal{A}_c \setminus \mathcal{F}$, $\mathcal{A}_{*u} \setminus \mathcal{F}$, respectively) is a σ-porous subset of $\mathcal{A}_*$ ($\mathcal{A}_{*l}$, $\mathcal{A}_{*c}, \mathcal{A}_{*u}$, respectively).*

10.9 Proof of Proposition 10.23

Let $x \in X$. For each natural number n there is $z_n \in X$ such that

$$v(z_n, z_n) \le \mu(v) + 2^{-n}. \tag{10.134}$$

Define a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ as follows:

$$x_0, x_1 = x,\ x_n = z_k,\ n = 2^k, \dots, 2^{k+1} - 1,\ k = 1, 2, \dots. \tag{10.135}$$

We show that

$$\limsup_{N\to\infty} N^{-1}\sum_{i=0}^{N-1} v(x_i,x_{i+1}) = \mu(v). \tag{10.136}$$

Clearly

$$\limsup_{N\to\infty} N^{-1}\sum_{i=0}^{N-1} v(x_i,x_{i+1}) \geq \mu(v). \tag{10.137}$$

Let $N \geq 9$ be a natural number. There is a natural number $k = k(N)$ such that

$$2^k \leq N < 2^{k+1}. \tag{10.138}$$

Set

$$p = N - 2^k. \tag{10.139}$$

Then by (10.135), (10.138), (10.139) and (10.134)

$$\sum_{i=0}^{N-1} v(x_i,x_{i+1}) = v(x,x) + v(x,z_1) + v(z_1,z_1)$$

$$+\sum_{j=2}^{k-1}[v(z_{j-1},z_j) + (2^j-1)v(z_j,z_j)] + (v(z_{k-1},z_k) + pv(z_k,z_k))$$

$$\leq 3||v|| + \sum_{j=2}^{k-1}[2^j v(z_j,z_j) + 2||v||] + [(p+1)v(z_k,z_k) + 2||v||]$$

$$\leq \sum_{j=2}^{k-1} 2^j v(z_j,z_j) + (p+1)v(z_k,z_k) + 2||v||(k+1)$$

$$\leq \sum_{j=2}^{k-1} 2^j(\mu(v) + 2^{-j}) + (p+1)[\mu(v) + 2^{-k}] + 2||v||(k+1)$$

$$= \mu(v)[\sum_{j=2}^{k-1} 2^j + (p+1)] + k + 2||v||(k+1)$$

$$\leq \mu(v)[2^k - 1 - 3 + (p+1)] + (k+1)(2||v|| + 1)$$
$$\leq \mu(v)(N-3) + (\log_2 N + 1)(2||v|| + 1). \tag{10.140}$$

This relation implies that

$$\limsup_{N\to\infty} N^{-1}\sum_{i=0}^{N-1} v(x_i,x_{i+1}) \leq$$

$$\limsup_{N\to\infty}[N^{-1}\mu(v)(N-3) + (2||v||+1)(\log_2 N + 1)N^{-1}] = \mu(v).$$

Proposition 10.23 is proved.

10.10 An auxiliary result for Theorem 10.24

Let
$$v \in \mathcal{A}_*, \ \gamma \in (0,1]. \tag{10.141}$$
Define
$$v_\gamma(x,y) = v(x,y) + \gamma \min\{1, \rho(x,y)\}, \ x, y \in X. \tag{10.142}$$
Clearly
$$v_\gamma \in \mathcal{A}_*, \ \mu(v_\gamma) = \mu(v), \tag{10.143}$$
if $v \in \mathcal{A}_{*l}$ ($\mathcal{A}_{*c}$, $\mathcal{A}_{*u}$, respectively), then $v_\gamma \in \mathcal{A}_{*l}$ ($\mathcal{A}_{*c}$, $\mathcal{A}_{*u}$, respectively).

Lemma 10.25. *Let $\delta > 0$, $u \in \mathcal{A}_*$ satisfy*
$$d(u, v_\gamma) \le \delta \tag{10.144}$$
and let $\{x_i\}_{i=0}^\infty \subset X$ satisfy
$$\limsup_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} u(x_i, x_{i+1}) \le \mu(u) + \delta. \tag{10.145}$$
Then for each $\epsilon \in (0,1]$,
$$\limsup_{N \to \infty} N^{-1} \mathrm{Card}\{i \in \{0, \dots, N-1\} : \ \rho(x_i, x_{i+1}) \ge \epsilon\} \le (3\delta)(\gamma\epsilon)^{-1}. \tag{10.146}$$

Proof: Let $\epsilon \in (0,1]$. By (10.144)
$$|\limsup_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} u(x_i, x_{i+1}) - \limsup_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} v_\gamma(x_i, x_{i+1})| \le \delta \tag{10.147}$$
and
$$|\mu(u) - \mu(v_\gamma)| \le \delta. \tag{10.148}$$
In view of (10.145), (10.147), (10.148) and (10.143)
$$\limsup_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} v_\gamma(x_i, x_{i+1}) \le \mu(v_\gamma) + 3\delta = \mu(v) + 3\delta. \tag{10.149}$$
It follows from (10.149), (10.142) and the definition of $\mu(v)$ that
$$\mu(v) + 3\delta \ge \limsup_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} [v(x_i, x_{i+1}) + \gamma \min\{1, \rho(x_i, x_{i+1})\}]$$

$$\geq \limsup_{N\to\infty} N^{-1}[\sum_{i=0}^{N-1} v(x_i, x_{i+1}) + \gamma\epsilon \mathrm{Card}\{i \in \{0, \ldots, N-1\} : \ \rho(x_i, x_{i+1}) \geq \epsilon\}]$$

$$\geq \limsup_{N\to\infty} N^{-1}\gamma\epsilon \mathrm{Card}\{i \in \{0, \ldots, N-1\} : \ \rho(x_i, x_{i+1}) \geq \epsilon\}$$

$$+ \liminf_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} v(x_i, x_{i+1})$$

$$\geq \limsup_{N\to\infty} \gamma\epsilon N^{-1}\mathrm{Card}\{i \in \{0, \ldots, N-1\} : \ \rho(x_i, x_{i+1}) \geq \epsilon\} + \mu(v). \quad (10.150)$$

This inequality implies that

$$\limsup_{N\to\infty} N^{-1}\mathrm{Card}\{i \in \{0, \ldots, N-1\} : \ \rho(x_i, x_{i+1}) \geq \epsilon\} \leq (3\delta)(\gamma\epsilon)^{-1}.$$

Lemma 10.25 is proved.

10.11 Proof of Theorem 10.24

For each natural number n denote by $\mathcal{F}_n$ the set of all $v \in \mathcal{A}_*$ which have the following property:

(P1) There exist $\delta > 0$ and a neighborhood $\mathcal{V}$ of v in $\mathcal{A}_*$ such that for each $u \in \mathcal{V}$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$\limsup_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} u(x_i, x_{i+1}) \leq \mu(u) + \delta \quad (10.151)$$

the following relation holds:

$$\limsup_{N\to\infty} N^{-1}\mathrm{Card}\{i \in \{0, \ldots, N-1\} : \ \rho(x_i, x_{i+1}) \geq n^{-1}\} \leq 1/n. \quad (10.152)$$

It is not difficult to see that $\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}_n$. In order to prove Theorem 10.24 it is sufficient to show that for each integer $n \geq 1$, $\mathcal{A}_* \setminus \mathcal{F}_n$ ($\mathcal{A}_{*l} \setminus \mathcal{F}_n$, $\mathcal{A}_{*c} \setminus \mathcal{F}_n$, $\mathcal{A}_{*u} \setminus \mathcal{F}_n$, respectively) is a porous subset of $\mathcal{A}_*$ ($\mathcal{A}_{*l}$, $\mathcal{A}_{*c}$, $\mathcal{A}_{*u}$, respectively).

Let n be a natural number. Set

$$\alpha = (32n^2)^{-1}. \quad (10.153)$$

Assume that

$$v \in \mathcal{A}_*, \ r \in (0, 1]. \quad (10.154)$$

Put

$$\gamma = 6\alpha r n^2 < r/4 \quad (10.155)$$

and define

$$v_\gamma(x,y) = v(x,y) + \gamma \min\{1, \rho(x,y)\},\ x, y \in X. \tag{10.156}$$

Clearly $v_\gamma \in \mathcal{A}_*$,

$$\mu(v_\gamma) = \mu(v) \tag{10.157}$$

and if $v \in \mathcal{A}_{*l}$ ($\mathcal{A}_{*c}$, $\mathcal{A}_{*u}$, respectively), then $v_\gamma \in \mathcal{A}_{*l}$ ($\mathcal{A}_{*c}$, $\mathcal{A}_{*u}$, respectively).

By (10.156) and (10.155)

$$d(v, v_\gamma) \le \gamma \le r/4. \tag{10.158}$$

Assume that

$$u \in \mathcal{A}_*,\ d(u, v_\gamma) \le 2\alpha r, \tag{10.159}$$

$$\{x_i\}_{i=0}^\infty \subset X,\ \limsup_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} u(x_i, x_{i+1}) \le \mu(u) + 2\alpha r. \tag{10.160}$$

In view of (10.158), (10.159) and (10.153)

$$d(u, v) \le r. \tag{10.161}$$

It follows from (10.159), (10.160), Lemma 10.25 (with $\delta = 2\alpha r$, $\epsilon = 1/n$) and (10.155) that

$$\limsup_{N\to\infty} N^{-1}\mathrm{Card}\{i \in \{0, \dots, N-1\} :\ \rho(x_i, x_{i+1}) \ge 1/n\}$$

$$\le (6\alpha r)(\gamma/n)^{-1} = 6\alpha r \gamma^{-1} n \le 1/n. \tag{10.162}$$

We have shown that each $u \in \mathcal{A}_*$ satisfying $d(u, v_\gamma) \le \alpha r$ belongs to $\mathcal{F}_n$ and satisfies (10.161). This completes the proof of Theorem 10.24.

10.12 Comments

In this chapter we consider infinite horizon minimization problems. It should be mentioned that the study of properties of solutions of these problems defined on infinite intervals has recently been a rapidly growing area of research. See, for example, [4, 11, 12, 62, 63, 64, 67, 69, 78, 89, 101, 102, 107, 130] and the references mentioned therein. These problems arise in engineering [130], in models of economic growth [67, 69, 89, 130], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 101] and in the theory of thermodynamical equilibrium for materials [64].

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